Ph. D. Thesis

FUNDAMENTAL COMPUTER SCIENCE
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presented by

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Subject

Combinatorial Algorithms for Non compact Lie groups relevant to physics

realized under the supervision of

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A la mémoire,
et en temoignage de gratitude
à mes grand-parents,
Madeleine, Odette,
Henri et Lucien

Thanks

When I was five years old, I asked my father to why the sky is blue. From this event I can give the date of my interest for science and it is sure that if the response was not at the level of a young child it was a nice mystery to discover. Like this phenomenom that I learned about many years later, doing a PhD thesis in combinatorics and physics was a continuation of this experiment where I was during months in a stimulatimg atmosphere to search and sometime discover solutions of questions and problems. But on this way to learn and to learn to learn, I discovered another dimension through my work, the human environment. As usual, it is more convenient to work with good human relations but I will say here that it is more that good relations that I found around me but a real family. I especially thank Jean-Yves Thibon and Brian G. Wybourne who accepted me as PhD student and spend so much time with me in various discussions, in France and in Poland. Also I want to thank Ronald C. King for his patient help and discussions that he provided me during meetings and a visit to Southampton. If these three persons represent the kernel of my research family I don't forget the *phalanstère de combinatoire* where I discovered a good ambiance to do research. I want to thank the members of this team, Jacques Désarmenien, Alain Lascoux, Bernard Leclerc, Pierre-André Picon and the PhD students, some of whom are now Doctor, Bruno Gauthier, Florent Hivert, Vincent Prosper, Bun Chan Vorac Ung for this period of common life.

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Certainly as Wim Wenders wrote "Am Ende unsere Geschichte fängt eine gerade an" but this story was one of my nicest.

Abstract

In many domains of science it is necessary to count objects and to do long computations to validate models or to improve conjectures. Theoretical computer science provides in this way different methods and algorithms to approach various domains of science and to help to solve problems. It is not surprising that combinatorics plays a crucial role and allows many scientists to count objects and exhibit new relations on these objects. Also the relation between algebraic combinatorics, symmetric functions and group theory conduces to examine applications of such methods in the fields of theoritical physics where it is necessary to produce enormous calculation on non-compact group representations. The high complexity of the algorithms requires in this case to develop new methods to calculate more easily the decomposition of classes of tensor products or plethysms that allow the construction of new algorithms that are more efficient and that can be use for practical calculation in physical models. This thesis provides a theoretical background to wards understanding the sequel in which is developed the techniques for dealing with infinite dimensional representations of certain non compact groups. Among these groups we have the real symplectic group for which we present two illustrative applications. In all cases we recall the connexion with practical examples computed with the program SCHUR that contains all the new features corresponding to the new presented results. Indeed we also give for certain cases a very simple Maple program to exhibit the simplicity of the new methods of calculation. New results are also presented for the irreducible representation of the non compact groups $SO^\star(2n)$, $Sp(2n,\mathbb{R})$ and $U(p,q)$ and for finite groups that embed in the exceptional group G_2.

Résumé

Dans de nombreux domaines scientifiques il est nécessaire de compter des objets ou de réaliser de longs calculs pour valider des modèles ou tester des conjectures. L'informatique théorique produit à cette fin de nombreux algorithmes et méthodes ayant pour vocation d'aider à résoudre ces problèmes. Il n'est pas surprenant que la combinatoire algébrique joue ici un grand rôle en offrant des moyens de compter des objets ou d'établir des relations sur ces objets. Aussi, les liens entre combinatoire algébrique, fonctions symétriques et théorie des groupes conduisent à examiner l'application de ses méthodes en physique théorique où il est nécessaire de produire des calculs de grande envergure sur les représentations des groupes non compacts. La grande complexité de ces algorithmes requiert le développement de nouvelles méthodes pour calculer plus facilement la décomposition de certaines classes de produit tensoriels et pléthysmes. Ceci permet alors la construction de nouveaux algorithmes plus efficaces utilisés pour des calculs sur les modèles physiques. Cette thèse présente après des rappels théoriques des techniques utilisées pour les calculs sur les représentations de dimension infinie de certains groupes non compacts, parmi lesquels le groupe symplectique réel. Pour ce groupe nous donnons deux exemples d'illustration. Plus généralement sont présentés des exemples de calcul réalisés avec SCHUR, programme augmenté des résultats présentés. Pour certains cas on donne des exemples de programme Maple montrant la simplicité des nouvelles méthodes de calculs. De nouveaux résultats sont aussi présentés pour les représentations irréductibles des groupes non compacts $SO^{\star}(2n)$, $Sp(2n,\mathbb{R})$ and $U(p,q)$ et pour les sous-groupes finis du groupe exceptionnel G_2.

Abstrakt

W wielu dziedzinach nauki koniecznym jest zliczanie obiektów oraz wykonywanie obliczeń mających na celu potwierdzenie poprawności modeli lub ulepszenie hipotez. Teoretyczne nauki komputerowe dostarczają różnych metod i algorytmów pomocnych w rozwiązywaniu problemów z różnych dziedzin nauki. Nie jest więc niczym zaskakującym, że kombinatoryka odgrywa kluczową rolę pozwalając wielu badaczom zliczać obiekty i odkrywać nowe związki między tymi obiektami. Związek między kombinatoryką algebraiczną, teorią funkcji symetrycznych i teorią grup przyczynia się również do badania przydatności tych metod w fizyce teoretycznej, gdzie koniecznym jest wykonywanie wielkich obliczeń dla reprezentacji grup niezwartych. Ze względu na wielką złożoność algorytmów konieczne jest rozwijanie nowych metod umożliwiających łatwiejsze rozkładanie iloczynów tensorowych lub pletyzmów na klasy, dając tym samym bardziej efektywne algorytmy umożliwiające praktyczne obliczenia w ramach różnych modeli fizycznych. Prezentowana praca daje teoretyczne podstawy, pozwalające na rozwinięcie i zrozumienie technik posługiwania się nieskończenie wymiarowymi reprezentacjami pewnych grup niezwartych. Wśród rozważanych grup znajduje się rzeczywista grupa symplektyczna, dla której przedstawiamy dwa przykładowe zastosowania. Wszystkie przykłady i praktyczne obliczenia zostały wykonane z wykorzystaniem programu SCHUR posiadającego wszystkie możliwości niezbędne do uzyskania nowo prezentowanych rezultatów. W niektórych przypadkach, dla zaprezentowania prostoty nowych metod obliczeniowych, podajemy bardzo proste programy napisane w języku Maple. Nowe wyniki prezentujemy również dla nieredukowalnych reprezentacji niezwartych grup $SO^\star(2n)$, $Sp(2n,\mathcal{R})$ i $U(p,q)$, oraz dla grup skończonych zawierających się w grupie G_2.

Contents

Introduction

Combinatorics is related to many fields of science in the sense that scientists need to enumerate collections of objects. The fact that we can use computers for numerical or symbolic calculations gives an enhancement to many domains of mathematics and physics. Also, the development of computing produces new problems to solve, part of them are strictly computer science problems like questions of complexity , calculability and also problems coming from establishing new conjectures on the basis of calculation.

Theoretical physics can be seen at the border of these two classes of problems. First because physicists very early were using computers and indeed built them. Then the production of results engaged a research to the best algorithms and to obtain the most powerful engine. On the other side one can consider that modern theoretical physics is using plenty of computer methods with numerical applications or formal languages to produce new results, verify hypotheses or test conjectures.

Algebraic combinatorics

At this stage we can remark that physicists after the creation of quantum mechanics were using more and more group theory to search and analyse the symmetry of problems, to develop methods of doing calculations in atomic and nuclear physics. At the same time there appeared in mathematics a great development of group theory and an expansion of new methods in representation theory based on diagrammatic methods using Young tableaux, the theory of partitions and of symmetric functions.

Moreover there appears in physics standard combinatorial objects like Dyck paths, graphs, Stirling numbers etc. that also occurs in representation theory but from a totally different point of view. Algebraic combinatorics has established links between these combinatorial objects and some algebraic structures. It is no longer surprising, but very helpful, to calculate Clebsch-Gordan coefficients for $SU(2)$ using binary trees or to recognize the link between non-crossed partitions, Penrose diagrams and non commutative probability.

Finally, the domain of algebraic combinatorics can be considered as the border between domains requiring deep calculations and the exploitation of algebraic structures underlying the problem considered.

Combinatorics and physics

Condensed matter is certainly a part of physics where people need to use a very large amount of group theory to develop models and make detailed calculations. The fact that we can describe the representations using partitions, Young tableaux and finally symmetric functions is really the basis of the collaboration between combinatorics and physics. In this way, we have the opportunity to develop in group theory questions having many different kinds of solutions such as, for example, tensor product decompositions being resolved in terms of characters that are themselves special cases of symmetric functions.

Indeed, important results like the Littlewood-Richardson rule, show the fundamental role that the study of symmetric functions has for determining characters, multiplicity in tensor powers and finally to give clearly the methods to compute such decompositions.

To go further it is necessary to develop symmetric function theory. By providing the plethysm operation, inner plethysm, reduced notation and techniques for generating series, the theory of symmetric functions provides a wide domain of application, extending its influence and giving algorithms to simplify calculations in various domains of mathematics.

The case of non-compact groups

The use of symmetric functions can be now be considered as a basic method for group character theory. There are now several programs or packages providing techniques involving symmetric functions to calculate character tables for the symmetric group or more generally the classical groups. In fact, if we consider classical groups we need to specify that they are compact groups and for this reason the representations are finite dimensional and it follows that the characters can be interpreted as symmetric functions.

The aim of this thesis is to show that we can extend symmetric function techniques to the non-compact case, of course being very careful, and to show that there exist methods and algorithms to compute with such groups. One will see that these groups, with certain constraints, can be treated with standard techniques where we replace symmetric functions by series of symmetric functions and finally plethysms are now available for what we will later denote as harmonic representations.

Why study non-compact groups?

In regard to physical applications, we certainly need the use of compact groups for which King and Wybourne performed important several works and developped algorithms for practical calculations. That is the case if one considers the $f-$shell configuration using the exceptional group G_2 and indeed we shall examine the group-subgroup structure of G_2 where the subgroup is a finite group rather than another compact Lie group such as SO_3 or SU_3. We will give in chapter 2 a characterisation of such a subgroup and also hint at a method to use it in physical problems.

But if we consider classical problems like the hydrogen atom or $n-$dimensional oscillators we find that the dynamical groups are all non-compact groups. In these cases we cannot apply directly the same techniques and the standard algorithms for characters as they no longer work. This I present in chapter 3 and following that is the treatment of such problems for the real symplectic groups and related non-compact groups, $SO^{\star}(2n)$ and $U(p,q)$ groups. Now if we will consider in this part of the thesis some mathematical aspects of the representation theory of such groups we will also give an illustration of some possible physical applications and algorithms to produce effective calculations.

We will finish the presentation of new results by regarding a very special case, the product of finite dimensional non-unitary representations of non-compact groups by the infinite dimensional unitary irreducible representations of these non-compact groups. We will recover there the link between the different families of groups and if it is not possible to give a general frame for such products we will specify a special case for which we can calculate the products and give some examples.

Now, we can begin. After a preview of non-compact groups, I give the background of symmetric functions and representation theory. The reader will find elementary Lie group and algebra theory that will be sufficient to understand the case of G_2. After that, an introduction to the metaplectic representation and the description of the role of the symplectic group would give elements that are necessary for the last chapters. But now, we begin with a small amount of symplectic geometry ...

Chapter 1

A preview of some non-compact groups

1.1 The symplectic group

One of the central objects in this thesis is the real symplectic group $Sp(2n, \mathbb{R})$. This group can be defined in terms of matrices satisfying certain conditions. We will first introduce some more general concepts to explain why the symplectic group plays a pivotal role in many physical problems.

1.1.1 Symplectic group and canonical transformations

One way to arrive at the symplectic group is to examine the Hamiltonian formulation of classical mechanics. Our aim here is not to give a detailed exposition (see e.g.[2, 29, 30]) but we will recall what the canonical transformations are and establish the link between these transformations and the symplectic group.

Before giving the expressions for canonical transformations, we briefly recall the basis of symplectic geometry in mechanics. First we need to establish the transition from Newton's second law for a particle moving in a Euclidean three-space $\mathbb{R}^3$ (or more generally in $\mathbb{R}^n$) under the influence of a potential $V(q)$,

$$F = m\frac{d^2q}{dt^2} \tag{1.1}$$

where $q \in \mathbb{R}^3$ is the position, and $F(q) = -\nabla\ V(q)$ is the force and m the mass of the particle. We assume that the force field is conservative. The introduction of kinetic energy

$$T = \frac{1}{2}m\left|\frac{dq}{dt}\right|^2 \tag{1.2}$$

is related to the rate of work by the equation

$$\frac{dT}{dt} = m\langle \dot{q}, \ddot{q}\rangle = \langle \dot{q}, F\rangle \tag{1.3}$$

where $\langle , \rangle$ denotes the inner product on $\mathbb{R}^3$, and $\dot{q}$ the derivative with respect to time t.

The Lagrangian is defined by

$$L(q, \dot{q}) = \frac{m}{2}|\dot{q}|^2 - V(q) \tag{1.4}$$

and the Euler-Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{1.5}$$

which are second order equations in the coordinates q_i of the vector q, are equivalent to Newton's law.

It is also easy to check that $\frac{dE}{dt} = 0$ if E is the total energy,

$$E = \frac{m}{2}|\dot{q}|^2 + V(q) = \frac{|\dot{p}|^2}{2m} + V(q) \tag{1.6}$$

where $p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}$ is the momentum. It is now possible to rewrite E as a function of p and q, the Hamiltonian

$$H(p, q) = \frac{|\dot{p}|^2}{2m} + V(q). \tag{1.7}$$

Using this formalism, Newton's law is equivalent to the *Hamilton's canonical equations*

$$\dot{q}_i = \frac{\partial H}{\partial p}, \quad \dot{p}_i = -\frac{\partial H}{\partial q}, \tag{1.8}$$

which are a first order system in the (p, q) space or phase space.
We can now introduce here another notion, the *Poisson bracket*. If we consider two *observables*, i.e., two functions f, g of the variables (p, q) then the Poisson bracket of f and g is given by

$$\{f, g\} = \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\ . \tag{1.9}$$

The evolution of any observable f along a trajectory is then described by the differential equation $\frac{df}{dt} = \{f, H\}$. Hamilton's equations (1.8) correspond to the particular cases $f = q$ and $f = p$.

The Poisson bracket has the following remarkable properties

$$\{u,v\} = -\{v,u\} \tag{1.10}$$
$$\{u,\{v,w\}\} + \{w,\{u,v\}\} + \{v,\{w,u\}\} = 0 \tag{1.11}$$
$$\{uv,w\} = u\{v,w\} + \{u,w\}v \tag{1.12}$$

Properties (1.10,1.11) define a Lie algebra. Adding (1.12) we obtain a Poisson algebra.

Reformulation of the above considerations in more abstract terms will lead us to the basic notions of symplectic geometry. Let E be a real vector space and $E^\star$ its dual space. We denote by $(e_1,\dots,e_n)$ a basis of E and by $(e^1,\dots,e^n)$ the associated dual basis of $E^\star$, defined by

$$\langle e^i, e_j\rangle = e^i(e_j) = \delta^i_j.$$

A vector v of E can be written $v = \sum_i v^i e_i = v^i e_i$ and a covector $w \in E^\star$ as $w = \sum_i w_i e^i = w_i e^i$. If $A : E \to F$ is a linear transformation then the matrix A relative to basis $e_1,\dots,e_n$ of E and $f_1,\dots,f_n$ of F is denoted by A^j_i and is defined by

$$A(e_i) = A^j_i f_j, \quad [A(v)]^j = A^j_i v^i. \tag{1.13}$$

The columns of the matrix A are $A(e_1),\dots,A(e_n)$. The upper index corresponds to the row index the lower index is the column index.
Consider now a bilinear form $B : E \times F \to \mathbb{R}$. The corresponding matrix B_{ij} is given by

$$B_{ij} = B(e_i, f_j), \quad B(u,v) = u^i B_{ij} v^j \tag{1.14}$$

We define the associated linear map $\tilde{B} : E \to F^\star$ by

$$\tilde{B}(u)(v) = B(u,v) \tag{1.15}$$

so that $\tilde{B}(e_i) = B_{ij} f^j$. Since $\tilde{B}(e_i)$ corresponds to the ith column of the matrix representing the linear map $\tilde{B}$, it follows that in the basis $e_1,\dots,e_n, f^1,\dots,f^n$, the matrix of $\tilde{B}$ is the transpose of B, i.e. $[\tilde{B}]_{ji} = B_{ij}$.

We can reconsider equation (1.8) as follows. We consider the operation

$$dH(z) = \left(\frac{\partial H}{\partial q_i}, \frac{\partial H}{\partial p_i}\right) \to \left(\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q_i}\right) = X_H(z)\ .$$

X_H is a vector field (called *Hamiltonian vector field*), obtained from the differential of H by the linear map

$$R : Z^\star \longrightarrow Z$$

where Z is the vector space of pairs $z = (q,p)$ and $dH(z)$ the differential of H. The matrix of R is

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$$

Thus

$$X_H(z) = R \cdot dH(z)$$

or equivalently

$$X_H = J\nabla H \ .$$

Let $B(\alpha, \beta) = \langle \alpha, R(\beta) \rangle$ be the bilinear form associated to R where $\langle \ , \ \rangle$ denotes the pairing between Z^* and Z. One calls the bilinear form B (or the associated linear R) the *Poisson structure.* The classical Poisson bracket defined previously can be expressed as

$$\{F, G\} = B(f, dg) = df \cdot J\nabla G \ .$$

The *symplectic structure* Ω is the bilinear form associated to $R^{-1} : Z \longrightarrow Z^\star$, that is

$$\Omega(u, v) = \langle R^{-1}(u), v \rangle = u^T Jv \ .$$

Now the pair (Z, Ω) forms a finite *dimensional symplectic vector space.* The linear maps T preserving Ω, i.e. such that $\Omega(Tu, Tv) = \Omega(u, v)$ form a subgroup of $GL(Z)$ called the *symplectic group* and denoted by $Sp(Z, \Omega)$. If we represent a map T by its matrix A it is easy to check that $T \in Sp(Z, \Omega)$ iff

$$^tAJA = J. \tag{1.16}$$

A matrix satisfying (1.16) is said to be *symplectic.* The subgroup of $GL(2n, \mathbb{R})$ formed by symplectic matrices is the *real symplectic group*, denoted by $Sp(2n, \mathbb{R})$

Now, with $z = (q, p)$,Hamilton's equations (1.8), can be rewritten as

$$\frac{dz}{dt} = J\nabla H(z)$$

Consider now

$$f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$$

and write $w = f(z)$. If $z(t)$ satisfies Hamilton's equation then $w(t) = f(z(t))$ satisfies

$$\frac{dw}{dt} = {}^tA\frac{dz}{dt}$$

where ${}^tA = \left(\frac{\partial w_i}{\partial z_j}\right)$. ${}^tA$ is the jacobian matrix of f. Now, we have also

$$\frac{dw}{dt} = {}^tAJ\nabla_z H(z) = {}^tAJA\nabla_w H(w(z)) \ .$$

Thus, the equations for $w(t)$ have the form of Hamilton's equation with the Hamiltonian $K(w) = H(z(w))$ if and only if ${}^tAJA = J$, that is if and only if A is symplectic. A nonlinear transformation is said to be *canonical* if and only if its Jacobian matrix is symplectic.

As a special case we can now consider a linear map $A \in Sp(2n, \mathbb{R})$. Consider $w = Az$ and suppose that H is quadratic of the form $H(z) = \langle z, B(z)\rangle/2$ where $B(z)$ is a symmetric $2n \times 2n$ matrix. Then,

$$\begin{aligned} \nabla H(z).\delta z &= \frac{1}{2}\langle \delta z, Bz\rangle + \langle z, B\delta z\rangle \\ &= \frac{1}{2}(\langle \delta z, Bz\rangle + \langle Bz, \delta z\rangle) \\ &= \langle \delta z, Bz\rangle \ , \end{aligned}$$

so that $\nabla H(z) = Bz$ and the equations of motion become the linear equations

$$\frac{dz}{dt} = JBz$$

Now

$$\frac{dw}{dt} = A\frac{dz}{dt} = AJBz = J(\,{}^tA)^{-1}Bz = J(\,{}^tA)^{-1}BA^{-1}Az = JB'w \ ,$$

where $B' = {}^tA^{-1}BA^{-1}$ is symmetric. Now for the new Hamiltonian we get

$$\begin{aligned} H'(w) &= \frac{1}{2}\langle w, \,{}^tA)^{-1}BA^{-1}w\rangle \\ &= \frac{1}{2}\langle A^{-1}w, BA^{-1}w\rangle \\ &= H(A^{-1}w) \\ &= H(z) \end{aligned}$$

Thus $Sp(2n, \mathbb{R})$ is the linear invariance group of classical mechanics.

1.1.2 The Symplectic Group and its Lie algebra

At this stage we can summarize our knowledge about the symplectic group. We have just verified that the set of linear canonical transformations or symplectic maps constitute a group for composition, called the real symplectic group. We can reformulate its definition in a more abstract way.

1. A *symplectic form* Ω on a vector space Z is a non-degenerate skew symmetric bilinear form on Z (therefore, Z must be of even dimension $2n$).

2. The pair (Z, Ω) is called a *symplectic vector space.*

3. If (Z, Ω) ia a symplectic vector space, the set of all linear mappings $T : E \to$ form a group under composition. It is called *the symplectic group* and is denoted by $Sp(Z, \Omega)$.

The last definition is equivalent to

$$g \in Sp(Z, \Omega) \longleftrightarrow g \in GL(Z), \text{ and } \forall x, y \ , \Omega(gx, gy) = \Omega(x, y) \ .$$

As we have remarked, the matrix M is symplectic if and only if ${}^t MJM = J$. From this last relation we can exhibit other relationships. If we write

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

then the fact that $M \in Sp(E)$ implies

$$B\ {}^t A \text{ and } C\ {}^t D \text{ are symmetric,} \tag{1.17}$$

$$A\ {}^t D - B\ {}^t C = I_n \text{where } I_n \text{is the identity matrix of rank } n \ . \tag{1.18}$$

In our case we want to focus attention on the real symplectic group $Sp(2n, \mathbb{R})$. The complexified form of the group is the group $Sp(2n, \mathbb{C})$ and its complexified Lie algebra is $\mathfrak{sp}(2n, \mathbb{C})$, the simple Lie algebra of type C_n. From this algebra we will construct the metaplectic representation, a central object for the physical applications of the real symplectic groups.

The symplectic group is a Lie group. The Lie algebra of $Sp(2n, \mathbb{R})$ is formed by the matrices $M = A'(0)$ where $A(t) \in Sp(2n, \mathbb{R})$ is a smooth curve with $A(0) = \mathrm{Id}$. We have

$$A(t) J\ {}^t A(t) = J \ \Rightarrow \frac{d}{dt}|_{t=0} A(t) J\ {}^t A(t) = J\ {}^t A'(0) + A'(0) J \ .$$

Since $J^2 = -I$ we see that $M = A'(0)$ has to satisfy ${}^t M = JMJ$. The algebra $sp(2n)$ can be viewed as the set of matrices such that

$$\mathfrak{sp}(2n) = \{M \in \mathfrak{gl}(2n) |\ {}^t M = JMJ\}$$

Important work was done by E. Cartan to classify the different categories of Lie algebra. He gave a complete classification of simple complex Lie algebras. The algebra $sp(2n)$ belongs to the type C_n that we will now discuss.

