A note on coloring powers of cycles *

Martín F. Jiménez † Mario Valencia-Pabon †

Abstract

Let G denotes the graph a-th power of the n-cycle C_n. In this note is given a simple and linear algorithm to proper color the vertices of G by using $\chi(G)$ colors.

Keywords: Graph coloring algorithms, independent sets, powers of cycles.

1 Introduction

A *coloring* (i.e. *proper coloring*) of a graph $G = (V, E)$ is an assignment of colors to the vertices of G, such that any two adjacent vertices have different colors. A k-coloring is one that uses k colors. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum integer k for which G has a k-coloring. The independence number of a graph G, denoted by $\alpha(G)$, is defined as the maximum number of pairwise non adjacent vertices in G. As is well known, the chromatic number of a graph G and its independence number are closely related via the inequality $\chi(G) \geq \lceil |G|/\alpha(G) \rceil$.

For positive integers n and a such that $a \leq n/2$, we denote by $C(n, a)$ the graph with vertex set $\{0, 1, \ldots, n-1\}$ and edge set $\{ij : i - j \equiv \pm k \mod n, 1 \leq k \leq a\}$; the graph $C(n, a)$ is the a-th power of the n-cycle $C(n, 1)$. It is easy to note that the clique number (i.e. the maximal size of a clique) of $C(n, a)$ is equal to $a + 1$.

Let $n = q(a + 1) + r$, with $q > 0$ and $0 \leq r \leq a$. Concerning the independence number of $C(n, a)$, we can deduce that $\alpha(C(n, a)) = q$. In fact, the subset $\{a + 1, 2(a + 1), 3(a + 1), \ldots, q(a + 1)\}$ (arithmetic operations are taken modulo n) is an independent set of $C(n, a)$ with size q. Moreover, let I be a maximal independent set in $C(n, a)$. As $C(n, a)$ is a vertex-transitive graph, we can assume that $0 \in I$. Consider the subsets of vertices $C_i = \{i(a + 1), i(a + 1) + 1, i(a + 1) + 2, \ldots, i(a + 1) + a\}$, for $i = 0, 1, \ldots, q - 1$, and let $C_q = \{q(a + 1), q(a + 1) + 1, \ldots, n - 1\}$. The subsets C_i, for $0 \leq i \leq q$, constitute a clique decomposition of $C(n, a)$, where for $0 \leq i < q$, the subset C_i has size $a + 1$, and the subset C_q has size r. Therefore I can contain at most one element of each such subsets. Now, $C_q \cup \{0\}$ is a clique. Therefore, I doesn't contain any element of C_q, and so $|I| \leq q$. From these results, we obtain that $\chi(C(n, a)) \geq a + 1 + \lceil r/q \rceil$.

Prowse and Woodall analyze in [3] a restricted coloring problem (*the list coloring problem*) on the graphs $C(n, a)$. In particular, they show the following result.

Theorem 1 *(Prowse-Woodall, [3]) Let n and a positive integers such that $n \geq 2a$ and $n = q(a + 1) + r$, with $q > 0$ and $0 \leq r \leq a$. Then, $\chi(C(n, a)) = a + 1 + \lceil r/q \rceil$.*

In this note, we present a simple and linear algorithm to color the graph $C(n, a)$ by using $\chi(C(n, a))$ colors.

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*This work was supported by the Facultad de Ciencias de la Universidad de los Andes, Bogotá, Colombia

†Departamento de Matemáticas, Universidad de los Andes, Cra. 1 No. 18A - 70, Bogotá, Colombia (ma-jimen@uniandes.edu.co, mvalenci@uniandes.edu.co).

2 The algorithm

A *circular arc family* is a set $F = \{A_1, A_2, \ldots, A_n\}$ of arcs on a circle. A circular arc family is *proper* if no arc is contained within another. A graph is a (*proper*) *circular arc graph* if there is a $1:1$ correspondence between the vertices of the graph and the arcs of a (proper) circular arc family such that two vertices of the graph are adjacent if an only if the corresponding arcs overlap.

It is easy to check that the graph $C(n,a)$ is a proper circular arc graph. In fact, let $x_0, x_1, \ldots, x_{n-1}$ be n different points ordered in the clockwise direction on a circle, and let $F = \{A_0, A_1, \ldots, A_{n-1}\}$ be a family of arcs on the circle such that $A_i = (x_i, x_j)$, where $j = i + a + 1 \mod n$, for $i = 0, 1, \ldots, n-1$. Thus, the circular arc graph associated with the family F is a proper circular arc graph which is isomorphic to $C(n,a)$.