To describe a semisimple Lie algebra we need its Dynkin diagram. From this diagram it is possible to reconstruct the Cartan matrix, and then to write down a presentation of the Lie algebra, in terms of the so called Chevalley generators. The Dynkin diagram of type C_n is

The Cartan matrix corresponding to the previous diagram is

$$\begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -2 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

This matrix is of size $n \times n$. The particular cases $n = 2$ and $n = 3$ are respectively

$$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} .$$

We know that each simple Lie algebra of rank n may be constructed from a set of generators $e_i, f_i, h_i : i = 1, 2, \cdots, n\}$, satisfying certain relations including $[h_i, h_j] = 0$ and

$$[h_i, e_j] = \alpha_j(h_i)e_j, \quad [h_i, j_j] = -\alpha_j(h_i)f_j, \quad [e_i, f_j] = \delta_{ij}h_i \ .$$

where the α_i are the simple roots, that is, for C_n

$$\{\varepsilon_i - \varepsilon_{i+1} \ \text{ for } \ i = 1, 2, \cdots, n-1 \ \text{ and } \ 2\varepsilon_n\}$$

where $\varepsilon_1, \cdots, \varepsilon_n$ is the standard basis of $\mathbb{R}^n$.

1.1.3 Introduction to the metaplectic representation

It is interesting to compare symplectic maps, which are those preserving an antisymmetric bilinear form, with orthogonal transformations, defines as those preserving a symmetric bilinear form. As is well known, orthogonal groups have spinor representations. The corresponding concept for symplectic groups is provided by the so called metaplectic representations. We will illustrate this representation in the case of the one-dimensional oscillator the more general case can be ound in Chapter 4.

1.1.4 Another presentation

Now we want to give another presentation of the real symplectic group but this time in terms of bosonic operators.

Here we will consider the basic operators describing a bosonic system with n degrees of freedom. They are the creation operators $a_j^\dagger$ and annihilation operators a_k with $i, j =$

$1, \cdots, n$. These operators act in the standard Hilbert space $\mathfrak{H}_B$ called the Fock space and satisfy the following commutation relations

$$[a_j, a_k^\dagger] = \delta_{ij}, \quad [a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0 \ .$$

Consider now the transformations preserving the commutation relations that are the canonical transformations that we have already encountered.

The simplest canonical transformations are the displacements $a_j \to a_j + \alpha_j$ where α_j are arbitrary complex numbers. More generally we can consider the transformations

$$\begin{aligned} a_i &\to \tilde{a}_i = u_{ij} a_j + v_{ik} a_k^\dagger, \\ a_i^\dagger &\to \tilde{a_i^\dagger} = \bar{u_{ij}} a_j + \bar{v_{ik}} a_k^\dagger, \end{aligned}$$

or in compact notation, in terms of $n \times n$ matrices

$$\begin{aligned} a &\to \tilde{a} = Ua + Va^\dagger, \\ a^\dagger &\to \tilde{a^\dagger} = \bar{U} a + \bar{V} a^\dagger. \end{aligned}$$

In fact the transformations are canonical if

$$UU^\dagger - VV^\dagger = I, \quad U\,{}^tV = V\,{}^tU \ .$$

where the sign $\dagger$ corresponds here to the Hermitian adjoint operator. The last condition is equivalent to a condition for the $2N \times 2N$ matrix

$$M = \begin{pmatrix} U & V \\ \bar{V} & \bar{U} \end{pmatrix}, \quad MK\,{}^tM = K, \quad K = \begin{pmatrix} I_n & 0_n \\ 0_n & -I_n \end{pmatrix} \ . \tag{1.19}$$

One can also obtain expression for the inverse matrix

$$M^{-1} = KM^\dagger K = \begin{pmatrix} U^\dagger & -\,{}^tV \\ -V^\dagger & {}^tU \end{pmatrix}$$

and the conditions are then rewritten

$$M^\dagger K M = K, \quad U^\dagger U - {}^tV\bar{V} = I, \quad U^\dagger V = {}^tV\bar{U} \ .$$

The existence of the inverse matrix implies that

$$U^{-1}V = {}^tV\,{}^tU^{-1}, \quad \bar{V}U^{-1} = {}^tU^{-1}V^\dagger \ .$$

This relation is equivalent to the symmetry of the matrices

$$Z = U^{-1}V \quad \text{and} \quad W = \bar{V}U^{-1} \ .$$

It follows that

$$1 - ZZ^\dagger = (U^\dagger U)^{-1}, \quad 1 - WW^\dagger = (UU^\dagger)^{-1} \ .$$

The matrices $1 - ZZ^\dagger$ and $1 - WW^\dagger$ are positive definite. The matrices M defined in (1.19) form a group G with respect to the standard matrix multiplication. This group is also a matrix realization of the *real symplectic group* that we have encountered as invariance group of classical mechanics.

1.1.5 Symplectic group and harmonic oscillator

It is time now to go back to physical applications. The relevance of the symplectic groups to quantum mechanics can be understood by analyzing one of its most basic systems, the one-dimensional harmonic oscillator.

The harmonic oscillator plays a crucial role in various problems, of theoretical or practical importance such as molecular vibrations and rotations, scattering phenomena, etc. Here, we regard it as an example of the use of Lie algebras to solve the Schrödinger equation. We will interpret the eigenstates of a one dimensionnal oscillator in terms of the representation theory of $Sp(2,\mathbb{R})$. The other symplectic groups $Sp(2n,\mathbb{R})$ will then be related in a similar way to the isotropic harmonic oscillator in $\mathbb{R}^n$.

The classical harmonic oscillator models a particle of mass m submitted to a potential of the form

$$V(x) = \frac{1}{2}kx^2$$

The Hamiltonian is therefore

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 \tag{1.20}$$

where $p = m\dot{x}$, using x instead of q.

In quantum mechanics, the x and p of the last example are replaced by observables (see [74, 19, 66]) X and P satisfying

$$[X, P] = i\hbar \tag{1.21}$$

The correspondence between the two systems is given by the following rules

$$q = x \to X$$

and

$$p \to -i\hbar\frac{\partial}{\partial q} \ .$$

The corresponding Hamiltonian is obtained by rewriting (1.20)

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2$$

and we are looking for the eigenvalues of this Hamiltonian. In other words, we want to solve

$$H|\phi\rangle = E|\phi\rangle$$

which can be rewritten as

$$[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2]\phi(x) = E\phi(x) \ . \tag{1.22}$$

As is well known, this problem can be solved in several ways. The one that will be of interest to us proceeds by merely exploiting the commutation relations (1.21).

Let us first introduce a dimensionless pair of variables $\hat{X}$ and $\hat{P}$ as follows

$$\begin{aligned} \hat{X} &= \sqrt{\frac{m\omega}{\hbar}} X \\ \hat{P} &= \frac{1}{\sqrt{m\hbar\omega}} P \end{aligned}$$

so that (1.21) becomes

$$[\hat{X}, \hat{P}] = i$$

and the Hamiltonian can be rewritten as

$$H = \hbar\omega \hat{H}, \quad \text{with} \quad \hat{H} = \frac{1}{2}(\hat{X}^2 + \hat{P}^2)$$

At this stage one can simplify the problem by introducing two other operators

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}(\hat{X} + i\hat{P}) \\ a^\dagger &= \frac{1}{\sqrt{2}}(\hat{X} - i\hat{P}) \ . \end{aligned}$$

One remarks that a and $a^\dagger$ are not Hermitian, but are mutually adjoint. If we calculate the commutator of these operators, one obtains

$$[a, a^\dagger] = 1 \ .$$

The three dimensional Lie algebra generated by a and $a^\dagger$ is the simplest case of a Heisenberg algebra. If we calculate now the product $a^\dagger a$, one obtains

$$a^\dagger a = \frac{1}{2}\left(\hat{X}^2 + \hat{P}^2 - 1\right) \ .$$

Then one can write

$$\hat{H} = a^\dagger a + \frac{1}{2} = a a^\dagger - \frac{1}{2}$$

which can be simplified by writing $N = a^\dagger a$, with $N^\dagger = a^\dagger (a^\dagger)^\dagger = a^\dagger a = N$,

$$\hat{H} = N + \frac{1}{2}$$

and the eigenvectors of N are the same as those of $\hat{H}$. Moreover, one can exhibit commutation relations between N, a and $a^\dagger$

$$\begin{aligned} [N, a] &= [a^\dagger a, a] = a^\dagger [a, a] + [a^\dagger, a] a = -a \\ [N, a^\dagger] &= [a^\dagger a, a^\dagger] = a^\dagger [a, a^\dagger] + [a^\dagger, a^\dagger] a = a^\dagger \end{aligned}$$

which lead to the solution of the problem. Indeed, these relations imply that if $|\phi\rangle$ is an eigenvector of N with eigenvalue ν, then $a^\dagger|\phi\rangle$ and $a|\phi\rangle$ are also eigenvectors, with eigenvalues $\nu+1$ and $\nu-1$. N is clearly a non negative selfadjoint operator, so that $\nu \geq 0$. By explicitly solving $a|\phi\rangle$, one can exhibit an eigenvector of N with $\nu = 0$, and then show that the $|\phi_n\rangle = (a^\dagger)^n|\phi\rangle$, which are eigenvectors with $\nu = n$, form a complete orthogonal system in $L^2(\mathbb{R})$, so that the spectrum of N is $\mathbb{N}$.

The eigenvalues of the harmonic oscillator Hamiltonian are then

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \tag{1.23}$$

Using now the fact that a and $a^\dagger$ are raising and lowering operators it is possible to construct all the eigenstates from a single one, the vector $|\phi_0\rangle$ satisfying $a|\phi_0\rangle = 0$ (corresponding to $E_0 = \frac{\hbar\omega}{2}$). From this one constructs the eigenvectors $|\phi_n\rangle$ associated with the values of (1.23). The wave function associated with $|\phi_n\rangle$ is

$$\phi_n(x) = \langle x|\phi_n\rangle \ .$$

We have

$$\phi_0(x) = \langle x|\phi_0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2}$$

then

$$\begin{aligned}
\phi_n(x) &= \langle x|\phi_n\rangle = \frac{1}{\sqrt{n!}}\langle x|(a^\dagger)^n|\phi_0\rangle \\
&= \frac{1}{\sqrt{n!}}\frac{1}{\sqrt{2^n}}\left[\sqrt{\frac{m\omega}{\hbar}}x - \sqrt{\frac{\hbar}{m\omega}}\frac{d}{dx}\right]^n e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2}
\end{aligned}$$

and finally

$$\begin{aligned}
\phi_n(x) &= \left[\frac{1}{2^n n!}\left(\frac{\hbar}{m\omega}\right)^n\right]^{\frac{1}{2}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \left[\frac{m\omega}{\hbar}x - \frac{d}{dx}\right]^n e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2} && (1.24) \\
&= C_n H_n\left(\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} x\right) e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2} && (1.25)
\end{aligned}$$

where H_n is a Hermite polynomial.

Another approach is to use the classical theory of differential equations.

Finding eigenvalues of H is equivalent to solving the differential equation (see [20], vol. 4 et 5)

$$y'' + (\lambda - x^2)y = 0 \ . \tag{1.26}$$

Writing

$$y = e^{-\frac{x^2}{2}} z \ ,$$

(1.26) becomes

$$z'' - 2xz' + (\lambda - 1)z = 0 \ . \tag{1.27}$$

If we look for a solution in the form of a series $\sum_n a_n x^n$, we obtain the following recurrence relation

$$(n+2)(n+1)a_{n+2} = (2n+1-\lambda)a_n \ . \tag{1.28}$$

If $\lambda \neq 2q+1, \ q \in \mathbb{N}$, one obtains for (1.28) even and odd solutions that are linearly independentand expandable in power series in the complex plane. These solutions are not acceptable, because of the singularity at infinity is too strong and $e^{-\frac{x^2}{2}}z \notin L^2(\mathbb{R})$.

If $\lambda = 2q+1, \ q \in \mathbb{N}$ then one of the previous solutions is a polynomial H_q of degree q and then $e^{-\frac{x^2}{2}}H_q$ is square integrable on the real axis.

One can say that the spectrum of the operator H is characterized by the real numbers

$$\lambda_n = 2n+1, \ n \in \mathbb{N}$$

the eigenfunctions of H are of the form

$$\Phi_n = e^{-\frac{x^2}{2}}H_n$$

where the H_n are Hermite polynomials such that

$$\int_{-\infty}^{+\infty} \Phi_n \Phi_m dx = \delta_{n,m}, \qquad n, m \in \mathbb{N}.$$

Now using

$$\langle Au, v\rangle = \int_{\mathbb{R}} \left(\frac{du}{dx}\frac{d\bar{v}}{dx} + x^2 u\bar{v} \right) dx \overset{def}{=} a(u,v) \tag{1.29}$$

where $\langle \ \rangle$ is the duality bracket between $\mathcal{D}(\mathbb{R})$ and $\mathcal{D}'(\mathbb{R})$, with $u, v \in \mathcal{D}(\mathbb{R})$, one can show that H is self-adjoint and thus diagonalisable on an orthonormal basis. The Φ_n constitute an orthonormal basis of $L^2(\mathbb{R})$.

One obtains the H_n by orthonormalization of the sequence $1, x, x^2, \cdots$ with respect to the scalar product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} \overline{f(x)} g(x) \mathrm{e}^{-x^2} dx \ .$$

The *Hermite functions* are defined by

$$\Phi_n(x) = \frac{(-1)^n}{(\sqrt{\pi}2^n n!)^{\frac{1}{2}}} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-x^2} \tag{1.30}$$

The Hermite polynomials are given by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{1.31}$$

and one has the generating function

$$e^{-t^2+2tx} = \sum_0^\infty \frac{t^n}{n!} H_n(x) \ ,$$

From this one can deduce

$$\begin{aligned} H_{n+1} - 2xH_n + 2nH_{n-1} &= 0 \\ H_n{}' &= 2nH_{n-1} \ . \end{aligned}$$

The Hermite functions possess other interesting properties, namely they are eigenfunctions of $\mathcal{F}$, the Fourier transform operator in $L^2(\mathbb{R})$ (see [96, 20]).

We will now exhibit the link between Hermite polynomials and the metaplectic representation in the simplest case, the representation of the group $Sp(2, \mathbb{R})$.

Actually, (1.17) implies that

$$Sp(2, \mathbb{R}) \simeq SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | ad - bc = 1 \right\} \ .$$

The Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ of $SL(2, \mathbb{R})$ is spanned by the three matrices

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

and a presentation of it is given by the relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Let $\mathcal{S}(\mathbb{R})$, be the Schwartz space over $\mathbb{R}$. One can define a representation ω of $\mathfrak{sl}(2, \mathbb{R})$ over $\mathcal{S}(\mathbb{R})$ by

$$\begin{aligned} \omega(h) &= x\tfrac{d}{dx} + \tfrac{1}{2} \\ \omega(e) &= \tfrac{i}{2}x^2 \\ \omega(f) &= \tfrac{i}{2}\tfrac{d^2}{dx^2} \end{aligned} \tag{1.32}$$

The fundamental point, which is in fact valid for $\mathfrak{sp}(2n, \mathbb{R})$ in general, is the following result:

Theorème 1.1.1 (Shale-Weil) *[65, 98, 104] The $\mathfrak{sl}(2)$-module $S(\mathbb{R})$ described by the equation (1.32) exponentiates to a unitary representation the double cover of $\widetilde{SL}(2, \mathbb{R}) = Mp(2)$ on $L^2(\mathbb{R})$.*

This theorem is a special case of a more general theorem which constructs a unitary representation of $Mp(2n) = \widetilde{Sp}(2n, \mathbb{R})$, two-fold covering of the real symplectic group $Sp(2n, \mathbb{R})$.

In general one obtains in this way a double valued projective representation of $Sp(2n, \mathbb{R})$ in $L^2(R)$, analogous to the spinor representation of $O(n)$, but infinite dimensional in our case.

This representation is called the metaplectic representation of $Sp(2, \mathbb{R})$. It is the sum of two subrepresentations of $\widetilde{SL}(2, \mathbb{R})$ formed by even and odd functions.

Consider now the operator defined by

$$\omega(k) = 2i\left(\omega(f) - \omega(e)\right) = x^2 - \frac{d^2}{dx^2}$$

where $k = 2i(e - f)$.
It is called a Hermite operator and we see that it is, up to a scalar factor, the harmonic oscillator Hamiltonian. Actually it is the generator of the maximum compact subgroup of $\widetilde{SL}(2, \mathbb{R})$, i.e. $\widetilde{SO}(2)$ (isomorphic to $SO(2)$).

As we have already seen, the functions Φ_n defined by (1.30) form an orthogonal basis of eigenvectors of the Hermite operator in $L^2(\mathbb{R})$.

One recovers here using representation theory the preceding results. One obtains also here the fact that energy levels of a harmonic oscillator are separated in two sets, even and odd states, forming two irreducible unitary representations of $Mp(2)$. These results for $SL(2, \mathbb{R}) \simeq Sp(2, \mathbb{R})$ are in fact the simplest that we can present for the metaplectic representation but illustrates how this representation arises in physics. In the sequel of this thesis we will obtain new results in the more general case of the metaplectic representation, linear representation of $Mp(2n) = \widetilde{Sp}(2n, \mathbb{R})$.

The isotropic N-dimensional oscillator

The Hamiltonian of the N-dimensional oscillator is

$$H = \sum_{i=0}^{N} H_i, \text{ with } H_i = \frac{1}{2}\left(P_i^2 + m\omega^2 x_i^2\right)$$

and one can express its dynamical state space as the tensor product of N copies of the dynamical state space associated to a one dimensional oscillator. One has

$$\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \cdots \otimes \mathcal{E}_N$$

and the vectors

$$|n_1 n_2 \cdots n_N\rangle = |n_1\rangle|n_2\rangle \cdots |n_N\rangle, \text{ with } (n_1 = 0, 1, 2, \cdots, \infty; \cdots ; \; n_N = 0, 1, 2, \cdots, \infty)$$

formed by the tensor product of N eigenvectors of the one-dimensional Hamiltonian, form an complete orthonormal system of $\mathcal{E}$. These vectors are the eigenvectors of H, and satisfy

$$H|n_1 n_2 \cdots n_N\rangle = \sum_{i=0}^{N} H_i |n_1 n_2 \cdots n_N\rangle \ ,$$

then

$$H|n_1 n_2 \cdots n_N\rangle = (n_1 + n_2 + \cdots + n_N + \frac{1}{2}N)\hbar\omega |n_1 n_2 \cdots n_N\rangle \ .$$

The fundamental state for this system is given by $|0 \cdots 0\rangle$ and the state $|n_1 n_2 \cdots n_N\rangle$ by

$$|n_1 n_2 \cdots n_N\rangle = (n_1! \cdots n_N!)^{-\frac{1}{2}} a^{\dagger n_1} \cdot a^{\dagger n_N} |0 \cdots 0\rangle \ .$$

An important special case is $N = 3$. Applying the last equations and introducing the angular momentum quantification one obtains the energy level of this oscillator as function of the three quantum numbers n, l, m.

$\mathcal{E}$ is therefore naturally a representation of $\mathfrak{sp}(2, \mathbb{R})$. We will see that in fact, this representation can be extended to $\mathfrak{sp}(2n, \mathbb{R})$ and that it can be exponentiated in a unitary representation of $Mp(2n)$, which is the direct sum of two irreducible representations, as in the one-dimensional case (see Chapter 4).

1.2 About some other groups

1.2.1 From symplectic group to pseudo-unitary groups

We have described here the link between oscillators and the symplectic group. In physics it is also frequent to refer to the group $U(2, 2)$, a $4-$fold covering of the conformal group of minkowski space, that occurs in the quantization of a zero mass particle system (see Todorov [103]), and which is the basis of Penrose's twistor theory (**??**).

Techniques to deal with $U(p, q)$ will be derived from those used in the case of the real symplectic groups that we will present in more details in chapter 3. These techniques are based on symmetric functions, and their relevance to $U(p, q)$ comes from the fact that exists an embedding $U(p, q) \subset Sp(2p+2q, \mathbb{R})$. In fact, in both cases we will use the fact that we can restrict our attention, under certain constraints, to its maximal compact subgroup.

The group $U(p, q)$ is formed by the matrices $M \in GL(p+q, \mathbb{C})$ preserving the sesquilinear form over $\mathbb{C}^{p+q}$

$$\phi(z_1, \ldots, z_{p+q}) = \sum_{i=1}^{p} |z_i^2| - \sum_{j=1}^{q} |z_{p+j}^2|$$

If we denote by $I_{p,q}$ the matrices

$$\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

with I_p and I_q the unit matrice of order p and q, then, the group matrices M satisfy

$${}^tMI_{p,q}\bar{M} = I_{p,q}$$

Moreover, as in the case of the symplectic group we can reduce many problems about the non-compact group $U(p,q)$ to question involving only its maximal compact subgroup. $SU(p,q) \cap U(p+q) = S(U(p) \times U(q))$. One shows (see [80])

$$S(U(p) \times U(q)) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where $A \in U(p), B \in U(q)$ and $\det A \ \det B = 1$.

Application of our method to the representations of $U(p,q)$, will be obtained from these kind of arguments and through an embedding into a symplectic group.

1.2.2 The group $SO^{\star}(2n)$

The group $SO^*(2n)$ occurs as a maximal non-compact subgroup of the metaplectic group, $Mp(2n)$, the double covering group for the non-compact symplectic group $Sp(2n,\mathbb{R})$. The group $SO^*(8)$ has been considered in the literature [68, 69] with the local isomorphism $SO(6,2) \sim SO^*(8)$ being exploited to show the existence of a complete set of $SU(3)$ tensor operators in the enveloping algebra of $SO^*(8)$. Here we wish to discuss the general case of the groups $SO^*(2n)$ which leave the skew Hermitean form

$$-z_1z^*_{n+1} + z_{n+1}z^*_1 - \ldots - z_nz^*_{2n} + z_{2n}z^*_n$$

invariant [37]. An account of the group $SO^*(2n)$ can be found [60]. This paper is largely devoted to the holomorphic discrete series and harmonic series of the non-compact groups $Sp(2n,\mathbb{R})$ and $U(p,q)$ with detailed derivations of appropriate branching rules and tensor products. Only scant attention was paid to $SO^*(2n)$ and the authors concluded their paper with the remark: "Little attention has been given to $SO^*(2n)$ but we suspect that comparable formulae can be derived in this case, it being merely necessary to change D to B and modify other rules appropriately". In this thesis we will explore in some detail the properties of the irreducible representations of $SO^*(2n)$ and obtain the non-trivial "merely necessary" changes. Recent work [35, 101, 102] has shed further light on the properties of $Sp(2n,\mathbb{R})$ and $U(p,q)$, and more recently [61] still it has been found convenient to introduce the notion of associate irreducible representations of $Sp(2n,\mathbb{R})$. While such a notion owes its origin to the existence of mutually associate pairs ([49, 55]) of irreducible representations of $SO(2k)$, such irreducible representations do not exist for $SO^*(2n)$. Nonetheless, we show herein that one can map self-associate finite sets of irreducible representations of $Sp(2n,\mathbb{R})$ into infinite sets of irreducible representations of $SO^*(2n)$.

Formulae will be given for the evaluation of branching rules, tensor products and plethysms for arbitrary unitary harmonic series irreducible representations of $SO^*(2n)$. These irreducible representations all appear as constituents of some power of the harmonic representation, H, of $SO^*(2n)$. The representation H is the restriction to $SO^*(2n)$ of the irreducible metaplectic representation, $\tilde{\Delta}$, of the metaplectic group $Mp(2n)$. H is itself reducible into a direct sum of an infinite number of fundamental irreducible representations in accordance with the decomposition:

$$H = \sum_{m=0}^{\infty} H_m = \sum_{m=0}^{\infty} [1(m)], \tag{1.33}$$

where it has been convenient to denote each of the infinite-dimensional fundamental irreducible representations H_m of $SO^*(2n)$ by $[1(m)]$. It is also convenient to write $H = H_+ + H_-$ with

$$H_+ = \sum_{k=0}^{\infty} [1(2k)] \text{ and } H_- = \sum_{k=0}^{\infty} [1(2k+1)]. \tag{1.34}$$

Relatively simple expressions are obtained for the branching rules, tensor products and plethysms involving H and its various constituents $H_\pm$ and H_m for $m = 0, 1, \ldots, \infty$. In particular, complete results are given for the terms in $H^2 = H \times H$ and for those in the symmetric and antisymmetric parts, $H \otimes \{2\}$ and $H \otimes \{1^2\}$, respectively, of H^2.

Chapter 2

Symmetric functions and characters of the classical groups

2.1 Introduction

The aim of this chapter is to recall the fundamental notions that are necessary to understand and interpret the calculations in the next chapters. We will define the basic objects like partitions, Young tableaux and symmetric functions.

After that we will show the role of symmetric functions as characters of symmetric and general linear groups. We will first recall some general group theory and then explain the role of symmetric functions in that area.

To proceed further we will need some properties of symmetric function series and special operations that are needed to deal with certain group representations. We will introduce the notion of plethysm and also technical tools like *vertex operators* that will be used to handle infinite series of symmetric functions. These tools will be applied to calculate explicitly tensor product decompositions of infinite dimensional representations of non-compact groups.

We will complete our toolbox by the introduction and illustration of the Frobenius characteristic and Littlewood's inner plethysm, two important notions on symmetric functions that are also involved in our calculations.

After this preliminary introduction we will recall some important results on non-compact groups and we will be ready to present our new results.

2.2 Schur functions

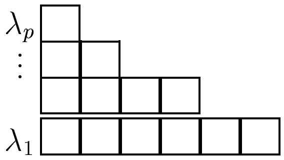
2.2.1 Integer partitions

Definition *A $p-$tuple of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ is a partition of n if*

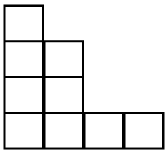
- $\lambda_1 + \lambda_2 + \cdots + \lambda_p = n$
- $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$

This will be written $|\lambda| = n$ or $\lambda \vdash n$. The length of the partition is the number of parts of the partition. It is denoted by $l(\lambda)$, or l_λ, and is equal to p.

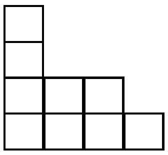
From a combinatorial point of view a partition can be seen as a planar object called a *Young or Ferrers diagram.* The basis of the object is represented by a row of boxes of length λ_1 and the top of the diagram is constituted by a row of λ_p boxes corresponding respectively to the first and last term of the $p-$tuple defining the partition, as in the following diagram



It is easy now to define the conjugate partition of a given partition λ, denoted by $\tilde{\lambda}$. It is the partition obtained by reading the Young diagram column by column. The diagram of partition $\lambda = (4, 2, 2, 1)$ is



and for its conjugate partition we obtain $\tilde{\lambda} = (4, 3, 1, 1)$ whose diagram is



A partition λ is self-conjugate if $\lambda = \tilde{\lambda}$.

It is also possible to write partitions in another way: the Frobenius notation. One associates with each partition a symbol

$$\begin{pmatrix} l_1 & l_2 & \cdots & l_p \\ c_1 & c_2 & \cdots & c_p \end{pmatrix}$$

where the l_i (respectively the c_i) are the number of boxes in the diagram contained between the end of the i-th row (resp. the i-th column) and the first diagonal. We continue with an example

$$\lambda = (4,2,2,1) = \begin{pmatrix} 3 & 0 \\ 3 & 1 \end{pmatrix}.$$

We remark that for a self-conjugate partition, in the Frobenius notation, $l_i = c_i$ for all i.

2.2.2 Young Tableaux

A *Young tableau* is an assignment of n numbers (or letters) to the n boxes of a frame F^λ corresponding to a partition $\lambda \vdash n$ according to some numbering sequence. A tableau is *standard* if the assignment of the numbers $1, 2, \ldots, n$ is such that the numbers are strictly increasing from left to right in rows and in columns from bottom to top, each letter appearing once.

A tableau is *semi-standard* if its rows are weakly increasing, and its columns strictly increasing.
We illustrate the standard tableaux by an example corresponding to the partitions of 4

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline \end{array}$$

$$\begin{array}{|c|cc}\cline{1-1} 4 & \multicolumn{2}{c}{} \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad \begin{array}{|c|cc}\cline{1-1} 3 & \multicolumn{2}{c}{} \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \qquad \begin{array}{|c|cc}\cline{1-1} 2 & \multicolumn{2}{c}{} \\ \hline 1 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}$$

$$\begin{array}{|c|c}\cline{1-1} 4 & \\ \cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{|c|}{2} \\ \hline \end{array} \qquad \begin{array}{|c|c}\cline{1-1} 4 & \\ \cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{|c|}{3} \\ \hline \end{array} \qquad \begin{array}{|c|c}\cline{1-1} 3 & \\ \cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{|c|}{4} \\ \hline \end{array}$$

$$\begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}$$

2.2.3 Symmetric functions

A function f is *symmetric* if f is invariant under all permutations of its variables:

$$f(x_1, x_2, x_3, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, \ldots) \tag{2.1}$$

Let Sym_n the ring of symmetric polynomials in n independent variables $x_1, x_2, \ldots, x_n$ with integer coefficients, i.e.

$$Sym_n = \mathbb{Z}[x_1, x_2, \ldots, x_n]^{S_n}.$$

For a given partition λ, one defines the monomial symmetric function m_λ by

$$m_\lambda(x_1, x_2, \ldots, x_n) = \sum x_{\sigma(1)}^{\lambda_1} \ldots x_{\sigma(n)}^{\lambda_n},$$

where the sum is over all distinct permutations of the monomial x_λ. If $l(\lambda) = p < n$, one sets $\lambda_{p+1} = \ldots = \lambda_n = 0$. Then the polynomials indexed by the partitions λ of length less or equal to n form a $\mathbb{Z}$-basis of Sym_n. In other terms it is equivalent to say that each symmetric function in Sym_n can be decomposed into a linear combination of m_λ with integer coefficients. Henceforth we will reserve the term *symmetric function* for a symmetric polynomial with an infinite number of variables and will denote by Sym the ring of symmetric functions.We will now define certain particular families of symmetric polynomials.

For each integer $n \geq 0$ the nth *elementary symmetric function* e_n is the sum of all products of n distinct variables x_i, so that $e_0 = 1$ and

$$e_n = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_1} \ldots x_{i_1} = m_{(1^n)} \tag{2.2}$$

Elementary symmetric functions, for $n \geq 1$, $e_n(X)$ are given by the generating function

$$\sum_{n=0}^{\infty} e_n(X) z^n = \prod_{i=1}^{\infty} (1 + x_i z). \tag{2.3}$$

It is also possible to associate a product of elementary symmetric functions with a partition $\lambda \vdash n$:

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_{l_\lambda}} \;.$$

The symmetric functions e_λ form a basis of Sym. This is the "fundamental theorem" of the theory of symmetric functions, ensuring that the e_n are algebraically independent. When the number of variables is finite, the nonzero e_k are algebraically independent.

In a similar way it is possible to define the *complete symmetric functions*. For each $n \geq 0$, the nth complete symmetric function is the sum of all monomials of total degree n in the variables $x_1, x_2, \ldots,$ so that

$$h_n = \sum_{|\lambda| = n} m_\lambda \;,$$

in particular, $h_0 = 1$ and $h_1 = e_1$.
The complete symmetric functions admit the generating function

$$\sigma_z(X) = \sum_{n=0}^{\infty} h_n(X)z^n = \prod_{i=1}^{\infty}(1 - x_i z)^{-1}. \tag{2.4}$$

As for the elementary ones, it is also possible to associate a product of complete symmetric functions to a given partition $\lambda \vdash n$:

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_{l_\lambda}} .$$

Elementary and complete symmetric functions satisfy the following relations, which are immediate from the generating functions:

$$\sum_{j=0}^{n}(-1)^j e_j(X)h_{n-j}(X) = 0. \tag{2.5}$$

For $n \geq 1$ the nth *power sum* is

$$p_n(X) = m_{(n)}(X) = \sum_{i=1}^{\infty} x_i^n. \tag{2.6}$$

A very important relation involving power sums is *Newton's formula* that we recall here in the form of a generating function

$$\sigma_z(X) = \prod_i (1 - x_i z)^{-1} = \exp\left(\sum_{n=1}^{\infty} \frac{p_n(x)}{n} z^n\right). \tag{2.7}$$

2.2.4 Schur functions

Another important basis of Sym is given by the Schur functions, also called $S-$*functions*, which may be defined in several ways. We will first give here a combinatorial definition and then recall two other definitions. It is important to remark that the equivalence of the different definitions is not obvious.

Let $X = \{x_1 < x_2 < \cdots < x_n\}$ be an alphabet of formal variables. To each tableau T can be associated a monomial X^T obtained by replacing each integer i by the variable x_i. For example to the tableau

4	6	7		
3	3	3	5	
1	1	2	4	5

is associated the monomial

$$x_1^2 x_2 x_3^3 x_4^2 x_5^2 x_6 x_7$$

Combinatorially, it is now possible to define the Schur function associated with a partiton λ as

$$s_\lambda(X) = \sum_T X^T, \tag{2.8}$$

where the sum is over all *semi-standard tableaux* of shape λ. We previously defined $h_n(X)$ as the sum of all monomials obtained from the tableaux of shape (n). For this reason we have $h_n = s_{(n)}$ and a similar remark implies that $e_n = s_{(1^n)}$.

We give now an example of this construction, the Schur function s_λ, $\lambda = (2,1)$, obtained with three variables, $X = \{x_1, x_2, x_3\}$. One constructs the following eight tableaux

$$\begin{array}{ll}\boxed{2} & \\ \boxed{1} & \boxed{1}\end{array} \qquad \begin{array}{ll}\boxed{3} & \\ \boxed{1} & \boxed{1}\end{array} \qquad \begin{array}{ll}\boxed{2} & \\ \boxed{1} & \boxed{2}\end{array} \qquad \begin{array}{ll}\boxed{3} & \\ \boxed{1} & \boxed{2}\end{array}$$

$$\begin{array}{ll}\boxed{2} & \\ \boxed{1} & \boxed{3}\end{array} \qquad \begin{array}{ll}\boxed{3} & \\ \boxed{1} & \boxed{3}\end{array} \qquad \begin{array}{ll}\boxed{3} & \\ \boxed{2} & \boxed{2}\end{array} \qquad \begin{array}{ll}\boxed{3} & \\ \boxed{2} & \boxed{3}\end{array}$$

To each tableau there corresponds a monomial X^T, and the associated Schur function is

$$s_{21}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + +x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2.$$

Remark if $l(\lambda) > |X|$, then $s_\lambda(X)$ is set to 0.

We can also give a determinantal definition of the Schur functions. It is in this way that they were first introduced by Jacobi and Cauchy.

Let λ be a partition of length $l(\lambda) \leq p$. One defines Δ_λ as the following deteminant

$$\Delta_\lambda = \begin{vmatrix} x_1^{\lambda_1+p-1} & x_1^{\lambda_2+p-2} & \dots & x_1^{\lambda_p} \\ x_2^{\lambda_1+p-1} & x_2^{\lambda_2+p-2} & \dots & x_2^{\lambda_p} \\ \vdots & \vdots & \ddots & \vdots \\ x_p^{\lambda_1+p-1} & x_p^{\lambda_2+p-2} & \dots & x_p^{\lambda_p} \end{vmatrix}. \tag{2.9}$$

Let $\Delta = \prod_{1 \leq i < j \leq p} (x_i - x_j)$ be the Vandermonde determinant. For a finite number of indeterminates $x_1, x_2, \dots, x_p$, one defines the Schur functions $s_\lambda(x_1, \dots, x_p)$ as

$$s_\lambda(x_1, x_2, \dots, x_p) = \frac{\Delta_\lambda}{\Delta}. \tag{2.10}$$

As can be checked on the example of $s_{21}(x_1, x_2, x_3)$, the calculation using this method is longer than the previous one. We can illustrate the simplest examples of Schur functions and see the link existing with other families of symmetric functions in the case of three variables $\{x_1, x_2, x_3\}$

$$\begin{array}{lcl} s_{11} & = & x_1x_2 + x_2x_3 + x_3x_1 = e_2 = h_1^2 - h_2 \\ s_2 & = & x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_3x_1 = h_2 = e_1^2 - e_2 \end{array}$$

In fact these relations are particular cases of determinantal relations that express Schur functions in terms of complete and elementary symmetric functions. These relations are the *Jacobi-Trudi formula*

$$s_\lambda = |h_{\lambda_i+j-i}| = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+k-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2+k-2} \\ \vdots & & & \\ h_{\lambda_k-k+1} & h_{\lambda_k-k+2} & \cdots & h_{\lambda_k} \end{vmatrix} \tag{2.11}$$

and *Nägelsbach's formula*

$$s_\lambda = |e_{\mu_i+j-i}| = \begin{vmatrix} e_{\mu_1} & e_{\mu_1+1} & \cdots & e_{\mu_1+k-1} \\ e_{\mu_2-1} & e_{\mu_2} & \cdots & e_{\mu_2+k-2} \\ \vdots & & & \\ e_{\mu_k-k+1} & e_{\mu_k-k+2} & \cdots & e_{\mu_k} \end{vmatrix} \tag{2.12}$$

where μ is the conjugate of the partition λ. We follow by an example of the Jacobi-Trudi formula :

$$s_{21} = \begin{vmatrix} h_2 & h_3 \\ 1 & h_1 \end{vmatrix} = h_2h_1 - h_3$$

We will see later a fundamental application of this formula to vertex operators. Finally it is possible to show that

$$s_\lambda(x_1, x_2, \ldots, x_p, 0, \ldots, 0) = \begin{cases} s_\lambda(x_1, x_2, \ldots, x_p) & \text{if } l(\lambda) \leq p \\ 0 & \text{if } l(\lambda) > p \end{cases} \tag{2.13}$$

which allows to define Schur functions with an infinite number of variables x.

One of the most important results of the theory of Schur function is that these functions form a $\mathbb{Z}$-basis of the ring Sym. We have now five different bases of Sym. We have already seen that m_λ, h_λ and e_λ each form a $\mathbb{Z}$-basis of Sym, while the $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_p}$ forms a $\mathbb{Q}$-basis.

Multiplication

A product of two Schur functions can be expressed as a linear combination of Schur functions.

$$s_\mu s_\nu = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda. \tag{2.14}$$

The theory of group representations shows that the $c_{\mu\nu}^{\lambda}$ are non-negative integers (see secton 2.6.2). The Littlewood-Richardson rule gives a combinatorial interpretation of this last result [26, 73, 25], and therefore a practical algorithm for computing $c_{\mu\nu}^{\lambda}$.

For example the product $s_{5311}s_{42}$, calculated with the program SCHUR developed by B. G. Wybourne (where the Schur functions s_λ are denoted by $\{\lambda\}$) is

```
Schur Function Mode
SFN>
o 5311, 42
      {951^2 } + {9421} + {941^3 } + {93^2 1} + {9321^2 }
       + {861^2 } + 2{8521} + 2{851^3 } + 2{8431} + {842^2 }
       + 3{8421^2 } + {841^4 } + {83^2 2} + 2{83^2 1^2 } + {832^2 1}
       + {8321^3 } + {7^2 1^2 } + 2{7621} + 2{761^3 } + 3{7531}
       + {752^2 } + 4{7521^2 } + {751^4 } + {74^2 1} + 2{7432}
       + 4{7431^2 } + 2{742^2 1} + 2{7421^3 } + {73^3 } + 2{73^2 21}
       + {73^2 1^3 } + {732^2 1^2 } + {6^2 31} + {6^2 2^2 }
       + 2{6^2 21^2 } + {6^2 1^4 } + 2{6541} + 2{6532} + 4{6531^2 }
       + 2{652^2 1} + 2{6521^3 } + {64^2 2} + 2{64^2 1^2 } + {643^2 }
       + 3{64321} + 2{6431^3 } + {642^2 1^2 } + {63^3 1}
       + {63^2 21^2 } + {5^3 1} + {5^2 42} + 2{5^2 41^2 }
       + {5^2 3^2 } + 2{5^2 321} + {5^2 31^3 } + {5^2 2^2 1^2 }
       + {54^2 21} + {54^2 1^3 } + {543^2 1} + {54321^2 }

SFN>
```

2.2.5 Characters of the symmetric group

These is a vast amount of literature about the symmetric group and its representations (see e.g. [70, 73, 25, 91]). We will here only recall certain important results that will be needed in the sequel.

The first fact is that Schur functions and power sums occur in the calculation of the characters of the *symmetric group* S_n, given by the Frobenius formula

$$p_\rho = \sum_\lambda \chi_\rho^\lambda s_\lambda, \qquad s_\lambda = \sum_\rho z_\rho^{-1} \chi_\rho^\lambda p_\rho, \tag{2.15}$$

where χ_ρ^λ is the character of the representation (λ) of S_n evalued on an element of the conjugacy class ρ, and $z_\lambda = \prod_i i^{m_i} m_i!$ for a partition with form $\lambda = (n^{m_n}, \ldots, 2^{m_2}, 1^{m_1})$. The characters satisfy the orthogonality relations

$$\sum_{p\vdash n} z_\rho^{-1} \chi_\rho^\mu \chi_\rho^\nu = \delta_{\mu\nu}, \qquad \sum_{\lambda\vdash n} \chi_\rho^\lambda \chi_\sigma^\lambda = z_\rho \delta_{\rho\sigma}. \tag{2.16}$$

2.2.6 Scalar product

The Frobenius correspondence between Schur functions and characters of the symmetric group suggests the definition of a scalar product on Sym by writing

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$$

Recall that we defined $p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_r}$. Developing the Newton formula (2.7), one obtains

$$\sum_\lambda \frac{1}{z_\lambda} p_\lambda(x) p_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1} \tag{2.17}$$

This formula can be compared to the *Cauchy identity* used by Frobenius to show the previous formulae:

$$\sum_\lambda s_\lambda(x) s_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1} \tag{2.18}$$

One can deduce that if (u_λ) and (v_λ) are bases of Sym, then the following relations are equivalent:

$$\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu}, \Longleftrightarrow \sum_\lambda u_\lambda(x) v_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1} \ . \tag{2.19}$$

Using the identities (2.17) and (2.19), one obtains

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu} \ .$$

With the same arguments one shows that

$$\langle h_\lambda, m_\mu \rangle = z_\lambda \delta_{\lambda\mu} \ .$$

Now it is time to introduce the Foulkes operators. For a symmetric function f, we denote by D_f the adjoint of the *multiplication by f*, i.e. D_f it is the operator such that for all $g, h \in Sym$

$$\langle fg, h \rangle = \langle g, D_f h \rangle \ . \tag{2.20}$$

It is easy to check that the adjoint of the operator multiplication by p_k is $k \frac{\partial}{\partial p_k}$. This implies that all D_f can be expressed in terms of differential operators for which the variables are the p_i [24, 73].

Vertex operators

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ a partition and s_λ the associated Schur function. One can add a part n to λ and we obtain a new partition $(n; \lambda) = (n, \lambda_1, \lambda_2, \dots)$. It is not necessary that $n \geq \lambda_1$ to ensure that $s_{(n;\lambda)}$ be defined. The *Jacobi-Trudi* formula (2.11), that expresses s_λ as a determinant in terms of the h_i permits to make sense of s_α for $\alpha \in \mathbb{Z}^p$. One can now use the generating function of these Schur functions, and using the development of the Jacobi-Trudi determinant along the first row, one obtains the following relation

$$\sum_{n\in\mathbb{Z}} s_{(n,\lambda)} z^n = \left(\sum_{n\geq 0} z^n h_n\right)\left(\sum_{m\geq 0}(-\frac{1}{z})^m D_{e_m}\right) s_\lambda, \tag{2.21}$$

that yields, using the Newton formulae

$$\sum_{n\in\mathbb{Z}} s_{(n,\lambda)} z^n = \exp\left\{\sum_{k\geq 1}\frac{z^k}{k}p_k\right\}\exp\left\{-\sum_{l\geq 1}\frac{z^{-l}}{l}D_{p_l}\right\} s_\lambda. \tag{2.22}$$

One defines now the vertex operator Γ_z as

$$\Gamma_z = \exp\left\{\sum_{k\geq 1}\frac{z^k}{k}p_k\right\}\exp\left\{-\sum_{l\geq 1}\frac{z^{-l}}{l}D_{p_l}\right\} \tag{2.23}$$

where, according to the fact that $D_{p_l} = l\frac{\partial}{\partial p_l}$,

$$\Gamma_z = \exp\left\{\sum_{k\geq 1}\frac{z^k}{k}p_k\right\}\exp\left\{-\sum_{l\geq 1}z^{-l}\frac{\partial}{\partial p_l}\right\} \tag{2.24}$$

Therefore

$$\sum_{n\in\mathbb{Z}} s_{(n,\lambda)} z^n = \Gamma_z s_\lambda. \tag{2.25}$$

This operator, that naturally occurs when one tries to write in terms of generating series some recurrences with characters was introduced for the first time in physics in the 70's. It appears in the dual model (a strong interaction model), forgotten for a while but revived as a basis for string theory. Vertex operators are also commonly used in the representation theory of affine Lie algebras and for integration of various hierarchies of soliton equations (see [45]).

2.3 Lie groups and Lie algebras

2.3.1 Lie groups

Young tableaux and some variants of them are the basis of many algorithms for practical calculations on representations of classical groups and algebras like ones used in many physical applications such as spectroscopy, atomic or nuclear physics, etc. [36, 71, 87].

In this section we will quickly recall some fundamental notions on Lie groups and Lie algebras and establish a link with classical combinatorial objects. We will also show how

Schur functions can be interpreted as characters of group representations.

A Lie group is a manifold G that has group structure consistent with its manifold structure in the sense that the group multiplication

$$\mu : G \times G \to G, \quad \mu \ : (x, y) \to xy$$

is a C^∞ map.

The simplest examples are the line of real numbers $\mathbb{R}^1$, and more generally the n-dimensional vector space $\mathbb{R}^n$ equiped with addition:

$$\mu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \ \mu(x, y) = x + y, \ I(x) = -x \ .$$

These groups are very special since they are commutative. Their representation theory essentially amounts to classical Fourier analysis.

More typical examples are the *general linear groups* $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$, formed by the $n \times n$ invertible matrices with coefficients respectively in $\mathbb{R}$ or $\mathbb{C}$. The multiplication is the usual matrix multiplication

$$\mu : GL(n, \mathbb{R}) \times GL(n, \mathbb{R}) \to GL(n, \mathbb{R}), \ \mu(A, B) = A \circ B, \ I(A) = A^{-1} \ .$$

The most important Lie groups are formed by subgroups of $GL(n, \mathbb{C})$, so that the general linear group occupies a very important place in the study of Lie groups. Special linear, symplectic, orthogonal and associated groups are the main topics of important body of literature in which the detailed analysis of representation theory is developed [105, 14, 26].

We present now other groups frequently used in geometry and physics. If we focus our attention on subgroup on $GL(n, \mathbb{R})$, we can find some interesting subgroups. The first that appears is the **real special linear group**, denoted $SL(n, \mathbb{R})$ set of matrices

$$SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) | \det A = 1\} \ .$$

Clearly $SL(n, \mathbb{R})$ is a closed subgroup of $GL(n, \mathbb{R})$. The tangent space to $SL(n, \mathbb{R})$ at $A \in SL(n, \mathbb{R})$ consists of the matrices B such that $\mathrm{trace}(A^{-1}B) = 0$. It follows that the Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ is the set of $n \times n$ matrices having trace zero with the bracket

$$[A, B] = AB - BA \ .$$

One can also show that $SL(n, \mathbb{R})$ is a non-compact connected and $(n^2 - 1)$-dimensional Lie group. A fundamental example that we have already seen is the group $SL(2, \mathbb{R})$ isomorphic to $Sp(2, \mathbb{R})$.