Orlin, Bonuccelli and Bovet give in [2] an $O(n^2)$ algorithm to k-color a family of n proper circular arcs (whenever possible). In [4] (see also [1]) Teng and Tucker improve the result of Orlin et al. by giving an $O(kn)$ algorithm to k-color a family of n proper circular arcs. In the particular case of a power of a cycle graph $C(n,a)$, we give in this section a very simple algorithm which efficient color such a graph using exactly n steps.

Definition 1 *Let k be a positive integer. A* **coloring modulo** *k of $C(n,a)$ is a color function which assigns to each vertex i of $C(n,a)$ the color $i \mod k$.*

Definition 2 *Let $V = \{0, 1, \ldots, n-1\}$ be the vertex set of the graph $C(n,a)$, and let $C \subseteq V$. C is called a* **consecutive clique** *of $C(n,a)$ if C is a clique of $C(n,a)$ and it is composed of a sequence of consecutive integers (addition is taken modulo n).*

The following lemma was proved by Orlin, Bonuccelli and Bovet [2] for proper circular arc graphs, which we rephrase in terms of consecutive clique in $C(n,a)$.

Lemma 1 *(Orlin-Bonuccelli-Bovet, [2]) Let n and k be positive integers such that k is a divisor of n. Then, $C(n,a)$ can be colored with k colors if and only if $C(n,a)$ has no consecutive clique of size $k+1$.*

Theorem 2 *There is a simple and linear algorithm to color $C(n,a)$ by using $\chi(C(n,a))$ colors.*

Proof : Let $n = q(a+1) + r$, with $q \geq 1$ and $0 \leq r \leq a$. We consider two cases:

- *Case 1:* $r = 0$. In this case, $a+1$ divides n and by Lemma 1, $C(n,a)$ can be colored using $a+1$ colors. Moreover, the clique number of $C(n,a)$ is equal to $a+1$.
- *Case 2:* $r \neq 0$. Let $k = \lceil \frac{r}{q} \rceil$ and let $t = \lfloor \frac{r}{k} \rfloor$. As $r > 0$, we have $k > 0$ and $t > 0$. Let $\chi = a + 1 + k$. Then,
  - *Case 2.1:* $q = t$. In this case, color $C(n,a)$ using a coloring modulo χ. Notice that, $q = t \leq \frac{r}{k} \leq \frac{r}{(\frac{r}{q})} = q$, and thus, $k = \frac{r}{q}$, which implies that $\chi = a + 1 + \frac{r}{q}$ and so, $q\chi = q(a+1) + r = n$. Therefore, $\chi | n$ and by Lemma 1, the coloring modulo χ is a proper coloring of $C(n,a)$.
  - *Case 2.2:* $q \neq t$. Let $w = a + 1 + r - kt$. So, in this case we color the vertices of $C(n,a)$ as follows:
    - Color each vertex $i \in \{0, 1, \ldots, t\chi + w - 1\}$ with color $i \mod \chi$.

∗ Color each vertex $i \in \{t\chi + w, \ldots, n-1\}$ with color $i - (t\chi + w) \mod (a+1)$.

Notice that $0 \le r - kt \le k$. In fact, on one hand, $r = k\left(\frac{r}{k}\right) \ge k\left\lfloor\frac{r}{k}\right\rfloor = kt$. On other hand, $r - k = k\left(\frac{r-k}{k}\right) = k\left(\frac{r}{k} - 1\right) \le k\left\lfloor\frac{r}{k}\right\rfloor = kt$. So, $r - k \le kt \le r$ which implies that $0 \le r - kt \le k$. Thus, we have $a + 1 \le w \le \chi$. As $t \le r/k \le r/(r/q) = q$ and $t \ne q$, then $t \le q - 1$. Moreover, it is easy to check that $n - t\chi - w = (q - t - 1)(a + 1) \ge 0$, that is, the cardinality of the subset of vertices $\{t\chi + w, \ldots, n-1\}$ is a multiple of $(a+1)$. Now, we should to prove that this coloring is a proper coloring of $C(n, a)$. By construction, vertex $t\chi + w - 1$ is colored with a color c such that $a \le c \le \chi - 1$, and vertex $n - 1$ is colored with color a. This proves that previous coloring is a proper coloring that uses at most χ colors. □

References

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