Now we will give another example, the **orthogonal group** $O(n)$. Consider that we use the standard inner product on $\mathbb{R}^n$

$$< x, y >= \sum_{i=1} x_i y_i \ , (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \ .$$

A map A is orthogonal if

$$< Ax, Ay >=< x, y > ,$$

for all x and y. Then one can deduce that $|Ax| = |x|$ for all x and in terms of the transpose ${}^tA$ defined by

$$< Ax, y >=< x, \ {}^tAy > ,$$

one sees that A is orthogonal if and only if $A\,{}^tA = I$. Then one can see that the orthogonal group is the group of elements A satisfying

$$1 = \det(A\,{}^tA) = \det(A)\det(\,{}^tA) = \det(A)^2$$

From this last relation one sees that $\det(A) = \pm 1$, then

$$O(n) = \{A \in GL(n, \mathbb{R}) | \det(A) = \pm 1\} .$$

The group orthogonal is a compact Lie group of dimension $\frac{1}{2}n(n-1)$ and one can show that it has a compact subgroup $SO(n)$

$$SO(n) = O(n) \cap SL(n, \mathbb{R}), \ \ SO(n) = \{A \in GL(n, \mathbb{R}) | \det(A) = 1\} .$$

that is the connected component of $O(n)$ containing the identityt I and has the same Lie algebra $\mathfrak{o}(n)$.

Examples of $SO(n)$ are the groups of rotation in the plane and space $SO(2)$ and $SO(3)$.

To finish with examples of Lie groups I will recall the existence of symplectic groups, pseudo-unitary or $SO^\star(2n)$ groups, that like the exceptionnal group G_2 we have already described in the introduction.

In fact, in the literature we will find frequently these groups or the complexified form of these groups with the following definition

- $SL(n, \mathbb{C})$: group of $n \times n$ matrices with determinant equal to 1. This group is also called A_{n-1} according to the Cartan classification.
- $SO(2n + 1)$: group of real matrices A such that $A^tA = \mathrm{Id}$ (also denoted B_n).
- $SO(2n)$: group of matrices A such that $A^tA = \mathrm{Id}$ (also called D_n).
- $Sp(2n)$: group of matrices satisfaying $A^tJA = J \ J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$, (also called C_n).

2.3.2 Lie algebras

A Lie algebra $\mathfrak{g}$ is a vector space over a field F equipped with a non-associative product $[,]$ (called Lie bracket) such that

1. $X, Y \in \mathfrak{g} \Longrightarrow [X, Y] \in \mathfrak{g}$ (closure by commutation)
2. $[X, \alpha Y + \beta Z] = \alpha[X, Y] + \beta[X, Z] \forall \alpha, \beta \in F, \ \forall X, Y, Z \in \mathfrak{g}$ (bilinearity)
3. $[X, Y] = -[Y, X]$ (skew-symetry)
4. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi identity)

If $F = \mathbb{R}$ then $\mathfrak{g}$ is called *real Lie algebra* and if $F = \mathbb{C}$ then $\mathfrak{g}$ is a *complex Lie algebra.* Any associative algebra can be equipped with a Lie algebra structure by taking for the Lie bracket the commutator $[X, Y] = XY - YX$.

One encounters many examples of Lie algebras in physics, for example Lie algebras of vector field. Indeed a vector field can be viewed as a first order differential operator $D_{\vec{V}}$ associated with $\vec{V}$. One can observe that $D_{\vec{V}} D_{\vec{W}} - D_{\vec{W}} D_{\vec{V}}$ is again a first order differential operator and corresponds to a vector field $D_{\vec{U}}$. Then one writes $\vec{U} = [\vec{V}, \vec{W}]$.

Of course, Lie algebras are related to Lie groups. We will attempt to make precise this link.

The Lie algebra $\mathfrak{g} = \mathfrak{L}(G)$ of a Lie group G is the tangent space to the identity of a Lie group G, endowed with a Lie algebra structure defined in the following manner. A tangent vector X to G at e (the group identity) can be considered as a left-invariant vector field over G, i.e. such that $X(g) = TLg(e)X(e)$, where $TLg(e)$ is the linear map tangent at the point e to the translation $Lg : x \mapsto gx$. One verifies that the bracket as previously defined of two left-invariant vector fields is left-invariant. It is sufficient to define the bracket on $\mathfrak{g}$. In the case where $G \subset GL(n, \mathbb{C})$ is a group of matrices, $\mathfrak{g}$ is formed by the derivatives $\frac{d}{dt}|_{t=0} A(t)$ of the curves $t \mapsto A(t) \in G$ such that $A(0) = 1$, and the bracket is the commutator.

We continue with an example : the group $SO(3)$ of space rotations around a point O. The group is generated by the rotations around the coordinate axes. The matrices representing the rotations are

$$R_1(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix},$$

$$R_2(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix},$$

$$R_3(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Their derivatives taken at the origin $L_1 = \dot{R}_1(0)$, $L_2 = \dot{R}_2(0)$, $L_3 = \dot{R}_3(0)$ are given by the matrices

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \; L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

These matrices form a basis for the Lie algebra $\mathfrak{so}(3)$. One can remark that these matrices are skew-symmetric (i.e verify the relation. relation ${}^tA + A = 0$). Also, one can check that the *Lie bracket* of the space rotations Lie algebra is equivalent to the cross product used in physics. Indeed, to each matrix can be associated a vector such that

$$X = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & o \end{pmatrix} \; X \to v_X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Then, if X and Y are two matrices of $\mathfrak{so}(3)$ one has $[X,Y] \leftrightarrow v_X \times v_Y$. To check this, it is sufficient to see the commutation relations

$$\begin{aligned} [L_1, L_2] &= L_1L_2 - L_2L_1 = L_3 \;, \\ [L_2, L_3] &= L_2L_3 - L_3L_2 = L_1 \;, \\ [L_3, L_1] &= L_3L_1 - L_1L_3 = L_2 \;. \end{aligned}$$

We can conclude that the axial vectors used in mechanics form the Lie algebra of the rotation group. With the same kind of arguments we can verify that the torsors constitute the Lie algebra of the Euclidean group.

$\mathfrak{so}(3)$ is an example of a *linear* Lie algebra, that is, a Lie algebra of matrices with $[X,Y] = XY - YX$.

The fundamental example of this sort of algebra is $\mathfrak{sl}_2(\mathbb{C})$. One can see this algebra as the set of traceless 2×2 matrices with coefficients in $\mathbb{C}$. A basis of $\mathfrak{sl}_2(\mathbb{C})$ is :

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.26}$$

The commutation relations between these matrices are:

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h. \tag{2.27}$$

There is an isomorphism between the complexified Lie algebra $\mathfrak{so}(3, \mathbb{C}) = \mathfrak{so}(3) \otimes \mathbb{C}$ and $\mathfrak{sl}(2, \mathbb{C})$ that we will illustrate in the next section. From this fact one can derive most of the classical theory of spherical harmonics. This is a typical example of the application of Lie theory to the analysis of the differential equations of mathematical physics.

It is time now to give more information about representations. It will be a very simple introduction and we will refer for more details to the literature (see e.g. [18, 26, 62, 14]).

2.4 Representation theory

A *representation* of a group G over a vector space V is a homomorphism $\rho : G \to GL(V)$ of G in the automorphism group of V. One says that such an application gives V a *G-module* structure.

Frequently we just call V the representation and write $gv \equiv \rho(g)(v)$. The dimension of V is called the degree of the representation.

One defines similarly a representation of a Lie algebra $\mathfrak{g}$ as a homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ where $\mathfrak{gl}(V)$ is the Lie algebra of all endomorphisms of V, equipped with the bracket $[u, v] = uv - vu$.

One says that the representation is *irreducible* if one cannot find a proper subspace invariant under the action of G (resp. of $\mathfrak{g}$). The classification of irreducible representations was one of the first problems in the representation theory. This question was the starting point of different works that have let very important results. For certain classes of groups and algebras, the understanding of the irreducible representations is enough:

1. Any representation over $\mathbb{C}$ of a finite group can be decomposed into a direct sum of irreducible representations (Maschke).

2. Any finite dimensional representation over $\mathbb{C}$ of a compact group can be decomposed into a direct sum of irreducible representations (Weyl).

3. Any finite-dimensional representation over $\mathbb{C}$ of a semi-simple Lie algebra can be decomposed into a direct sum of irreducible representations (Cartan).

For other classes, e.g. infinite discrete groups, or non-compact Lie groups, it is also necessary to study indecomposable representations, which is usually a much more difficult problem.

2.4.1 Example : $\mathfrak{sl}(2, \mathbb{C})$

As announced in the preceding section we will illustrate the definitions by the representations of $\mathfrak{sl}_2(\mathbb{C})$. One takes $V = \mathbb{C}^2 = \mathbb{C}x \bigoplus \mathbb{C}y$. Recall that a basis of $\mathfrak{sl}_2(\mathbb{C})$ is given by (2.26) and that the elements of this basis satisfy the commutation relations (2.27).

It is clear that:

- e acts on V by $\begin{cases} x \to 0 \\ y \to x \end{cases}$ i.e. as the differential operator $x\frac{\partial}{\partial y}$.

- f acts on V by $\begin{cases} x \to y \\ y \to 0 \end{cases}$ i.e. as $y\frac{\partial}{\partial x}$.

- and h acts on V by $\begin{cases} x \to x \\ y \to -y \end{cases}$ i.e. as $x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$.

Denote now by V_n the space of homogenous polynomials of degree n in x, y

Then one can verify that if we define ρ by $\rho(e) = x\frac{\partial}{\partial y}, \rho(f) = y\frac{\partial}{\partial x}, \rho(h) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ then we have

- $[\rho(e), \rho(f)] = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} = \rho(h)$
- $[\rho(h), \rho(e)] = y\frac{\partial}{\partial x} + y\frac{\partial}{\partial x} = 2\rho(e)$
- $[\rho(h), \rho(f)] = -x\frac{\partial}{\partial y} - x\frac{\partial}{\partial y} = -2\rho(f)$

Consequently, ρ is a representation of $\mathfrak{sl}_2(\mathbb{C})$ in $\mathbb{C}[x, y]$, leaving stable each homogenous component V_n. Moreover, one can check that V_n is an irreducible representation, and that any finite dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to some V_n. We can go further by making some remarks establishing the isomorphism already announced between $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{so}(3) \otimes \mathbb{C}$. This isomorphism is given by $L_k \leftrightarrow -\frac{i}{2}\sigma_k$ where the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ are constructed from the matrices e, f, h. These matrices form a basis of $\mathfrak{sl}(2, \mathbb{C})$ and can be written

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.28}$$

that is,

$$\sigma_1 = e + f, \ \sigma_2 = -i(e - f), \ \sigma_3 = h$$

Consider now a rotation α, element of $SO(3)$. This rotation can be expressed by means of the matrices L_1, L_2, L_3. We can also use the Pauli matrices to express the same rotations. To show that we need first to introduce the unitary unimodular group $SU(2)$ to see the relationships between $SU(2)$ and $SO(3)$.

2.4.2 The unitary unimodular group $SU(2)$

The unitary group $U(2)$ is the set of complex 2×2 matrices satisfying $M^\dagger M = MM^\dagger = I$ where $M^\dagger$ is the adjoint matrix of M and I the identity matrix.

From this relation we can write the matrices of $U(2)$ in the form

$$M = \begin{pmatrix} \mathrm{e}^{\mathrm{i}u}\cos\lambda & \mathrm{e}^{\mathrm{i}v}\sin\lambda \\ -\mathrm{e}^{\mathrm{i}w}\sin\lambda & \mathrm{e}^{\mathrm{i}w+v-u}\cos\lambda \end{pmatrix} . \tag{2.29}$$

One sees that the group $U(2)$ is a four parameter group.

The unitary unimodular group $SU(2)$ is the subgroup of $U(2)$ formed by matrices of determinant equal to 1 .

From relation (2.29) one deduces $v = -w \pm 2k\pi$. The group $SU(2)$ is a three parameter group like $SO(3)$. The generic form for $SU(2)$ matrices is

$$M = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} , \quad |a^2| + |b^2| = 1 \tag{2.30}$$

where z^* is the complex conjugate of z. The space $\mathbb{C}^2$ on which $SU(2)$ acts is called the Pauli spinor space. A rotation of angle θ around a unit vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ in the three dimensional space can be written as an element of $SU(2)$

$$R(\alpha, \frac{\theta}{2}) = \begin{pmatrix} \cos\frac{\theta}{2} - \mathrm{i}\alpha_3\sin\frac{\theta}{2} & -(\mathrm{i}\alpha_1 + \alpha_2)\sin\frac{\theta}{2} \\ (-\mathrm{i}\alpha_1 + \alpha_2)\sin\frac{\theta}{2} & \cos\frac{\theta}{2} + \mathrm{i}\alpha_3\sin\frac{\theta}{2} \end{pmatrix}$$

From this last equation one deduces the form of the infinitesimal rotation matrix in $SU(2)$

$$X(\alpha) = \begin{pmatrix} -\mathrm{i}\alpha_3 & -(\mathrm{i}\alpha_1 + \alpha_2) \\ (-\mathrm{i}\alpha_1 + \alpha_2) & \mathrm{i}\alpha_3 \end{pmatrix} .$$

The last relation can be rewritten using the Pauli matrices

$$X(\alpha) = -i(\alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3) \ .$$

For example, a rotation of $\frac{\pi}{2}$ around the x axis will be given by the infinitesimal matrix

$$X(O_x) = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -\frac{i}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

This shows the relationship between $SU(2)$ and $SO(3)$. If $M \in SU(2)$ corresponds to a rotation, then $-M$ also corresponds to the same rotation. Thus we have two $SU(2)$ elements for each element of $SO(3)$.

2.4.3 Characters

Now to the notion of a representation we can associate the notion of a character that is defined as follows for finite dimensional representations.

Let V be a representation of a group G. Then the character χ of the representation is a complex function defined on the group by

$$\chi_V(g) = \mathrm{tr}(g|_V)$$

where tr is the trace operator. For a linear Lie group $G \subset GL(n,\mathbb{C})$, one can show that $g \mapsto \mathrm{tr}\rho(g)$ is a symmetric function $\chi(x_1, \cdots, x_n)$ of the eigenvalues of the matrix g. For example the irreducible polynomial representations V_λ of $GL(n,\mathbb{C})$ are indexed by the partitions with no more than n parts, and the character of V_λ is the Schur function $s_\lambda(x_1, \cdots, x_n)$.

One can give three properties of characters

1. $\chi_V(1) = \mathrm{dim}V$.
2. $\chi_{V\oplus W} = \chi_V + \chi_W$.
3. $\chi_{V\otimes W} = \chi_V \cdot \chi_W$.

The last identity involves the tensor product of representations, of which we will now say a few words (for more about characters see e.g. [71, 26, 18]).

2.4.4 Tensor product of representations

The tensor product $V_1 \otimes V_2$ of two vector spaces V_1 and V_2 with bases (e_i) and (f_j) can be regarded as the vector space spanned by the formal products $e_i \otimes f_j$.
Therefore, if E and F are finite dimensional then $\dim(E \otimes F) = \mathrm{dim}E \cdot \mathrm{dim}F$.

Now, if G is a group and R_1 and R_2 two representations of this group having V_1 and V_2 for representation spaces, then

$$(R_1 \otimes R_2)(g)(v_1 \otimes v_2) \overset{def}{=} R_1(g)v_1 \otimes R_2(g)v_2$$

defines a representation of G over $V_1 \otimes V_2$ called tensor product $R = R_1 \otimes R_2$ of R_1 and R_2.

Let $M_1(g)$ and $M_2(g)$ the matrices corresponding to the operators $R_1(g)$ and $R_2(g)$, and denote by M the matrix of R, on the basis of the vector space $V_1 \otimes V_2$. We say that the matrix M is the matrix tensor product of the matrix $M_1(g)$ by the matrix $M_2(g)$ and we write $M(g) = M_1(g) \otimes M_2(g)$. Other designations, such as direct product or Kronecker products of matrices can be also found in the literature.

In fact, more generally if we consider two matrices $A = (a_{ij})$ and $B = (b_{kl})$, of respective orders (m_1, n_1) and (m_2, n_2), their tensor product is the matrix $C = A \otimes B$ defined by

$$C = A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n_1}B \\ a_{21}B & a_{22}B & \cdots & a_{2n_1}B \\ \cdots & \cdots & \cdots & \cdots \\ a_{m_11}B & a_{m_12}B & \cdots & a_{m_1n_1}B \end{pmatrix}$$

and each inner matrix of the form $a_{ij}B$ possesses elements of the form $a_{ij}b_{kl}$ such that the elements of the matrix C can be written $c_{ij,kl} = a_{ij}b_{kl}$.
If A, B, C and D are matrices such that the products AB and CD exist then

$$AB \otimes CD = (A \otimes C)(B \otimes D)$$

More, if A and B are invertible then

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

Finally, one has $\mathrm{tr}(A \otimes B) = \mathrm{tr}A \cdot \mathrm{tr}B$. Thus, in particular we have for the character of a tensor product

$$\chi_{A\otimes B} = \chi_A \cdot \chi_B$$

We can now define in a similar manner the tensor product of Lie algebra representations.

Consider two modules V_1 and V_2 of a Lie algebra $\mathfrak{g}$, and R_1 and R_2 the corresponding representations. Then $\mathfrak{g}$ can also be represented on the tensor product vector space $V_1 \otimes V_2$ via

$$R_1 \otimes R_2 : \quad ((R_1 \otimes R_2(g))(v_1 \otimes v_2) \stackrel{def}{=} (R_1(g)v_1) \otimes v_2 + v_1 \otimes (R_2(g)v_2)$$

This representation is called the tensor product of the $\mathfrak{g}$-representations R_1 and R_2.

We can remark that the $V_1 \otimes V_2$ and $V_2 \otimes V_1$ are different vector spaces and hence also the module are different However, it turns out that the two modules are isomorphic

$$R_1 \otimes R_2 \simeq R_2 \otimes R_1 \ .$$

Indeed, the tensor product is commutative up to isomorphism and is also associative up to isomorphism

$$(R_1 \otimes R_2) \otimes R_3 \simeq R_1 \otimes (R_2 \otimes R_3) \ .$$

and some problems in physics considt in describing this isomorphism explicitely.

We now illustrate the tensor product by three examples familiar from elementary quantum quantum mechanics.

Particle in a two-dimensional potential well

Consider a particle of mass m confined in a square box of side a, in the xOy plane. The potential energy of the particle becomes infinite if the variables x and y go out of the interval $[0, a]$. The Hamiltonian of the sytem is

$$H = \frac{1}{2m}\left(P_x^2 + P_y^2\right) + V_\infty(x) + V_\infty(y)$$

where

$$P_x = -i\hbar\frac{\partial}{\partial x},\ P_y = -i\hbar\frac{\partial}{\partial y}$$

and

$$\begin{cases} V_\infty(u) &= \quad 0 \ \text{ if } \quad 0 \leq u \leq a \\ &= +\infty \ \text{ if } \ u \leq 0 \text{ or } u \geq a \end{cases}$$

This Hamiltonian can be rewritten as a sum of two identical Hamiltonians with only one degree of freedom

$$\begin{aligned} H_x &= \frac{1}{2m}P_x^2 + V_\infty(x) \\ H_y &= \frac{1}{2m}P_y^2 + V_\infty(y) \end{aligned}$$

The problem now is to diagonalize the operator H, to obtain the eigenvalues and eigenvectors. One can separate the problem and express the solution on the space $\mathcal{E}$, tensor product of the spaces $\mathcal{E}_x$ and $\mathcal{E}_y$.

We begin with the diagonalization of H_x and H_y. To obtain the eigenstates of the two hamiltonians we write the equations

$$\begin{aligned} H_x|\phi\rangle_x = E_x|\phi\rangle_x \quad \text{with} \quad |\phi\rangle_x \in \mathcal{E}_x \\ H_y|\phi\rangle_y = E_y|\phi\rangle_y \quad \text{with} \quad |\phi\rangle_y \in \mathcal{E}_y \end{aligned}$$

where H_x and H_y corresponds to the observables in $\mathcal{E}_x$ and $\mathcal{E}_y$, state spaces on x and y. Separation of variables can here be applied because H_x is constructed only with the X and P_x operators. Eigenvalues of $H|\Phi\rangle = E|\Phi\rangle$ will be $E = E_x + E_y$. Finally the solution of the one-dimensional system yields H leads to $E_{n,p} = \frac{1}{2ma}(n^2 + p^2)\hbar^2$ where n and p are positive integers. To these states correspond eigenfunctions $|\Phi_{n,p}\rangle$ expressed on the tensor product $\mathcal{E}_x \otimes \mathcal{E}_y$

$$|\Phi_{n,p}\rangle = |\phi_n\rangle_x \otimes |\phi_p\rangle_y \ \in \mathcal{E}_x \otimes \mathcal{E}_y$$

and the associated normalized wave functions are of the form

$$\Phi_{n,p} = \langle x|\phi_n\rangle_x \langle y|\phi_p\rangle_y = \frac{2}{a}\sin\frac{n\pi x}{a}\sin\frac{p\pi y}{a}$$

Angular momentum operator

In quantum mechanics, the total angular momentum can be expressed in the following manner [1]

$$J_i = L_j \otimes I_{\mathbb{C}^n} + I_{L^2(\mathbb{R}^3)} \otimes S_j, \ j = 1, 2, 3$$

acting in $\mathfrak{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^n$. In this expression expression, $I_{\mathbb{C}^n}$ and $I_{L^2(\mathbb{R}^3)}$ are respectively the identity operators for $\mathbb{C}^n$ and $L^2(\mathbb{R}^3)$ and L_j is the operator

$$L_j = -i\hbar \left(x_k \frac{\partial}{\partial x_l} - x_l \frac{\partial}{\partial x_k} \right), \ l, k \in \{1, 2, 3\}$$

called orbital component of the total angular momentum and S_i is an operator on $\mathbb{C}^n$, called spin component of the momentum such that the three operators $S_j, \ j = 1, 2, 3$ satisfy the commutation relations

$$S_j S_k - S_k S_j = i\hbar S_l,$$

where j, k, l are circular permutations of $1, 2, 3$.

In other words, the operators $\frac{L_j}{i\hbar}$ form a basis of the Lie algebra $\mathfrak{so}(3)$, acting on the left factor on $L^2(\mathbb{R}^3)$. The operators $\frac{S_j}{i\hbar}$ span a representation of $\mathfrak{so}(3)$ acting on the right factor $\mathbb{C}^n$. The irreducible representation $\mathbb{C}^n$ is the representation of $\mathfrak{sl}(2, \mathbb{C}) \sim \mathfrak{so}(3, \mathbb{C})$ over the homogenous polynomials of degree $n - 1$ that we have seen previously.

The half degree $s = \frac{n-1}{2}$ corresponds to the spin. The $J_j = L_j + S_j$ are the operators of the representation $L^2(\mathbb{R}^3) \otimes \mathbb{C}^n$, tensor product of the two representations.

A particle for which $s = 0$ is a spinless particle and for this case, $n = 1$ and we have for associated Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C} = L^2(\mathbb{R}^3)$. Now for a particle with spin $s = \frac{1}{2}$ $(n = 2)$, we associate the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$. In this case the spin operators can be written in terms of Pauli matrices already described here

$$S_1 = \frac{\hbar}{2}\sigma_1, \ S_2 = \frac{\hbar}{2}\sigma_2, \ S_3 = \frac{\hbar}{2}\sigma_3 \ .$$

N-particle system

Consider a system formed by N distinct particles (discernable) and let h_i the Hilbert space of the ith particle.Then the Hilbert space associated with the whole system of N particles is

$$\mathcal{H} = h_1 \hat{\otimes} h_2 \hat{\otimes} \cdots \hat{\otimes} h_n$$

[1] This expression can be more commonly found written as a simple sum $J_j = L_j + S_j$.

that is the associated with the algebraic tensor product $h_1 \otimes h_2 \otimes \cdots \otimes h_n$ for the hermitian form (,) such that

$$(x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_n) = \prod_i (x_i, y_i), \ \forall x_i, y_i \in h_i$$

If all of the particles are identical then the space to consider is not the tensor product

$$h^{\hat{\otimes} N} = h \hat{\otimes} h \hat{\otimes} \cdots \hat{\otimes} h = \hat{\otimes}^N h$$

but a certain subspace of $h^{\hat{\otimes} N}$ that we will now make precise. Before that we need to recall some definitions.
Let ψ an element of $h^{\hat{\otimes} N}$. It can be written

$$\psi_{\alpha_1, \cdots, \alpha_N} = \Phi_{\alpha_1}(x_1) \otimes \Phi_{\alpha_2}(x_2) \otimes \cdots \Phi_{\alpha_N}(x_N) \ ,$$

where $\Phi_{\alpha_i} \in h_i, \ i = 1, \cdots, N$. One considers now a permutation operator P^π acting on ψ.
For example if π is the permutation $(2, 1, \cdots, m)$ one has

$$P^\pi \left(\Phi_{\alpha_1}(x_1) \otimes \Phi_{\alpha_2}(x_2) \otimes \cdots \otimes \Phi_{\alpha_N}(x_N) \right) = \Phi_{\alpha_2}(x_2) \otimes \Phi_{\alpha_1}(x_1) \otimes \cdots \otimes \Phi_{\alpha_m}(x_m) \ .$$

Thus it is possible to define two types of elements

$$\psi_+ \stackrel{def}{=} k_+ \sum_\pi P^\pi \left(\Phi_{\alpha_1}(x_1) \otimes \Phi_{\alpha_2}(x_2) \otimes \cdots \Phi_{\alpha_N}(x_N) \right)$$

and

$$\psi_- \stackrel{def}{=} k_- \sum_\pi P^\pi (-1)^{P^\pi} \left(\Phi_{\alpha_1}(x_1) \otimes \Phi_{\alpha_2}(x_2) \otimes \cdots \Phi_{\alpha_N}(x_N) \right)$$

where the coefficient k_- and k_+ are normalization coefficients and $(-1)^{P^\pi}$ is the signature of the permutation.

The elements ψ_+ and ψ_- are respectively called symmetric and skew-symmetric of $h^{\hat{\otimes} N}$. The Hilbert space generated in $h^{\hat{\otimes} N}$ by the symmetric elements (resp. skew-symmetric) is called symmetric Hilbert subspace and is denoted by $h_+^{\hat{\otimes} N}$ (resp. skew-symmetric Hilbert subspace and is noted $h_-^{\hat{\otimes} N}$)

Then, to a N-particle system with integer spin (bosons), we associate the symmetric Hilbert subspace $h_+^{\hat{\otimes} N}$, that is a subspace of a tensor product $h^{\hat{\otimes} N} = \hat{\otimes}^N h$ and to a N-particles system with half-integer spin (fermions) we associate the skew-symmetric subspace $h_-^{\hat{\otimes} N}$ of the same tensor product.

It is possible to find other notations for $h_+^{\hat{\otimes} N}$ and $h_-^{\hat{\otimes} N}$. Indeed, $h_+^{\hat{\otimes} N} = \hat{S}^N h$, is the completed of the symmetric product of order N, $S^N h$ and $h_-^{\hat{\otimes} N} = \hat{\wedge}^N h$, is the completed of the skew-symmetric product of order N, $\wedge^N h$ (see [20] vol. 5)

One can also remark that the indiscernability of particles implies that the operators associated with observables need to commute with the permutation operators. It is equivalent to write

$$AP^{\pi} = P^{\pi}A, \ \forall \pi$$

and this implies that the subspaces $h_{+}^{\hat{\otimes}N}$ et $h_{-}^{\hat{\otimes}N}$ are invariant under the action of the operators A. (The symmetric group approach to decompose the wave function can be found in ref. [87]).

2.5 Schur and Littlewood series

We want to present now some formal series of symmetric functions that will be useful in the next chapters (see [67, 70, 8]). Let $X = \{x_1, x_2, \ldots, x_n\}$. Then we have Schur's identity

$$\prod_i \frac{1}{1-x_i} \prod_{i<j} \frac{1}{1-x_i x_j} = \sum_{\lambda} s_{\lambda}(X)$$

and the following are due to Littlewood

$$B = \prod_{i<j} \frac{1}{1-x_i x_j} = \sum_{\beta} s_{\beta}(X)$$

where β runs over the set of partitions with even columns, and

$$D = \prod_{i\leq j} \frac{1}{1-x_i x_j} = \sum_{\delta} s_{\delta}(X)$$

where δ runs over the set of partitions with even rows. We will also need the inverses of these series, that are written by using the Frobenius notation for partitions defined earlier.

Hence the series $C = \frac{1}{D}$ is given by

$$C = \frac{1}{D} = \sum_{\nu} (-1)^{\frac{\rho}{2}} s_{\nu}(X)$$

with $\rho = |\nu|$ and $\nu = \begin{pmatrix} a_1+1 & \ldots & a_k+1 \\ a_1 & \ldots & a_k \end{pmatrix}$

similarly,

$$A = \frac{1}{B} = \sum_{\alpha} (-1)^{\frac{|\alpha|}{2}} s_{\alpha}(X)$$

with α of the form:

$$\alpha = \begin{pmatrix} a_1 & \ldots & a_k \\ a_1+1 & \ldots & a_k+1 \end{pmatrix}$$

These series can be expressed in terms of the plethysm operator, that we will now define.

2.5.1 Plethysm

It is a composition law between symmetric functions written $f[g] = f \circ g$, or $g \otimes f$ in Littlewood's notation ([70]), not to be confused with the tensor product sign. By definition, $f[g] = f(m_1, m_2, \dots)$ where the m_i are the monomials of g supposed with positive integer coefficients, each repeated exactly a number of times equal to its coefficient.

One sees that if f is expressed as a function of the power sums by $f = F(p_1, p_2, \cdots)$ then $f[g] = F(p_1[g], p_2[g], \cdots)$. Moreover the operators $\psi_k \ : \ g \mapsto p_k[g]$ are linear and multiplicative, and the definition gives directly $\psi_k(p_n) = p_k[p_n] = p_{kn}$. This permits to extend the definition in the case where the coefficients of g are not positive integers.

This operation represents the composition of polynomial representations of the linear groups at the level of characters.

Let $\rho : GL(n) \longrightarrow GL(N)$ a polynomial representation, One can take another representation $\eta : GL(N) \longrightarrow GL(M)$ and compute the composition $\eta \circ \rho : GL(n) \longrightarrow GL(M)$. The character of ρ is

$$\chi_\rho = \mathrm{tr}\ \rho \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} .$$

For example, if $\rho = \Lambda^2$ (the second exterior power), this is $e_2(X) = x_1x_2 + x_1x_3 + x_2x_3$. But $\rho(X) = \Lambda^2(X)$ is again diagonal and is:

$$\begin{pmatrix} x_1x_2 & & \\ & x_1x_3 & \\ & & x_2x_3 \end{pmatrix}$$

If the character of η is χ_η, then the one of $\eta \circ \rho$ is

$$\chi_{\eta\circ\rho} = \mathrm{tr} \quad \eta \begin{pmatrix} x_1x_2 & & \\ & x_1x_3 & \\ & & x_2x_3 \end{pmatrix} = \chi_\eta(x_1x_2, x_1x_3, x_2x_3) = \chi_\eta[\chi_\rho] \ .$$

The series of Schur and Littlewood can be now rewritten in terms of plethysms. Indeed,

$$\prod_i \frac{1}{1-x_i} \prod_{i<j} \frac{1}{1-x_ix_j} = \sigma_1(X) \cdot \sigma_1[e_2(X)] = \sigma_1[e_1(X) + e_2(X)] \ .$$

We have also the following relations:

$$\begin{cases} B = \sigma_1[e_2] & D = \sigma_1[h_2] \\ A = \lambda_{-1}[e_2] & C = \lambda_{-1}[h_2] \end{cases}$$

2.6 Characters of representations

We now come back now to the notion of a character and make more precise the connections with symmetric functions.

2.6.1 The symmetric group representations

The irreducible representations of the symmetric group S_n are indexed by the partitions of n. These representations are denoted by V_λ and satisfy

$$\dim V_\lambda = \chi^\lambda(\mathrm{id}) =< s_\lambda, p_1^n >= f^\lambda \ ,$$

where f^λ is the number of standard tableaux of shape λ. The bases of V_λ are labelled by the standard tableaux $E \in ST(\lambda)$. For example, for $V_{(2,1)}$, we have

$$\begin{ytableau} 2 \\ 1 & 3 \end{ytableau} \qquad \begin{ytableau} 3 \\ 1 & 2 \end{ytableau} \qquad .$$

The first construction of the representations was given by Young. There is a simple presentation due to Specht (1934), that we will describe now. Here V_λ is realized as a subspace of $\mathbb{C}[x_1, \cdots, x_n]$. Consider now each tableau as decomposed into the product of his columns. For example

$$\begin{ytableau} 8 \\ 4 & 9 \\ 3 & 5 & 7 \\ 1 & 2 & 6 \end{ytableau} = \begin{ytableau} 8 \\ 4 \\ 3 \\ 1 \end{ytableau} \quad \begin{ytableau} 9 \\ 5 \\ 2 \end{ytableau} \quad \begin{ytableau} 7 \\ 6 \end{ytableau}$$

where each column is interpreted as a Vandermonde determinant

$$\begin{ytableau} i_k \\ \none[] \\ i_1 \end{ytableau} = \prod_{r>s}(x_{i_r} - x_{i_s}) = \begin{vmatrix} 1 & x_{i_1} & x_{i_1}^2 & \cdots & x_{i_1}^{k-1} \\ 1 & x_{i_2} & x_{i_2}^2 & \cdots & x_{i_2}^{k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{i_k} & x_{i_k}^2 & \cdots & x_{i_k}^{k-1} \end{vmatrix} \ .$$

For example

$$\begin{ytableau} 5 \\ 3 & 4 \\ 1 & 2 & 6 \end{ytableau} = (x_3 - x_1)(x_5 - x_1)(x_5 - x_3) \cdot (x_4 - x_2) \cdot 1$$

with $\boxed{i} = |i| = 1$. Let us give an example in S_3, with the permutation $\sigma_1 = (213)$. For the partition (21), we have the following tableaux and Specht polynomials

$$\begin{ytableau} 2 \\ 1 & 3 \end{ytableau} = x_2 - x_1, \quad \begin{ytableau} 3 \\ 1 & 2 \end{ytableau} = x_3 - x_1 \ .$$

If we apply now σ_1 to the tableaux, we obtain

$$\sigma_1\left(\begin{array}{cc}\boxed{2} & \\ \boxed{1} & \boxed{3}\end{array}\right) = \begin{array}{cc}\boxed{1} & \\ \boxed{2} & \boxed{3}\end{array} = -\begin{array}{cc}\boxed{2} & \\ \boxed{1} & \boxed{3}\end{array}$$

and

$$\sigma_1\left(\begin{array}{cc}\boxed{3} & \\ \boxed{1} & \boxed{2}\end{array}\right) = \begin{array}{cc}\boxed{3} & \\ \boxed{2} & \boxed{1}\end{array} = x_3 - x_2 = \begin{array}{cc}\boxed{3} & \\ \boxed{1} & \boxed{2}\end{array} - \begin{array}{cc}\boxed{2} & \\ \boxed{1} & \boxed{3}\end{array}.$$

Finally we have

$$\rho_{21}(\sigma_1) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$$

2.6.2 Representations of the linear group $GL(n, \mathbb{C})$

The irreducible polynomial representations of the linear group $GL(n, \mathbb{C})$ are constructed from the symmetric group representations. These representations are parametrized by the partitions having no more than n rows, and their bases are indexed by by semi-standard tableaux on the alphabet $\{1, \cdots, n\}$.

We first give two particular examples of great importance.

Exterior powers

The kth exterior power $\Lambda^k(A)$ of a matrix A is formed by the $k \times k$ minors of A. The Binet-Cauchy formula

$$\Lambda^k(AB) = \Lambda^k(A)\Lambda^k(B)$$

shows that $A \mapsto \Lambda^k(A)$ is a representation of $GL(n, \mathbb{C})$. This representation is clearly polynomial. Moreover the character of this representation is given by

$$\text{trace}\ \Lambda^k \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} = e_k(x_1, \cdots, x_n) = s_{1^k}(x_1, \cdots, x_n)$$

Symmetric powers

Let V a vector space. The symmetric power $S^n(V)$ is the space of homogenous polynomials of degree n in the elements of V.

Consider now the symmetric power representation defined by $A \mapsto S^k(A)$. The character of this representation is

$$\operatorname{trace} S^k \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} = h_k(x_1, \cdots, x_n) = s_k(x_1, \cdots, x_n) \ .$$

These representations are irreducible. However, there are other irreducible representations. For $GL(n)$, one can find a representation for each partition λ in no more than n parts, $S_\lambda(\mathbb{C}^n)$. The corresponding character is the Schur function $s_\lambda(x_1, \cdots, x_n)$ (see e.g. [25]).

It is possible to realize $S_\lambda(\mathbb{C}^n)$ as a subspace of $\mathbb{C}[a_{ij};\ 1 \leq i;\ j \leq n]$.

In the Specht representation, one interprets $\begin{array}{|c|}\hline 6 \\ \hline 3 \\ \hline 2 \\ \hline \end{array}$ as the minor on the columns $2, 3, 6$ and the three first rows of the Vandermonde matrix

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^k & \vdots & \vdots & \vdots & x_n^k \end{vmatrix} .$$

For the Schur representation of $GL(n, \mathbb{C})$, one does exactly the same but but one interprets the same tableau as the corresponding minor in the matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} .$$

For example, for $GL(3, \mathbb{C})$, we have

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} .$$

and then

$$\begin{array}{|c|c}\hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{|c|}{1} \\ \hline \end{array} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} a_{11} \ ,$$

$$\begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{|c|}{2} \\ \hline \end{array} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} a_{12} \ .$$

One defines $S_\lambda(\mathbb{C}^n)$ as the subspace generated by the semi-standard tableaux of shape λ over $\{1,\cdots,n\}$. Then one defines the following action

$$\rho(B)\,(f(a_{ij})) = f(a'_{ij}),\;\; A' = AB$$

given by

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{pmatrix} = \begin{pmatrix} x_1 a_{11} & x_1 a_{12} & x_1 a_{13} \\ x_2 a_{21} & x_2 a_{22} & x_2 a_{23} \\ x_3 a_{31} & x_3 a_{32} & x_3 a_{33} \end{pmatrix}$$

then

$$\left(\begin{array}{|c|c} \hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{|c|}{1} \\ \hline \end{array} \right)' = \begin{vmatrix} x_1 a_{11} & x_1 a_{12} \\ x_2 a_{21} & x_2 a_{22} \end{vmatrix} x_1 a_{11} = x_1^2 x_2 \; \begin{array}{|c|c} \hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{|c|}{1} \\ \hline \end{array}$$

One sees that each standard tableau T is multiplied by the monomial x^T obtained by reading its entries. One can verify that the character of the representation is the Schur function s_λ.

2.6.3 Characters of $O(n, \mathbb{C})$

One has $O(n,\mathbb{C}) \subset GL(n,\mathbb{C})$. Let V_λ (also denoted by $\{\lambda\}$) an irreducible polynomial representation of $GL(n,\mathbb{C})$ indexed by the partition λ. One can restrict this representation to $O(n, C)$. This restriction is not irreducible in general.

The eigenvalues of an element of $O(n)$ come in pairs $z, \frac{1}{z}$. An element of $O(n)$ is similar to a diagonal matrix of the form

$$\text{for } O(2n) : z = \begin{pmatrix} z_1 & & & & \\ & \frac{1}{z_1} & & & \\ & & \ddots & & \\ & & & z_n & \\ & & & & \frac{1}{z_n} \end{pmatrix}$$

and

$$\text{for } O(2n+1) : z = \begin{pmatrix} z_1 & & & & & \\ & \frac{1}{z_1} & & & & \\ & & \ddots & & & \\ & & & z_n & & \\ & & & & \frac{1}{z_n} & \\ & & & & & 1 \end{pmatrix}$$

The character of a representation ρ is by definition the Laurent polynomial $\chi_\rho(z_1,\cdots,z_n) = \text{trace}\rho(z)$. One can show that it is of the form $f(z_1,\frac{1}{z_1},\ldots,z_n,\frac{1}{z_n})$ or $f(z_1,\frac{1}{z_1},\ldots,z_n,\frac{1}{z_n},1)$

for a symmetric function f.

In this case it is especially interesting to work with Littlewood's method i.e. to work as long as possible with symmetric functions that we can specialize at the end of the calculation.

Now we consider the irreducible polynomial tensor representations of $O(n)$. They are indexed by partitions λ with length satisfying

$$l \leq \lfloor \frac{n}{2} \rfloor \ .$$

One has two representations for each partition : $[\lambda]$ and $[\lambda]^*$ that verify $\rho^*_\lambda = \det \otimes \rho_\lambda$
For example $[0]$ = trivial representation, $[0]^*$ = determinant : $A \mapsto \det A = \pm 1$.
Now, all partitions are not admissible. One needs to supply some constraints as given by Newell [82] that we recall here.

For $O(2n)$ all the $[\lambda]$ verifying $l(\lambda) > n$ are equal to zero except the $[\lambda_1, \cdots, \lambda_n, \rho]$ with

$$\rho = \begin{pmatrix} a+1 & b+1 & c+1 & \cdots \\ a & b & c & \cdots \end{pmatrix} = (2), (31), (411), \cdots$$

If r is the rank of ρ, i.e. the number of columns in the Frobenius notation, then we have the following expression

$$[\lambda_1, \cdots, \lambda_n, \rho] = (-1)^{\frac{\rho}{2}} \varepsilon^r [\lambda_1, \cdots, \lambda_n] \ .$$

In the case of $O(2n+1)$, $[\lambda] = 0$ except for $[\lambda_1, \cdots, \lambda_n, \eta]$ with $\tilde{\eta} = \eta$, where $\tilde{\eta}$ is the conjugate of η. With these conditions one can then write

$$[\lambda_1, \cdots, \lambda_n, \eta] = (-1)^{\frac{|\eta|-r}{2}} \varepsilon^{|\eta|} [\lambda_1, \cdots, \lambda_n]$$

We now come back to characters. $[\lambda]$ has for character $o_\lambda(z_1, \frac{1}{z_1}, \ldots)$ where o_λ is a symmetric function, called an *orthogonal Schur function.*

Following Littlewood, working in Sym with an infinite set of variables, one can express the orthogonal Schur functions as:

$$o_\lambda = D_{\lambda_{-1}[h_2]} s_\lambda$$

In the traditional notations of physicists we have $\lambda_{-1}[h_2] = C$ and the expression $D_{\lambda_{-1}[h_2]} s_\lambda$ is written $\{\lambda / C\}$.

Thus,

$$s_\lambda = D_{\sigma_1[h_2]} o_\lambda$$

and

$$s_\lambda = D_{\sigma_1[h_2]} o_\lambda = D_{\sigma_1[h_2]} D_{\lambda_{-1}[h_2]} s_\lambda = D_{\lambda_{-1}[h_2]} \cdot D_{\sigma_1[h_2]} s_\lambda \ .$$

Then,

$$D_{\sigma_1[h_2]} s_\lambda = \sum_\delta s_{\lambda/\delta} = \sum c^\mu_\lambda s_\mu$$

with $c^\mu_\lambda \geq 0$.

Finally

$$s_\lambda = \sum_\mu c^\mu_\lambda D_{\lambda_{-1}[h_2]} s_\mu = \sum_\mu c^\mu_\lambda o_\mu$$

One obtains in this way the branching rule for $GL(n)$ or $U(n) \downarrow O(n)$.
By restriction from $U(n)$ to $O(n)$, the irreducible representation $\{\lambda\}$ is decomposed in

$$\{\lambda\} \to \sum_\mu c^\mu_\lambda [\mu]$$

where the c^μ_λ are the coefficients of the expansion of $\sum_\delta s_{\lambda/\delta}$ over the basis (s_μ).

We now give some examples of this development, obtained for the restriction $U(6) \downarrow O(6)$. We have performed this computation with the program SCHUR of B. G. Wybourne.

```
U(6) to O(6)
BRM>
21
      [21] + [1]
BRM>
42
      [42] + [4] + [31] + [2^2 ] + 2[2] + [0]
BRM>
521
      [521] + [51] + [42] + [41^2 ] + [4] + [321] + 2[31]
       + [2^2 ] + [21^2 ] + [2] + [1^2 ]
BRM>
11111
      [1]#
BRM>
444222
      [2^3 ] + [2^2 ] + [2] + [0]
BRM>
8331
      [832] + [83]# + [821] + [81]# + [73^2 ] + 2[731]
       + [72^2 ] + [72]# + [72] + 2[71^2 ] + 2[632] + 2[63]#
       + [63] + 3[621] + 2[61]# + [61] + [53^2 ] + 3[531]
       + [52^2 ] + [52]# + 2[52] + 3[51^2 ] + 2[432] + 2[43]#
       + [43] + 3[421] + 2[41]# + [41] + [3^3 ] + 3[3^2 1]
```

```
       + [32^2 ] + [32]# + 2[32] + 3[31^2 ] + [2^2 1] + [21]#
       + [21] + [1^3 ]
BRM>
```

The # symbol corresponds to the symbol " *" of the text.

2.7 Inner plethysm

Inner plethysm is a fundamental operation on representations of the symmetric group, that occurs frequently in the calculation of branching rules for classical groups.

Let $\rho : S_n \to GL(m)$ be a representation of S_n of characteristic $g \in Sym_n$, and $\eta : GL(m) \to GL(N)$ a representation of character $f \in Sym$.

The inner plethysm of g by f, denoted $\hat{f}[g] \in Sym_n$, is the characteristic of $\eta \circ \rho : S_n \to GL(N)$

This operation is difficult to evaluate but there are a few closed formulas in the fundamental cases where $\eta = S^k$ and $\eta = \Lambda^k$.

2.7.1 Case $\eta = \Lambda^k$

Let $\chi(\sigma) = \mathrm{tr}\rho(\sigma)$ be the character of ρ. The character of $\eta \circ \rho$ is given by $\sigma \mapsto \mathrm{tr}(\Lambda^k \rho(\sigma))$ where the $\mathrm{tr}(\Lambda^k \rho(\sigma))$ are the coefficient of the characteristic polynomial of the matrix $\rho(\sigma) \in GL(m)$.

Indeed, more generally one can write

$$\det(I - xA) = \sum_{k=0}^{\dim(A)} (-1)^k \mathrm{tr}(\Lambda^k A) \cdot x^k$$

Now it is important to know how it is possible to calculate in terms of $\alpha \vdash n$, the trace $\mathrm{tr}(\Lambda^k \rho(\sigma))$ for a permutation of cycle type α. One can do this calculation when $\rho : S_n \to GL(n)$ is the permutation matrix representation. It is equivalent to calculating the characteristic polynomial of a permutation matrix of type α . Suppose that $\alpha = (1^{m_1} 2^{m_2} \ldots n^{m_n})$ where $\alpha \vdash n$. One can choose in the class α a permutation σ having the following cyclic structure:

$$\sigma = \underbrace{(1)(2)\ldots(m_1)}_{\text{fixed points}} \quad \underbrace{(m_1+1, m_1+2)\ldots}_{2-\text{cycles}} \quad \cdots$$

Then, $\rho(\sigma)$ is block diagonal, and

$$\begin{aligned}\det(I - x\rho(\sigma)) &= (1-x)^{m_1}\det\left(I_2 - x\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)^{m_2}\cdots\\ &= \prod_i [P_i]^{m_i}\end{aligned}$$

where $P_i(x)$ is the characteristic polynomial of the i-cycle $(12\ldots i)$ matrix that corresponds to the matrix

$$\begin{pmatrix}0 & 0 & \ldots & 1\\ 1 & 0 & \ldots & 0\\ 0 & 1 & \ldots & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \ldots & 0\end{pmatrix}.$$

In particular, one verifies that $P_i = 1 - x^i$ and finally one obtains

$$\det(I - x\rho(\sigma)) = \prod_i (1 - x^i)^{m_i}$$

Now it is possible to rewrite this polynomial in another manner. One can write, using the operator notation (see [97]) that

$$p_\alpha(1-x) \stackrel{def}{=} \prod p_{\alpha_i}(1-x)$$

where, $p_{\alpha_i}(1-x) \stackrel{def}{=} 1 - x^{\alpha_i}$.

Indeed, by definition one has

$$\sigma_t(1-x) = \frac{\sigma_t(1)}{\sigma_t(x)} = \frac{\frac{1}{(1-t)}}{\frac{1}{(1-tx)}} = \frac{1-tx}{1-t}$$

Recall now the Cauchy formula :

$$h_n((1-x)X) = \sum_{\lambda\vdash n} p_\lambda(1-x)\frac{p_\lambda}{z_\lambda} = \sum_{\lambda\vdash n} s_\lambda(1-x)s_\lambda\ .$$

If σ is of type α, then tr $\left(\Lambda^k\rho(\sigma)\right)$ corresponds to the coefficient of $(-x)^k$ in $p_\alpha(1-x) = \langle p_\alpha, h_n\left((1-x)X\right)\rangle$. In fact, the Frobenius characteristic F of a representation ρ is defined by the property

$$\mathrm{tr}\rho(\sigma) = \langle p_\alpha, F\rangle\ .$$

Then the characteristic of $\Lambda^k\rho$ is equal to the coefficient of $(-x)^k$ in $h_n\left((1-x)X\right)$.

Moreover this is $\hat{e}_k\left[h_1 h_{n-1}\right]$ where e_k is the character of Λ^k and $h_1 h_{n-1}$ is the characteristic of ρ. Thus

$$\sum_{k=0}^{n}(-x)^k \hat{e}_k\left[h_1 h_{n-1}\right] = h_n\left((1-x)X\right)$$

or

$$\hat{\lambda}_{-x}\left[h_1 h_{n-1}\right] = h_n\left((1-x)X\right)$$

and $h_n\left((1-x)X\right)$ represents the exterior algebra of the representation ρ.

2.7.2 The case $\eta = S^k$

In the same way, using

$$\det(1-xA)^{-1} = \sum_{k\geq 0} x^k \mathrm{tr}(S^k A)$$

one obtains

$$\sum_{k\geq 0} \hat{h}_k\left[h_1 h_{n-1}\right] x^k = h_n\left(\frac{X}{1-x}\right)$$

Then to decompose into irreducibles the kth symmetric power $S^k(\rho)$, it is needed to take the coefficient of x^k in the development of $h_n\left(\frac{X}{1-x}\right)$ and write it on the Schur function basis.

2.7.3 General case

In this subsection we will introduce a new notion, the internal product that is needed for further developments.

We have seen that to an irreducible character χ^λ of the symmetric group, one associates the Schur function s_λ. The linear map $\mathcal{F} : \chi^\lambda \mapsto s_\lambda$ is called Frobenius characteristic. The inner product is the operation on symmetric functions corresponding to the product of characters, i.e. to the tensor product of representations (see [73, 25]). This product is defined by :

$$s_\lambda * s_\mu = \mathcal{F}\left(\chi^\lambda \chi^\mu\right)\ .$$

The Frobenius formula $p_\mu = \sum_\lambda \chi^\lambda_\mu s_\lambda$ and the orthogonality relations imply

$$p_\mu * p_\nu = \delta_{\mu\nu} z_\nu p_\nu\ .$$

One can deduce a generalisation of the Cauchy formula

$$s_\lambda(XY) = \sum_\mu \left(s_\lambda * s_\mu\right)(X) s_\mu(Y)\ ,$$

or more generally

$$f(XY) = \sum_\mu \left(f * u_\mu\right)(X) v_\mu(Y)\ ,$$

where (u_μ) and (v_μ) are any pair of adjoint bases.

We are now interested in the inner plethysm by the power sum p_k. If g is the characteristic of ρ, $\hat{p}_k[g]$ cannot correspond in general to a true representation (because it is not positive on the s_λ) but corresponds to a certain central function. If σ is of type α then

$$\langle p_\alpha, \hat{p}_k[g]\rangle = \mathrm{tr}(\rho(\sigma))^k = \mathrm{tr}\ \rho(\sigma^k)\ .$$

(One remarks that $\sigma \mapsto \rho(\sigma^k)$ is not a representation because in general $\rho\left((\sigma\tau)^k\right) \neq \rho(\sigma^k)\rho(\tau^k)$).

But tr $\rho(\sigma^k)$ is the sum of the kth powers of the roots of the characteristic polynomial of $\rho(\sigma)$. By the Newton formulae, one can obtain the coefficients of the characteristic polynomial, that are $\pm\mathrm{tr}(\Lambda^k)\rho(\sigma)$.

For any f, one can calculate $\hat{f}[g]$ using the development of f on the (p_μ) basis. Then :

$$f = \sum c_\mu p_\mu \Rightarrow \hat{f}[g] = \sum c_\mu \hat{p}_\mu[g]$$

and

$$\hat{p}_\mu[g] = \hat{p}_{\mu_1}[g] * \hat{p}_{\mu_2}[g] * \cdots * \hat{p}_{\mu_n}[g]$$

In general,

$$\left(\widehat{f_1 f_2}\right)[g] = \hat{f}_1[g] * \hat{f}_2[g]$$

Now the next step is to calculate $\hat{p}_k[g]$. One has the following relations

$$\hat{p}_k[g_1 + g_2] = \hat{p}_k[g_1] + \hat{p}_k[g_2]$$

and

$$\hat{p}_k[g_1 * g_2] = \hat{p}_k[g_1] * \hat{p}_k[g_2]$$

One can see what is given by these relations for $g = p_\alpha$.

Let $\hat{\phi}_k$, be the adjoint of the linear operator $g \mapsto \hat{p}_k[g]$ for the standard scalar product, i.e.:

$$\langle \hat{\phi}_k[g_1], g_2\rangle = \langle g_1, \hat{p}_k[g_2]\rangle$$

It is easy to calculate its effect on $p_\mu = p_1^{m_1} p_2^{m_2} \ldots$

$$\langle \hat{\phi}_k[p_\alpha], p_\beta\rangle = \langle p_\alpha, \hat{p}_k[p_\beta]\rangle\ .$$

But we have $p_\beta \approx K_\beta$, the characteristic function of the conjugacy class of β. Indeed,

$$K_\beta(\sigma) \stackrel{def}{=} \begin{cases} 1 & \text{if } \sigma \text{ is of type } \beta \\ 0 & \text{else} \end{cases}$$

has for characteristic $\frac{p_\beta}{z_\beta}$ and it follows that

$$K_\beta(\sigma) = \langle p_\alpha, \frac{p_\beta}{z_\beta}\rangle$$

where α is of type σ. One can continue by writing

$$\begin{aligned}\langle \hat{\phi}_k[p_\alpha], p_\beta\rangle &= z_\beta\langle p_\alpha, \hat{\phi}_k[\frac{p_\beta}{z_\beta}]\rangle \\ &= z_\beta K_\beta(\sigma^k)\end{aligned}$$

thus

$$\hat{\phi}_k[p_\alpha] = p_\gamma$$

where γ is the cycle type of σ^k.
Also one can show that if $\alpha = 1^{m_1}2^{m_2}\ldots$ and if $d_i = \gcd(i,k)$, the cycle type of σ^k is

$$\hat{\phi}_k[p_\alpha] = \prod_i \left(\frac{p_i}{d_i}\right)^{m_i d_i} \quad .$$

Comparing $\hat{\phi}_k[p_\alpha p_\beta]$ with $\hat{\phi}_k[p_\alpha]\hat{\phi}_k[p_\beta]$, one sees that

$$\hat{\phi}_k[fg] = \hat{\phi}_k[f]\hat{\phi}_k[g]$$

which shows that $\hat{\phi}_k$ is a homomorphism for the ordinary product and therefore is relatively easy to calculate. Now we return to the calculation of $\hat{f}[g]$ in a more general case. One has to calculate $\hat{p}_k[g]$. Then we obtain from the previous calculations

$$\begin{aligned}\hat{p}_k[g] &= \sum_{\alpha\vdash n}\langle \hat{p}_k[g], p_\alpha\rangle\frac{p_\alpha}{z_\alpha} \\ &= \sum_{\alpha\vdash n}\langle g, \hat{\phi}_k[p_\alpha]\rangle\frac{p_\alpha}{z_\alpha} \ .\end{aligned}$$

Now, concerning $\hat{s}_\lambda[h_1h_{n-1}]$, we have the reciprocity formula, with $\sigma_1 = \sum h_n$,

$$\langle \hat{s}_\lambda[h_1\sigma_1], s_\mu\rangle = \langle s_\lambda, s_\mu[\sigma_1]\rangle \ .$$

This is a compact version of the Littlewood formula implemented in SCHUR (see [97]).

2.8 Reduced notation

For the symmetric groups S_n, $<\lambda>$ is the set of all the characters of the form $(n - |\lambda|, \lambda_1, \lambda_2, \ldots, \lambda_r)$ in formulae that are independent of the first part. In fact it is equivalent to associate to the symbol $<\lambda>$ the formal sum of the associated characteristics, i.e.

$$<\lambda> = \sum_{n\in\mathbb{Z}} s_{n-|\lambda|,\lambda}$$

An example :

$$\begin{aligned} <21> &= \cdots + s_{\bar{3}21} + s_{\bar{2}21} + s_{\bar{1}21} + s_{021} + s_{121} + s_{221} + s_{321} + s_{421} + \ldots \\ &= \Gamma_1(s_{21}) \\ &= \sigma_1 D_{\lambda_{-1}} s_{21} \\ &= \sigma_1(X) s_{21}(X-1) \end{aligned}$$

with $\lambda_{-1} = \sum(-1)^n e_n$. One sees through this example that is possible to use the vertex operators to give another interpretation of this reduced notation. Littlewood (1958) gives a formula for the calculation of $<\lambda> * <\mu>$. It is equivalent to evaluating

$$(\Gamma_1 s_\lambda) * (\Gamma_1 s_\mu) = ((\sigma_1 s_\lambda(X-1)) * ((\sigma_1 s_\mu(X-1)) \enspace .$$

We are led to the calculation of the expression $(\sigma_1 f) * (\sigma_1 g)$ and we obtain now a solution of the form $\sigma_1 \cdot F$, where F is to be determined. Using the Hopf algebra structure of symmetric functions we can obtain

$$(\sigma_1 f) * (\sigma_1 g) = \sigma_1 \sum_{\alpha,\beta} (D_{v_\alpha} f)(D_{v_\beta} g)(u_\alpha * u_\beta)$$

where (u_α) and (u_α) are any two adjoint bases of Sym. Two examples are important (see [70, 81, 100]) :

$$\begin{cases} u_\alpha = v_\alpha = s_\alpha & \text{(Littlewood)} \\ u_\alpha = p_\alpha, \quad v_\alpha = \frac{p_\alpha}{z_\alpha} & \text{(Murnaghan)} \end{cases}$$

2.8.1 An example

We want to establish the following result

$$\hat{e}_k[s_{n-1,1}] = s_{n-k,1^k}$$

One has

$$\begin{aligned} \sum(-x)^k \hat{e}_k[h_1 h_{n-1}] &= \sum(-x)^k \hat{e}_k[s_{n-1,1} + s_n] \\ &= \hat{\lambda}_{-x}[h_1 h_{n-1}] \\ &= h_n\left((1-x)X\right) \\ &= \hat{\lambda}_{-x}[s_{n-1,1} + s_n] \\ &= \hat{\lambda}_{-x}[s_{n-1,1}] * \hat{\lambda}_{-x}[s_n] \\ &= (1-x)\hat{\lambda}_{-x}[s_{n-1,1}] \end{aligned}$$

and we have

$$\hat{\lambda}_{-x}[s_{n-1,1}] = \frac{1}{1-x} h_n\left((1-x)X\right)$$

which gives

$$h_n\left((1-x)X\right) = \sum_{\alpha \vdash n} s_\lambda(1-x) s_\lambda(X)\ . \tag{2.31}$$

To go further we need the formula

$$s_\lambda(1-x) = \begin{cases} 0 & \text{if } \lambda \text{ is not a hook,} \\ (-x)^k(1-x) & \text{if } \lambda = (n-k,1)^k. \end{cases}$$

Indeed,

$$\begin{aligned} \sum_\lambda s_\lambda(1-x)s_\lambda(X) &= \sigma_1\left((1-x)X\right) \\ &= \sigma_1(X)\sigma_1(-xX) \\ &= \frac{\sigma_1(X)}{\sigma_x(X)} \\ &= \sigma_1(X)\lambda_{-x}(X) \\ &= \sum_{k\geq 0} x^k s_k \sum_{l \geq 0} (-x)^l s_{1^l} \end{aligned}$$

which gives the result. We now come back to the calculation. Equation (2.31) yields

$$\begin{aligned} \frac{h_n\left((1-x)X\right)}{1-x} &= \frac{\sum_{k=0}^{n-1}(-x)^k(1-x)s_{n-k,1^k}(X)}{1-x} \\ &= \sum_{k=0}^{n-1}(-x)^k s_{n-k,1^k} \\ &= \sum (-x)^k \hat{e}_k[s_{n-1,1}]\ , \end{aligned}$$

which concludes the proof.

Now, we can rewrite this last result in reduced notation

$$\langle 1 \rangle \otimes \{1^k\} = \langle 1^k \rangle$$

Moreover we can write

$$\sum_{n\in\mathbb{Z}} \hat{e}_k[s_{n-1,1}] = \hat{e}_k[\sum_{n\in\mathbb{Z}} s_{n-1,1}]$$

because $f * g = 0$ if f and g are homogenous of different degrees. More generally one can write

$$\hat{e}_k[\sum f_n] = \sum \hat{e}_k\,[f_n]$$

Finally the reduced notation is compatible with the vertex operators. In fact, we can identify $< 1 > \otimes\{1^k\}$ with $\hat{e}_k\,[\Gamma_1 s_1]$.
Now a problem is that we can obtain results in a simple manner only in the case

$$\langle 1\rangle \otimes f = \hat{f}\,[\sigma_1 s_1(X-1)]$$

To realize more complicated operations like for example $< \lambda > \otimes f$, it is necessary to express $< \lambda >$ as a linear combination of the form

$$\sum_i < 1 > \otimes g_i = \langle 1\rangle \otimes g\ , \quad (\ g = \sum_i g_i)$$

in this case

$$\langle\lambda\rangle \otimes f = \langle 1\rangle \otimes \underbrace{(f \otimes g)}_{\text{ordinary plethysm}} \quad .$$

2.9 Applications

In this chapter we have briefly presented symmetric functions and their role in representation theory. We will now illustrate these techniques in the case of the exceptionnal group G_2 and after this first application we will define more precisely the role of symmetric function in the case of non-compact Lie groups, and explore tensor products and plethysms for this class of groups.

Chapter 3

A finite subgroup of G_2

In this chapter one will find new results on the exceptional group G_2, obtained by means of techniques of symmetric function and representation theory. This work is the subject of a joint paper with R. C. King and B. G. Wybourne, paper submitted to Journal of Physics A : Mathematical and General.

3.1 Introduction

The exceptional group G_2, while known to mathematicians since Cartan's thesis [13] of 1894 was not introduced to physicists until some fifty years ago in Racah's [90] 1949 paper "The Theory of Complex Spectra IV" in which he used the group G_2 in his classification of the states of the f^n electron configurations and in the simplification of the calculation of the matrix elements of the Coulomb and spin-orbit interactions[1]. Racah was able to exploit the fact that G_2 occurs as a subgroup of the rotation group in seven dimensions, $SO(7)$, and furthermore, that the physical rotation group $SO(3)$ associated with the angular momentum of electrons in $f-$orbitals occurs as a natural subgroup of G_2. Racah's work has been well described by Judd [43]. In nuclear physics [23] it was also possible to exploit the group G_2 in the classification of the states of the $f-$shell. The group $SU(3)$ also occurs as a subgroup of G_2, a fact that has been exploited in the interacting boson model of nuclei [78].

Judd [44] has suggested that it might be useful to find and exploit some non-trivial finite subgroups of G_2 noting in particular the subgroups of the symmetric group S_8 listed by Littlewood [70]. This possibility is encouraged by the occurrence of irreducible representations of dimension 7 in Littlewood's groups of order 1344 and 168. The group of order 168 is of particular interest herein, it contains not only a dimension 7 irreducible representation but also a complex pair of irreducible representationsof dimension 3 which

[1]The introduction of G_2 in physics is said to have arisen in one of Racah's lectures in Jerusalem where he spoke about the problem of labelling for f and d electrons. He discussed the power of group theory in physics in general and spectroscopy in particular. One thing led to another and then he suddenly left the class. Later he told students that he was so consumed by the idea and that he did not notice leaving.....

is suggestive of an embedding in $SU(3)$. We shall designate Littlewood's order 168 group as L_{168} noting that it appears in the Atlas of Finite Groups [17] as $L_3(2) \sim L_2(7)$, meaning that it is isomorphic to $GL_3(2)$, the general linear group in 3-dimensions over a field of order 2, and to $PSL_2(7)$, the projective special linear group in 2-dimensions over a field of order 7. In addition the Atlas indicates that it is isomorphic to $PSU_2(7)$ and $O_3(7)$. The Atlas indicates that L_{168} may be specified in terms of generators and relations by $\langle R, S, T, \mid R^2 = S^3 = T^7 = RST = (TSR)^4 = 1\rangle$. It can then be realised as a subgroup of the symmetric group S_8 through the identification $R = (18)(27)(34)(56)$, $S = (128)(375)$ and $T = (1234567)$. Since these permutations are all of even parity, L_{168} is also a subgroup of the alternating group A_8.

In the Atlas account of L_{168} nothing is said about G_2, nor indeed $O(7)$, the orthogonal group in 7 dimensions. However, a search of the literature reveals [16, 31, 32] that L_{168} is indeed a subgroup of G_2. Moreover, it has been established [31] that any two embeddings of a finite group in G_2 are conjugate if and only if they afford the same character on the natural 7-dimensional representation of $SO(7)$. In this paper it is confirmed by character theoretic and other methods that there exist two non-conjugate embeddings of L_{168} in G_2. These embeddings are explored in some detail along with the corresponding branching rules.

We first establish in section 2 some simple properties of the group-subgroup pair $S_8 \supset L_{168}$ that follow directly from their respective character tables and then explicitly identify L_{168} as a finite subgroup of both G_2 and $SU(3)$ by means of techniques based on the consideration of various plethysms, or symmetrised tensor and indeed spinor products. This is augmented in section 4 by an alternative derivation of the explicit form of the embeddings that is based on the use of a rather simple necessary and sufficient condition for any finite group H to be a subgroup of G_2. This derivation involves a consideration not just of the characters of representations, but of the eigenvalues of group elements in those representations.

We then compute in section 5 some explicit branching rules for $G_2 \to L_{168}$ and for $SU(3) \to L_{168}$ giving sufficient data for possible future applications to such things as a study of the properties of f-electrons in an octahedral environment. To this end the branching rules for L_{168} to the octahedral point group O are tabulated in section 6.

3.2 L_{168} as a subgroup of S_8

The characters of the group L_{168} as given by Littlewood [70] are reproduced in Table 1. We label the 6 irreducible representations of L_{168} by the letter sequence a, b, h, j, k, l. In the notation of the Atlas [17] the corresponding characters are denoted by $\chi_1, \chi_4, \chi_5, \chi_6, \chi_2, \chi_3$, respectively. In particular the irreducible representation a with character χ_1 is the trivial, identity representation of L_{168}.

Making use of the character table [70] for S_8 together with Table 1 readily gives the $S_8 \to L_{168}$ branching rules given in Table 2. Since under $S_8 \to L_{168}$ the decompositions of irreducible representations labelled by conjugate partitions are the same we give only the decomposition in the case of one partition of each conjugate pair.

Table 1. Character table for L_{168}.

cycles	(1^8)	(2^4)	$(1^2 3^2)$	(4^2)	(17)	(17)
order	1	21	56	42	24	24
a	1	1	1	1	1	1
b	6	2	0	0	-1	-1
h	7	-1	1	-1	0	0
j	8	0	-1	0	1	1
k	3	-1	0	1	$\frac{1}{2}(-1+i\sqrt{7})$	$\frac{1}{2}(-1-i\sqrt{7})$
l	3	-1	0	1	$\frac{1}{2}(-1-i\sqrt{7})$	$\frac{1}{2}(-1+i\sqrt{7})$

Table 2. Branching rules for $S_8 \to L_{168}$

Dim	$S_8 \quad \to$	L_{168}
1	$\{8\}$	a
7	$\{71\}$	h
20	$\{62\}$	$2b + j$
21	$\{61^2\}$	$h + j + k + l$
28	$\{53\}$	$2h + j + k + l$
64	$\{521\}$	$2b + 2h + 4j + k + l$
35	$\{51^3\}$	$a + 2b + 2h + j$
14	$\{4^2\}$	$2a + 2b$
70	$\{431\}$	$2b + 4h + 3j + k + l$
56	$\{42^2\}$	$a + 4b + h + 3j$
90	$\{421^2\}$	$2b + 4h + 4j + 3k + 3l$
42	$\{3^2 2\}$	$2h + 2j + 2k + 2l$

Next we need to know the decomposition of Kronecker products of all the irreducible representations of L_{168}, and in the case of Kronecker squares their resolution into their symmetric and antisymmetric parts. Again these decompositions may be readily calculated from the character tables. The non-trivial products and their resolution are given in Table 3.

Table 3. The resolution of Kronecker products and plethysms for the non-trivial irre-

ducible representations of L_{168}.

Product	*Resolution*
$b \times h$	$b + 2h + 2j + k + l$
$b \times j$	$2b + 2h + 2j + k + l$
$b \times k$	$h + j + l$
$b \times l$	$h + j + k$
$h \times j$	$2b + 2h + 3j + k + l$
$h \times k$	$b + h + j$
$h \times l$	$b + h + j$
$j \times k$	$b + h + j + k$
$j \times l$	$b + h + j + l$
$k \times l$	$a + j$

Plethysm	*Resolution*
$b \otimes \{2\}$	$a + 2b + j$
$b \otimes \{1^2\}$	$h + j$
$h \otimes \{2\}$	$a + 2b + h + j$
$h \otimes \{1^2\}$	$h + j + k + l$
$j \otimes \{2\}$	$a + 2b + h + 2j$
$j \otimes \{1^2\}$	$2h + j + k + l$
$k \otimes \{2\}$	b
$k \otimes \{1^2\}$	l
$l \otimes \{2\}$	b
$l \otimes \{1^2\}$	k

It can be deduced from Table 3 that all irreducible representations other than the trivial representation a are faithful representations of L_{168}. This is a consequence of the fact that a representation λ_H of a finite group H is faithful if and only if all the irreducible representations of H appear in the decomposition of some Kronecker power[2] of λ_H. The same conclusion is reached by noting that L_{168} is simple.

Furthermore, an irreducible representation λ_H of a group H is orthogonal if and only if the symmetrised square $\lambda_G \otimes \{2\}$ contains the trivial, identity representation of H. From Table 3 it can be seen, therefore, that b, h, j are all orthogonal, while k and l are not orthogonal. This can also be deduced of course from the fact that their characters are complex. Indeed k and l constitute a mutually complex conjugate pair of unitary irreducible representations of L_{168}. This is sufficient to conclude that their direct sum $k + l$ is orthogonal.

3.3 L_{168} as a subgroup of G_2

It is well-known that the symmetric group S_{n+1} occurs as a subgroup of the full orthogonal group $O(n)$ and hence $O(7) \supset S_8$. In such an embedding the vector irreducible representation [1] of $O(7)$ decomposes irreducibly into the $\{71\}$ irreducible representation of S_8. The irreducible representation $\{71\}$ is orthogonal but not unimodular [49] and hence while S_8 may be embedded in $O(7)$ it cannot be embedded in $SO(7)$. The irreducible representations of G_2 are all orthogonal and unimodular, including the defining 7-dimensional irreducible representation. It follows that G_2 is subgroup of $SO(7)$. This is sufficient to show that S_8 is certainly not a subgroup of G_2.

In order to show that L_{168} is a subgroup of G_2 the key observation is to note the characterisation [6] of G_2 as the subgroup of $SO(7)$ which leaves a spinor invariant and to determine whether any unimodular, orthogonal, faithful 7-dimensional representation

[2] we use here Littlewood's notation for plethysm (cf 2.5.1)

of L_{168} shares this characteristic. In what follows two such representations are found, corresponding to two distinct embeddings of L_{168} in G_2. Since $SU(3)$ is a subgroup of G_2 the question arises as to whether or not an embedding of L_{168} in G_2 corresponds to an embedding of L_{168} in $SU(3)$. This can be established through the identification, if possible, of some unimodular, unitary, faithful 3-dimensional representation of L_{168}. One of the two embeddings identified below is of this type.

It is straightforward to dispose of the question of unimodularity. Quite generally any irreducible representation λ_H of a group H having dimension N is unimodular if and only if the Nth-fold antisymmetrised Kronecker power, $\lambda_H \otimes \{1^N\}$ is just the trivial, identity representation of H. Since $\lambda_H \otimes \{1^N\}$ necessarily has dimension 1, it follows that every irreducible representation λ_H of H is unimodular if H possesses no 1-dimensional irreducible representation other than the identity representation. This is true, for example, of the alternating group A_n, and as can be seen from Table 1 it is also true of L_{168}. It follows that every irreducible representationof L_{168} is unimodular. Moreover, since every representation is a direct sum of irreducible representations, all representations of L_{168} are unimodular.

Recalling the conclusions regarding the orthogonality of representations of L_{168} which we drew from Table 3, we can conclude that the only 7-dimensional unimodular orthogonal representations of L_{168} are h, $a+b$, $a+k+l$ and $7a$. Of these all are faithful except, of course, $7a$. Each of the others defines an embedding of L_{168} in $SO(7)$.

Before proceeding further we must address the labelling of the irreducible representations of G_2. Two schemes exist in the literature, the traditional scheme of Racah [90] and that based upon [52] the $SU(3)$ subgroup of G_2. In the latter scheme the irreducible representations of G_2 are labelled by (λ_1, λ_2) where λ_1 and λ_2 are non-negative integers such that $\lambda_1 \geq 2\lambda_2$. Racah's labels (u_1, u_2) are such that $u_1 = \lambda_1 - \lambda_2$ and $u_2 = \lambda_2$. Throughout this paper we shall use the $SU(3)$ labelling scheme with separational commas and trailing zeros normally omitted.

The defining representation of G_2 is the fundamental 7-dimensional irreducible representation (1). Separating the Kronecker square of this irreducible representation of G_2 into its symmetric and antisymmetric parts gives

$$(1) \otimes \{2\} = (2) + (0) \qquad \text{and} \qquad (1) \otimes \{1^2\} = (21) + (1).$$

It follows that if G_2 contains H as a subgroup with $(1) \to \lambda_H$, where λ_H is a representation, not necessarily irreducible, of H, then it follows that the identity representation, 1_H, of H must occur in the decomposition of the symmetric part $\lambda_H \otimes \{2\}$ of the Kronecker square of λ_H, and that λ_H itself must occur in the decomposition of the antisymmetric part $\lambda_H \otimes \{1^2\}$ of its Kronecker square, ie.

$$\langle \lambda_H \otimes \{2\}, 1_H \rangle \geq 1 \qquad \text{and} \qquad \langle \lambda_H \otimes \{1^2\}, \lambda_H \rangle \geq 1. \tag{3.1}$$

These are necessary, though not sufficient, conditions for H to be a subgroup of G_2.

The first of these conditions is guaranteed if λ_H is orthogonal, but this is not enough to guarantee the satisfaction of the second condition. For example, the fact that for S_8 we have

$$\{71\} \otimes \{2\} = \{8\} + \{71\} + \{62\} \qquad \text{and} \qquad \{71\} \otimes \{1^2\} = \{61^2\}$$

immediately precludes the possibility of S_8 being a subgroup of G_2 with an embedding such that $(1) \to \{71\}$. In the case of L_{168} and our three candidate 7-dimensional representations h, $a+b$ and $a+k+l$, it can be deduced from Table 3 that:

$$h \otimes \{2\} = a + 2b + h + j; \tag{3.2}$$
$$h \otimes \{1^2\} = h + j + k + l; \tag{3.3}$$
$$(a+b) \otimes \{2\} = 2a + 3b + h + j; \tag{3.4}$$
$$(a+b) \otimes \{1^2\} = b + h + j; \tag{3.5}$$
$$(a+k+l) \otimes \{2\} = 2a + 2b + j + k + l; \tag{3.6}$$
$$(a+k+l) \otimes \{1^2\} = a + j + 2k + 2l. \tag{3.7}$$

It follows that consistency with (3.1) is violated in the case of the representation $a+b$, but that both h and $a+k+l$ satisfy the requisite consistency conditions.

It remains now to determine whether or not in the reduction of the 8-dimensional spin irreducible representation Δ of $SO(7)$ to L_{168} a spinor is left invariant, just as it is in the case of the reduction from $SO(7)$ to G_2. This defining property [6] of G_2 is such that $\Delta \to (0) + (1)$, or equivalently

$$(1) \otimes \Delta = (0) + (1). \tag{3.8}$$

It follows that in the case of any faithful, unimodular, orthogonal 7-dimensional representation λ_H of H, a necessary and sufficient condition for H to be a subgroup of G_2 is that

$$\lambda_H \otimes \Delta = 1_H + \lambda_H. \tag{3.9}$$

Using the algebra of plethysms, the known plethysm of the spin representation Δ of $SO(7)$ [70] and (3.9) we have:

$$\begin{aligned}(\lambda_H \otimes \Delta) \otimes \{2\} &= \lambda_H \otimes (\Delta \otimes \{2\}) \\ &= \lambda_H \otimes \{1^3\} + 1_H;\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}(\lambda_H \otimes \Delta) \otimes \{2\} &= (1_H + \lambda_H) \otimes \{2\} \\ &= 1_H + \lambda_H + \lambda_H \otimes \{2\}.\end{aligned} \tag{3.11}$$

Comparison of (3.10) and (3.11) then implies that for H to be a subgroup of G_2 it is necessary that

$$\lambda_H \otimes \{1^3\} = \lambda_H + \lambda_H \otimes \{2\}\ . \tag{3.12}$$

The fact that, conversely, (3.12) implies ((3.9) is dealt with in section 5. By virtue of ((3.1) we have

$$\langle \lambda_H \otimes \{1^3\}, 1_H \rangle \geq 1. \tag{3.13}$$

This coincides with the condition (ii) of Lemma 2 listed by Cohen and Wales [16] as necessary for H to be a subgroup of G_2. Condition (3.12) is stronger, and when coupled with the orthogonality and unimodularity conditions on λ_H, is sufficient to ensure that H is a subgroup of G_2.

Remarkably, the plethysm identity (3.12) can only be true if

$$Dim(\lambda_H) = 7, \tag{3.14}$$

which is the case here.

Explicit evaluation of plethysms then yields the results:

$$\begin{aligned}
h \otimes \{1^3\} &= a + 2b + 2h + j, && (3.15)\\
h + (h \otimes \{2\}) &= a + 2b + 2h + j; && (3.16)\\
(a+b) \otimes \{1^3\} &= 3h + j + k + l; && (3.17)\\
(a+b) + ((a+b) \otimes \{2\}) &= 3a + 4b + j; && (3.18)\\
(a+k+l) \otimes \{1^3\} &= 3a + 2b + j + 2k + 2l, && (3.19)\\
(a+k+l) + ((a+k+l) \otimes \{2\}) &= 3a + 2b + j + 2k + 2l. && (3.20)
\end{aligned}$$

It follows that (3.12) is satisfied in the case of both h and $a + k + l$, but not in the case of $a + b$. This is entirely in line with the conclusion following (3.2-3.7).

We conclude that L_{168} is a subgroup of G_2. Indeed there exist two distinct embeddings of L_{168} in G_2 defined by

$$G_2 \to L_{168} : \quad (1) \to h \quad \text{and} \quad (1) \to a + k + l. \tag{3.21}$$

These two embeddings are analogous to the two embeddings of $SO(3)$ in G_2 defined by the restrictions $G_2 \to SO(3)$ with $(1) \to [3]$ and $G_2 \to SU(3) \to SO(3)$ with $(1) \to \{1\}+\{1^2\}+\{0\} \to 2[1]+[0]$. In fact the analogy can be extended further by noting that the second of our two embeddings of L_{168} in G_2 defined in (3.21) is such that $G_2 \supset SU(3) \supset L_{168}$ with

$$(1) \to \{1\} + \{1^2\} + \{0\} \to k + l + a. \tag{3.22}$$

This follows from the fact that the irreducible representation k of L_{168} is a faithful, unimodular, unitary 3-dimensional representation. It therefore defines an embedding of L_{168} in $SU(3)$ through the restriction

$$SU(3) \to L_{168} : \quad \{1\} \to k. \tag{3.23}$$

Since $\{1^2\}$ is complex conjugate to $\{1\}$ and l is the complex conjugate of k, we also have $\{1^2\} \to l$, as required to give (3.22).

3.4 The use of eigenvalues of representation matrices of L_{168}

As noted earlier, the group denoted here by L_{168} can be defined in terms of generators and relations by $\langle R, S, T, \mid R^2 = S^3 = T^7 = RST = (TSR)^4 = 1\rangle$. It can be realised as a subgroup of the symmetric group S_8. The cycle structure of elements in its 6 conjugacy classes are given by $(1^8), (2^4), (1^2 3^2), (4^2), (17), (17)$, so that the elements themselves have order $1, 2, 3, 4, 7, 7$, respectively. This implies that the eigenvalues of the matrices representing these elements can only be powers of the corresponding primitive roots of unity. This observation is sufficient in almost all cases to write down the complete set of eigenvalues of each irreducible matrix representation simply from a knowledge of the characters in Table 1, which are of course the sums of the eigenvalues. The only additional observation that is required in order to complete this exercise is that the cycle structure of TSR is (4^2) and that of $(TSR)^2$ is (2^4). The corresponding matrices can be simultaneously diagonalised so that in any given irreducible representation the eigenvalues in the conjugacy class labelled by (2^4) are the squares of those in the conjugacy class labelled by (4^2). The results for each of the irreducible representations of L_{168} are displayed in Tables 4a and 4b, where $\omega = e^{i2\pi/3}$ and $\eta = e^{i2\pi/7}$.

Table 4a. Eigenvalues of group elements in each conjugacy class of L_{168} in the irreducible representations a, b and h.

Irrep	a	b	h
Dim	1	6	7
(1^8)	1	$(1,1,1,1,1,1)$	$(1,1,1,1,1,1,1)$
(2^4)	1	$(1,1,1,1,-1,-1)$	$(1,1,1,-1,-1,-1,-1)$
$(1^2 3^2)$	1	$(1,1,\omega,\omega,\omega^2,\omega^2)$	$(1,1,1,\omega,\omega,\omega^2,\omega^2)$
(4^2)	1	$(1,1,-1,-1,i,-i)$	$(1,-1,-1,i,i,-i,-i)$
(17)	1	$(\eta,\eta^2,\eta^3,\eta^4,\eta^5,\eta^6)$	$(1,\eta,\eta^2,\eta^3,\eta^4,\eta^5,\eta^6)$
(17)	1	$(\eta,\eta^2,\eta^3,\eta^4,\eta^5,\eta^6)$	$(1,\eta,\eta^2,\eta^3,\eta^4,\eta^5,\eta^6)$

Table 4b. Eigenvalues of group elements in each conjugacy class of L_{168} in the irreducible representations j, k and l.

Irrep	j	k	l
Dim	8	3	3
(1^8)	$(1,1,1,1,1,1,1,1)$	$(1,1,1)$	$(1,1,1)$
(2^4)	$(1,1,1,1,-1,-1,-1,-1)$	$(1,-1,-1)$	$(1,-1,-1)$
$(1^2 3^2)$	$(1,1,\omega,\omega,\omega,\omega^2,\omega^2,\omega^2)$	$(1,\omega,\omega^2)$	$(1,\omega,\omega^2)$
(4^2)	$(1,1,-1,-1,i,i,-i,-i)$	$(1,i,-i)$	$(1,i,-i)$
(17)	$(1,1,\eta,\eta^2,\eta^3,\eta^4,\eta^5,\eta^6)$	(η,η^2,η^4)	(η^3,η^5,η^6)
(17)	$(1,1,\eta,\eta^2,\eta^3,\eta^4,\eta^5,\eta^6)$	(η^3,η^5,η^6)	(η,η^2,η^4)

By taking products of the tabulated eigenvalues it is easy to confirm, as stated in section 3, that each irreducible representation of L_{168} is unimodular. These tabulations may also be used to confirm the orthogonality of each representation. Here, however, we wish to show that the tabulations offer a very quick way to determine which of the 7-dimensional representations h, $a+b$ and $a+k+l$ define embeddings of L_{168} in G_2.

Proposition *Let λ_H be a faithful, orthogonal, unimodular, 7-dimensional represention of a compact or finite group H. If the eigenvalues of every group element in the representation λ_H are of the form $\{1,x,y,z,x^{-1},y^{-1},z^{-1}\}$ for some x, y and z such that $|x|=|y|=|z|=1$ and $xyz=1$, then H is a subgroup of G_2 with an embedding defined by $(1)\to\lambda_H$.*

Proof First it should be noted that since λ_H is orthogonal, unimodular and 7-dimensional the image of each group element in the representation λ_H is an element of $SO(7)$. All such elements necessarily have eigenvalues $\{1,x,y,z,x^{-1},y^{-1},z^{-1}\}$, where x, y and z are each of the form $e^{i\phi}$ for some real ϕ.

With this notation the characters of λ_H and $\lambda_H\otimes\Delta$ are given by

$$\begin{aligned}\lambda_H &= 1+x+y+z+x^{-1}+y^{-1}+z^{-1}; &(3.24)\\ \lambda_H\otimes\Delta &= \left(x^{\frac12}+x^{-\frac12}\right)\left(y^{\frac12}+y^{-\frac12}\right)\left(z^{\frac12}+z^{-\frac12}\right). &(3.25)\end{aligned}$$

Now let

$$f(x,y,z)=\lambda_H\otimes\Delta-1_H-\lambda_H. \tag{3.26}$$

Substituting (3.24) and (3.25) into (3.26) gives

$$\begin{aligned}f(x,y,z) &= (x^{\frac12}+x^{-\frac12})(y^{\frac12}+y^{-\frac12})(z^{\frac12}+z^{-\frac12})-(2+x+y+z+x^{-1}+y^{-1}+z^{-1})\\ &= x^{-1}(x^{\frac12}y^{\frac12}z^{\frac12}-1)(x^{\frac12}y^{\frac12}z^{-\frac12}-1)(x^{\frac12}y^{-\frac12}z^{\frac12}-1)(x^{\frac12}y^{-\frac12}z^{-\frac12}-1) \qquad (3.27)\end{aligned}$$

It follows that $f(x,y,z)=0$ if and only if

$$x^\alpha y^\beta z^\gamma=1 \quad \text{for some } \alpha,\beta,\gamma\in\{1,-1\}. \tag{3.28}$$

Since the identification of x, y and z amongst the set $\{x,y,z,x^{-1},y^{-1},z^{-1}\}$ is entirely a matter of choice, it follows from (3.26) that

$$\lambda_H\otimes\Delta=1_H+\lambda_H \tag{3.29}$$

if and only if there exists some choice of x, y and z such that $xyz=1$. To complete the proof of the Proposition it is merely necessary to note that (3.29) coincides with the necessary and sufficient condition identified in (3.9). •

To avoid the use of spinor characters it is possible to arrive at the same result through the use of the necessary and sufficient condition (3.12). This can be seen by noting that

$$\lambda_H\otimes\{1^3\}-\lambda_H-\lambda_H\otimes\{2\}=f(x^2,y^2,z^2). \tag{3.30}$$

The condition (3.12) then takes the form $f(x^2,y^2,z^2)=0$. The same factorisation as noted in (3.27) coupled with an appropriate identification of x, y and z, then leads, as before, to the condition $xyz=1$.

The condition $xyz = 1$ is nothing other than a re-statement of condition (iv) of Lemma 2 of Cohen and Wales [16], from which can be derived their conditions (ii), (v) and (vi).

In the case of $H = L_{168}$ this can be illustrated for each of the faithful, orthogonal, unimodular 7-dimensional representations h, $a+b$ and $a+k+l$ by trying to identify from the Tables 4a and 4b a suitable choice of x y and z for each conjugacy class. They are not necessarily unique since any given eigenvalue may be associated with either one of a pair such as x and x^{-1}. However, the corresponding values of the character Δ are unique and these are displayed in Table 5 for λ_H given by h, $a+b$ and $a+k+l$ alongside one particular choice of (x,y,z) in each conjugacy class.

Table 5. Eigenvalues (x,y,z) for 7-dimensional orthogonal representations λ_H of L_{168} and the corresponding characters of $\lambda_H \otimes \Delta$.

Irrep	$\lambda_H = h$		$\lambda_H = a+b$		$\lambda_H = a+k+l$	
Class	(x,y,z)	Δ	(x,y,z)	Δ	(x,y,z)	Δ
(1^8)	$(1,1,1)$	8	$(1,1,1)$	8	$(1,1,1)$	8
(2^4)	$(1,-1,-1)$	0	$(1,1,-1)$	0	$(1,-1,-1)$	0
$(1^2 3^2)$	$(1,\omega,\omega^2)$	2	$(1,\omega,\omega^2)$	2	$(1,\omega,\omega^2)$	2
(4^2)	$(-1,i,i)$	0	$(1,-1,i)$	0	$(1,i,-i)$	4
(17)	(η,η^2,η^4)	1	(η,η^2,η^4)	1	(η,η^2,η^4)	1
(17)	(η^3,η^5,η^6)	1	(η^3,η^5,η^6)	1	(η^3,η^5,η^6)	1

Examination of this table indicates that $xyz = 1$ for all cases except those of the conjugacy classes (2^4) and (4^2) of the representation $a+b$. In these two cases there is no choice of x, y and z such that $xyz = 1$. Thus only h and $a+k+l$, but not $a+b$, define embeddings of L_{168} in G_2.

Comparison of the results of Table 5 with the character table of L_{168} given in Table 1 also shows that:

$$h \otimes \Delta \quad = a + h; \tag{3.31}$$

$$(a+b) \otimes \Delta \quad = a + h; \tag{3.32}$$

$$(a+k+l) \otimes \Delta \quad = a + (a+k+l).. \tag{3.33}$$

Since $a = 1_H$ for $H = L_{168}$ this confirms, yet again, that the necessary and sufficient condition (3.9) is satisfied in the cases $\lambda_H = h$ and $a+k+l$, but not $a+b$.

3.5 Branching rules for $G_2 \to L_{168}$ and $SU(3) \to L_{168}$

Branching rules for the decomposition of irreducible representations under the restrictions $G_2 \to L_{168}$ with $(1) \to h$ and $SU(3) \to L_{168}$ with $\{1\} \to k$ may be calculated quite readily. In the former case we may evaluate the decomposition for an arbitrary irreducible representation of G_2 by simply working along the group chain $G_2 \uparrow O(7) \downarrow S_8 \downarrow L_{168}$ to produce the results given in Table 6.

Table 6. Branching rules for the restriction $G_2 \to L_{168}$ defined by $(1) \to h$.

Dim	$G_2 \to$	L_{168}
1	(00)	a
7	(10)	h
27	(20)	$2b+h+j$
14	(21)	$j+k+l$
77	(30)	$a+2b+4h+3j+2k+2l$
64	(31)	$2b+2h+4j+k+l$
182	(40)	$3a+8b+7h+8j+3k+3l$
189	(41)	$6b+9h+9j+3k+3l$
77	(42)	$a+4b+2h+4j+k+l$
378	(50)	$a+12b+17h+18j+7k+7l$
448	(51)	$2a+16b+18h+22j+8k+8l$
286	(52)	$2a+10b+12h+123j+6k+6l$
714	(60)	$6a+28b+30h+33j+11k+11l$
924	(61)	$5a+32b+37h+45j+18k+18l$
729	(62)	$5a+28b+30h+35j+11k+11l$
273	(63)	$2a+8b+13h+12j+6k+6l$
1254	(70)	$8a+42b+54h+59j+24k+24l$
1728	(71)	$10a+62b+72h+82j+31k+31l$
1547	(72)	$8a+54b+65h+74j+28k+28l$
896	(73)	$6a+32b+38h+42j+16k+16l$
2079	(80)	$15a+78b+84h+99j+36k+36l$
3003	(81)	$15a+106b+126h+144j+53k+53l$
2926	(82)	$20a+108b+120h+139j+51k+51l$
2079	(83)	$11a+72b+88b+99j+38k+38l$
748	(84)	$6a+30b+28h+36j+13k+13l$
3289	(90)	$18a+114b+141h+155j+60k+60l$
4928	(91)	$28a+176b+204h+236j+88k+88l$
5103	(92)	$30a+180b+213h+243j+93k+93l$
4096	(93)	$26a+146b+172h+194j+73k+73l$
2261	(94)	$11a+80b+94h+109j+40k+40l$
5005	(10 0)	$33a+184b+208h+237j+86k+86l$

In the latter case one may rapidly build up a table by resolving products in $SU(3)$ and L_{168}. This yields the results listed in Table 7. The results for the contragredient partners to those listed in Table 7 may be found by simply making the interchange $k \leftrightarrow l$. Branching rules for the restriction $G_2 \to L_{168}$ using the alternative embedding such that $(1) \to a+k+l$ can be obtained by exploiting the group chain $G_2 \downarrow SU(3) \downarrow L_{168}$, the first stage of which is well documented [53].

Table 5. Branching rules for the restriction $SU(3) \to L_{168}$ defined by $\{1\} \to k$.

Dim	$SU(3) \quad \to$	L_{168}
1	$\{0\}$	a
3	$\{1\}$	k
3	$\{1^2\} = \{\bar{1}\}$	l
6	$\{2\}$	b
8	$\{21\}$	j
10	$\{3\}$	$h + l$
15	$\{31\}$	$h + j$
15	$\{4\}$	$a + b + j$
24	$\{41\}$	$b + h + j + k$
27	$\{42\}$	$2b + h + j$
21	$\{5\}$	$h + j + k + l$
35	$\{51\}$	$b + h + 2j + k + l$
42	$\{52\}$	$b + 2h + 2j + k + l$
28	$\{6\}$	$a + 2b + h + j$
48	$\{61\}$	$2b + 2h + 2j + k + l$
60	$\{62\}$	$a + 3b + 2h + 3j + l$
64	$\{63\}$	$a + 2b + 3h + 3j + k + l$

3.6 The octahedral subgroup of L_{168}

In terms of potential applications it is worth noting that L_{168} contains the symmetric group S_4 as a subgroup [17] which itself is isomorphic to the crystallographic point group O, commonly known as the octahedral group [43].

For f-electrons in a weak crystal field of octahedral symmetry it is common to consider the group-subgroup chain segment $G_2 \supset SO(3) \supset O$. An alternative group-subgroup chain segment would be $G_2 \supset L_{168} \supset O$, with $SO(3)$ being lost as an approximate speherical symmetry group.

The branching rules for $L_{168} \to S_4 = O$ may be readily found from a knowledge of the characters of L_{168} over the classes of its subgroup S_4 as shown in Table 8. It should be noted that on restriction from L_{168} to S_4 the surviving sets of elements in the classes (1^8), $(1^2 3^2)$ and (4^2) constitute the classes (1^4), (13) and (4), respectivley, of S_4, while the set of those in the class (2^4) splits so as to constitute the two classes $(1^2 2)$ and (2^2) of S_4.

Table 8. Characters of the irreducible representations of L_{168} evaluated on the conjugacy classes of the subgroup S_4.

	(1^4)	$(1^2 2)$	(2^2)	(13)	(4)
a	1	1	1	1	1
b	6	2	2	0	0
h	7	-1	-1	1	-1
j	8	0	0	-1	0
k	3	-1	-1	0	1
l	3	-1	-1	0	1

Comparison of this with the character table [70] of S_4 or, equivalently, that [43] of O, immediately leads to the $L_{168} \to S_4 \sim O$ branching rules displayed in Table 9.

Table 9. Branching rules for $L_{168} \to S_4 \sim O$.

Dim	L_{168}	S_4	O
1	a	$\{4\}$	Γ_1
6	b	$\{4\} + \{31\} + \{2^2\}$	$\Gamma_1 + \Gamma_4 + \Gamma_3$
7	h	$\{31\} + \{21^2\} + \{1^4\}$	$\Gamma_4 + \Gamma_5 + \Gamma_2$
8	j	$\{31\} + \{2^2\} + \{21^2\}$	$\Gamma_4 + \Gamma_3 + \Gamma_5$
3	k	$\{21^2\}$	Γ_5
3	l	$\{21^2\}$	Γ_5

Chapter 4

Symmetric functions and non-compact groups

4.1 Introduction

Classically, symmetric function techniques are applied to the finite dimensional representations of compact groups, which can be constructed by means of polynomial representations of matrix groups. In this chapter we want to show how to extend such techniques to the case of certain projective representations of non-compact groups. These are unitary representations constructed from the metaplectic representation of the real symplectic groups.

We will continue the exposition of Chapter 1 and then extend the applications of symmetric functions presented in Chapter 2 to certain infinite dimensional representations of the symplectic group. The next chapters will be devoted to the presentation of new results on the tensor product and plethysms for the metaplectic representation of $Sp(2n,\mathbb{R})$, and its restrictions to $U(p,q)$ and $SO^\star(2n)$.

4.2 The real symplectic group $Sp(2n,\mathbb{R})$

The real symplectic group is the first group for which we will make calculations on representations and characters. After that, we will apply the same techniques to solve similar problems for the groups $U(p,q)$ and $SO^\star(2n)$.

We will first recall the group subgroup structure of $Sp(2n,\mathbb{R})$ which is significant in many physical applications.

4.2.1 Symplectic group and physics

In the introduction we explained the relevance of the metaplectic representation to the problem of the harmonic oscillators. In fact we have treated the question in the simplest case, the group $Sp(2,\mathbb{R})$, which is isomorphic to $SL(2,\mathbb{R})$.

Now, if we refer to the physical literature ([15, 38, 39, 60, 59]), the real symplectic group $Sp(2n,\mathbb{R})$ is often presented as the dynamical group of the n-dimensional isotropic harmonic oscillator. In fact the symplectic group is not really a dynamical group, because all the eigenstates are not generated from only one state but from two states (the definition of a dynamical group can be found for example in Souriau [99]).

For a better understanding of the interest of the symplectic group, we will now investigate the description of an N-particle system, in which algebraic relations are involved between the different entitites describing the system.
Any one of the N particles can be described by its coordinates and its momenta. The rth particle ($r = 1 \ \cdots \ n$) is described by the pair x_{ri} and p_{ri} where the i are the dimensional indices ($i = 1 \ \cdots \ d$). The associated operators denoted by X_{ri} and P_{ri} obey the commutation rules of the Heisenberg algebra

$$\begin{aligned}
[X_{ri}, X_{sj}] &= 0 \\
[X_{ri}, P_{sj}] &= i\hbar\delta_{rs}\delta_{ij} \\
[P_{ri}, P_{sj}] &= 0
\end{aligned}$$

with N commuting distinct copies of the one-dimensional Heisenberg algebra. Using particle indices and position indices in the same manner, one sees that a d-dimensional N-particle system can be regarded as a Nd quanta system of dimension one. Then it is possible to consider a set of $(2Nd)^2$ bilinear operators closed under commutation, given by

$$\{X_{ri}X_{sj}, X_{ri}P_{sj}, P_{ri}X_{sj}, P_{ri}P_{sj}\} \ .$$

Now the second commutation rule implies that there can be only $(2Nd+1)Nd$ independent generators for this Heisenberg algebra associated with the symplectic group $Sp(2Nd,\mathbb{R})$. One can give the corresponding generators

$$\begin{aligned}
Q_{ri}^{sj} &= \frac{1}{2}\{X_{ri}, X_{sj}\} \\
V_{ri}^{sj} &= \frac{1}{2}\{X_{ri}, P_{sj}\} \\
K_{ri}^{sj} &= \frac{1}{2}\{P_{ri}, P_{sj}\}
\end{aligned}$$

We will now examine which subalgebras it is possible to generate starting with the generators X and P. One first remarks that the X commute among themselves and so do the P, i.e. they form two abelian subalgebras.

Now the V form a closed set under commutation and generate a subalgebra isomorphic to $\mathfrak{gl}(Nd,\mathbb{R})$. It is clear that is possible to form other subalgebras by restriction on the number of particles or on the dimension of the space.
We will now construct certain subalgebras of $\mathfrak{sp}(2Nd,\mathbb{R})$. Consider in particular the two generator systems

$$\begin{aligned} Q_{ij} &\equiv X_{ri}X_{rj} \\ L_{ij} &\equiv X_{ri}P_{rj} - X_{rj}P_{ri} \\ K_{ij} &\equiv P_{ri}P_{rj} \\ T_{ij} &\equiv \frac{1}{2}(X_{ri}P_{rj} + X_{rj}P_{ri} + P_{ri}X_{rj} + P_{rj}X_{ri}) \end{aligned}$$

and

$$\begin{aligned} Q_{rs} &\equiv X_{ri}X_{si} \\ L_{rs} &\equiv X_{ri}P_{si} - X_{si}P_{ri} \\ K_{rs} &\equiv P_{ri}P_{si} \\ T_{rs} &\equiv \frac{1}{2}(X_{ri}P_{si} + X_{si}P_{ri} + P_{ri}X_{si} + P_{si}X_{ri}) \end{aligned}$$

They are closed under commutation and form respectively the Lie algebras $\mathfrak{sp}(2d,\mathbb{R})$ and $\mathfrak{sp}(2N,\mathbb{R})$. These two algebras cannot commute with each other and allow us to separate $\mathfrak{sp}(2Nd,\mathbb{R})$ using the position and particle indices.

If we regard now the generator sets $\{L_{ij}\}$ and $\{L_{rs}\}$, one can verify that they are also separately closed under commutation,

$$\begin{aligned} [L_{ij}, L_{kl}] &= i\hbar(L_{ik}\delta_{jl} - L_{il}\delta_{jk} + L_{jk}\delta_{il} - L_{jl}\delta_{ik}) \\ [L_{rs}, L_{tu}] &= i\hbar(L_{rt}\delta_{su} - L_{ru}\delta_{st} + L_{st}\delta_{ru} - L_{su}\delta_{rt}) \ . \end{aligned}$$

They form the subalgebras $\mathfrak{o}(d)$ and $\mathfrak{o}(N)$. Moreover the generators of $\mathfrak{sp}(2N,\mathbb{R})$ (resp. of $\mathfrak{sp}(2d,\mathbb{R})$) commute with those of $\mathfrak{o}(d)$ (resp. those of $\mathfrak{o}(N)$). Finally one obtains a very important result for the sequel: the products $\mathfrak{sp}(2N,\mathbb{R}) \times \mathfrak{o}(d)$ and $\mathfrak{sp}(2d,\mathbb{R}) \times \mathfrak{o}(N)$ are subalgebras of $\mathfrak{sp}(2Nd,\mathbb{R})$. Therefore, the restriction to $\mathfrak{o}(d)$ describes the angular momentum in a $d-$dimensional system and the group subgroup structure of $O(N)$ ($O(N) \supset O(N-1) \supset S_N$) allows us to use the symmetric group techniques for the N-particles system (see [38, 39, 87, 35]).

We can also remark that the algebra of $Sp(2,\mathbb{R})$ is contained in those of $Sp(2N,\mathbb{R})$ and

$Sp(2d, \mathbb{R})$. One can write

$$\begin{aligned} Sp(2N, \mathbb{R}) &\supset Sp(2, \mathbb{R}) \times O(N) \\ Sp(2d, \mathbb{R}) &\supset Sp(2, \mathbb{R}) \times O(d) \end{aligned}$$

and by extension, verifying that the generators of $Sp(2, \mathbb{R})$ and those of $O(Nd)$ commute, they generate the algebra of $Sp(2, \mathbb{R}) \times O(Nd)$.
Consequently it is possible to extract from these observations a group-subgroup chain that will be of importance in the forthcoming chapters

$$Sp(2Nd, \mathbb{R}) \supset Sp(2, \mathbb{R}) \times O(N) \times O(d)$$

An example of this decomposition is illustrated in [108] for the classification of the spectroscopical states of a 12-particle system.

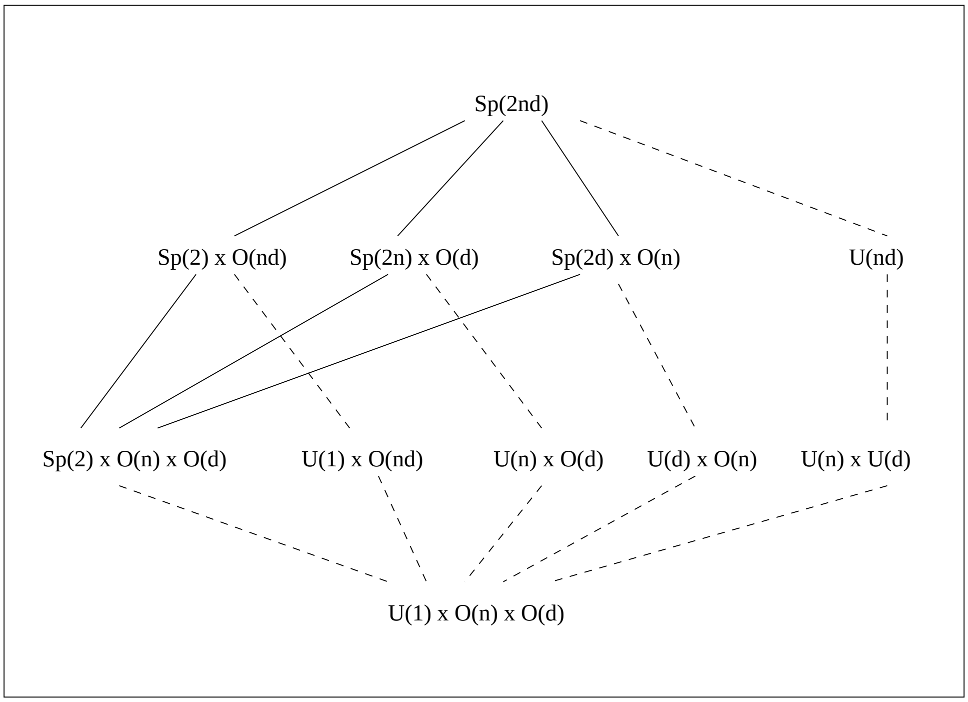
We continue now by an example. The hamiltonian of an isotropic harmonic oscillator contains the term corresponding to the kinetic energy, $\frac{1}{2m} P_{ri} P_{ri} = \frac{K}{2m}$ and also the potential energy term $\frac{m\omega^2}{2} Q$. Looking now at the expression of the Landau Hamiltonian of the classical Hall effect (see [19, 66]), that is

$$H_L = \frac{1}{2m} K + \frac{B^2 q^2}{8mc^2} Q - \frac{qB}{2mc} L_{12}$$

where q, m are the charge and mass of the electron, B the homogenous magnetic field, in which the system is embedded and c the speed of light. This Hamiltonian contains the kinetic and potential energy part corresponding to $Sp(2, \mathbb{R})$ (potential energy modified by the magnetic field) and also a term involving the angular momentum coming from the coupling between the local field and external field B. This corresponds to an action of the group $O(d)$.

By the preceding development and example, one understands that the study of the representations of the real symplectic groups will use the group subgroup structure especially when one wants to calculate tensor product or plethysms of such representations. By these operations using the restrictions to $U(n)$ and $O(d)$ it is possible to interpret the electronic states like singlet or triplet states.

We finish this section by illustrating the group subgroup chain for the real symplectic group with a diagram.



Use of this group-subgroup structure is illustrated through physical examples in Appendices A and B.

4.2.2 The metaplectic representation of $Sp(2n, \mathbb{R})$

Now in the sequel we will write $Sp(2n)$ for $Sp(2n, \mathbb{R})$ (it is also possible to find $Sp(n)$ in the literature). To obtain interesting representations of the symplectic group, it is necessary to introduce the metaplectic group $Mp(2n)$.

In fact we will work with the projective representations of the symplectic group. They are true linear representations of the metaplectic group, the two-fold covering of the symplectic group ($Mp(2n)$ plays the same role for the symplectic group as $Spin(2n)$ for $SO(2n)$). These representations come from linear representations of the Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$.

In the introductory chapter we already mentioned the metaplectic representation and we have constructed explicitly for the one-dimensional oscillator. At the level of the Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$ (or $\mathfrak{sp}(2n, \mathbb{C})$), this representation is the representation previously described in $\mathbb{C}[x_1, \cdots, x_n]$ with $\tilde{\Delta}_+ = \{\text{even egree polynomials}\}$ and $\tilde{\Delta}_- = \{\text{odd degree polynomials}\}$.

This metaplectic representation is also called the harmonic representation and can be decomposed as follows

$$\tilde{\Delta} \to \tilde{\Delta}_+ + \tilde{\Delta}_- = \langle \frac{1}{2}(0) \rangle + \langle \frac{1}{2}(1) \rangle$$

We have also announced that the study of $Sp(2n)$ representations is going on through their restriction to $U(n)$, its maximum compact subgroup. $\tilde{\Delta}$ is a projective representation of $Sp(2n, \mathbb{R})$. By restriction to $U(n)$ one obtains a two-valuated representation. Now, if we introduce a formal square root $\varepsilon^{\frac{1}{2}}$ of the determinant $\varepsilon : g \mapsto \det g$, one can represent the character of the restriction $\tilde{\Delta}$ to $U(n)$ by the formal series $\varepsilon^{\frac{1}{2}} M = \varepsilon^{\frac{1}{2}} \sum_{n \geq 0} h_n$.

Indeed, following the appropriate group-subgroup chain and the associated branching rules [79], one can show that $\tilde{\Delta}^k$ for $Sp(2n)$ can be obtained by restriction of the $\tilde{\Delta}$, representation of $Sp(2kn)$, to $Sp(2n) \times O(k)$ and then to $Sp(2n) \times 1$.

One obtains a multiplicity free decomposition ($k = 2\sigma$) and we write it

$$\tilde{\Delta} = \langle \frac{1}{2}(0) \rangle + \langle \frac{1}{2}(1) \rangle \downarrow^{Sp(2nk)}_{Sp(n) \times O(k)} = \sum_{\lambda \in S} \langle \sigma(\lambda) \rangle \times [\lambda]$$

where the set of λ is to be determined. In particular one needs to look for the restriction from $\langle \sigma(\lambda) \rangle$ to $U(n)$ and to find which symmetric functions correspond to the terms $\langle \sigma(\lambda) \rangle$. The representations $\langle \sigma(\lambda) \rangle$ form the *harmonic series* of $Sp(2n, \mathbb{R})$.

Character of $\tilde{\Delta}^k$ in the restriction $Sp(2n) \downarrow U(n)$

Consider a set of variables $X = \{x_1, \ldots, x_n\}$. The restriction that we are looking for can be expressed in the following form

$$\left(\varepsilon^{\frac{1}{2}} M\right)^k = \varepsilon^{\frac{k}{2}} M^k = \varepsilon^{\frac{k}{2}} [\sigma_1(X)]^k$$

where $M = \sigma_1(X) = \sum_{m \geq 0} h_m$ is the character of n-variable polynomials. Then

$$\begin{aligned} \left(\varepsilon^{\frac{1}{2}} M\right)^k &= \varepsilon^{\frac{k}{2}} \sigma_1(X) \sigma_1(X) \ldots \sigma_1(X) \\ &= \varepsilon^{\frac{k}{2}} \sigma_1(X + X + \cdots + X) \\ &= \varepsilon^{\frac{k}{2}} \sigma_1(kX) \\ &= \varepsilon^{\frac{k}{2}} \sum_{l(\lambda) \leq min(n,k)} s_\lambda(X) s_\lambda(k) \end{aligned}$$

where $s_\lambda(k)$ are the multiplicities.

It is difficult to go further but we can write down the following restriction

$$\tilde{\Delta}^k_{Sp(2n)} \simeq \tilde{\Delta}_{Sp(2nk)} \downarrow Sp(2n) \times O(k) \downarrow Sp(2n) \times 1$$

In this case one can associate a product of alphabets, corresponding to $Sp(2n)$ and $O(k)$, to the original alphabet associated to $Sp(2nk)$,

$$X = \{x_1, \ldots, x_n\} \to YZ = \{y_1, \ldots, y_n\} \cdot \{z_1, \ldots, z_n\}$$

Now, we can express the character of $\tilde{\Delta}$ for $Sp(2nk) \downarrow U(n) \times U(k)$ as

$$\begin{aligned} \varepsilon^{\frac{1}{2}}(YZ) \cdot \sigma_1(YZ) &= \varepsilon^{\frac{1}{2}}(YZ) \sum_{l(\lambda) \leq min(n,k)} s_\lambda(X) s_\lambda(Z) \\ &= \varepsilon^{\frac{1}{2}}(YZ)(1 \otimes D_{\lambda_{-1}[h_2]})(1 \otimes D_{\sigma_1[h_2]}) \sum_\lambda s_\lambda \otimes s_\lambda|_{Y \times Z} \end{aligned}$$

where $D_{\lambda_{-1}[h_2]}$ is the operator $s_\lambda \mapsto o_\lambda$

To go further we need the following lemma
Lemma *Let δ the coproduct $f \mapsto \delta f \simeq f(XY)$, $(p_n \mapsto p_n \otimes p_n)$ then one has, for all f*

$$(1 \otimes D_f)\delta\sigma_1 = (f \otimes 1)\delta\sigma_1$$

For any g and $h \in Sym$

$$\begin{aligned} < (1 \otimes D_f)\delta\sigma_1, g \otimes h > &= < \delta\sigma_1, (1 \otimes f) \cdot g \otimes h > \\ &= < \delta\sigma_1, g \otimes fh > \\ &= < \sigma_1, g * (fh) > \\ &= < \sigma_1 * g, fh > \\ &= < g, fh > \\ &= < D_f g, h > \\ &= < \sigma_1, D_f g * h > \\ &= < \delta\sigma_1, (D_f g) \otimes h > \\ &= < \delta\sigma_1, (D_f \otimes 1) \cdot g \otimes h > \\ &= < (f \otimes 1)\delta\sigma_1, g \otimes h > \end{aligned}$$

which completes the proof.

From the lemma

$$\begin{aligned}
\varepsilon^{\frac{1}{2}}(YZ)\cdot\sigma_1(YZ) &= \varepsilon^{\frac{1}{2}}(YZ)(1\otimes D_{\lambda_{-1}[h_2]})(1\otimes D_{\sigma_1[h_2]})\sum_\lambda s_\lambda\otimes s_\lambda|_{Y\times Z}\\
&= \varepsilon^{\frac{1}{2}}(YZ)(1\otimes D_{\lambda_{-1}[h_2]})\sum_\lambda(\sigma_1[h_2]s_\lambda)\otimes s_\lambda|_{Y\times Z}\\
&= \varepsilon^{\frac{1}{2}}(YZ)\sum_\lambda(\sigma_1[h_2]s_\lambda)\otimes o_\lambda|_{Y\times Z}\\
&= \varepsilon^{\frac{1}{2}}(YZ)\sum_{l(\lambda)\leq min(n,k)}(\sigma_1[h_2]s_\lambda)(Y)o_\lambda(Z)
\end{aligned}$$

but $\varepsilon^{\frac{1}{2}}=\sqrt{\det(\mathrm{diag}(Y)\otimes\mathrm{diag}(Z))}$ then one finally obtains the following result

$$\varepsilon^{\frac{1}{2}}(YZ)\cdot\sigma_1(YZ)=\varepsilon^{\frac{N}{2}}(Y)\varepsilon^{\frac{k}{2}}(Z)\cdot\sum_\lambda\{\lambda D\}\times[\lambda]\ ,$$

where $\{\lambda D\}=s_\lambda\sigma_1(h_2)$ in respect of the notation of chapter 2, which implies the required branching rule.
Indeed, the character of the restriction of $\tilde{\Delta}$ from $Sp(Nk)$ to $Sp(N)\times O(k)$ is

$$\varepsilon^{\frac{N}{2}}\varepsilon^{\frac{k}{2}}\sum_{l(\lambda)\leq\min(N,k)}\{\lambda D\}\times[\lambda]$$

where $\varepsilon^{\frac{N}{2}}=(\pm1)^{\frac{N}{2}}$ is the determinant over $O(k)$, $\varepsilon^{\frac{k}{2}}$ is the determinant over $Sp(N)$ and the terms in the sum represent the representations product of $U(n)\times O(k)$.

Now, to finish we can remark that if $N>\frac{k}{2}$, the sum contains partitions λ that are not standard for the representations of $O(k)$.

Finally one obtains the following branching rule $Sp(2n,\mathbb{R})\downarrow U(n)$, for $\langle\frac{1}{2}k(\lambda)\rangle$, given by ([95] Eq.(3.9), [60] Eq.(5.6))

$$\langle\frac{1}{2}k(\lambda)\rangle\to\varepsilon^{\frac{k}{2}}\{\{\lambda_s\}_N^k D_N\}_N \tag{4.1}$$

where $N=\min(n,k)$, and $\{\lambda_s\}_N^k$ is the *signed sequences* of terms $\pm\{\rho\}$ and with $\pm[\rho]$ equivalent to $[\lambda]$ from the Newell modification rule for $O(k)$ that we explained in Chapter 2. For more details see [95].

This signed sequence can be calculated by hand for small examples. For N equal to 4 or 5, it is really useful to possess a program like SCHUR, that allows to give safely and quickly results. For example for the group $O(3)$, one obtains

```
REP>
gr o3
Group is O(3)
REP>
signseq3,0
      [0]

REP>
signseq3,1
      [1]

REP>
signseq3,11
      [1^2 ]

REP>
signseq3,1^3
      [1^3 ]

REP>
signseq3,2
       - [2^3 ] + [2]

REP>
signseq3,3
       - [32^2 ] + [3]

REP>
signseq3,21
       - [2^2 1] + [21]

REP>
signseq3,2
       - [2^3 ] + [2]
```

We recover here the results from table 1 in [95].

It is now time to give some examples of the application of the branching rule. For example, the terms $\langle 1(m)\rangle$ arising in the right hand side of (5.1)-(5.4) have for restriction to $U(n)$, for $m \geq 2$

$$\langle 1(m)\rangle \to \varepsilon\left\{(\{m\}-\{m,2\})_2 \cdot D_2\right\}_2 \tag{4.2}$$

and for $m = 1$

$$\langle 1(1)\rangle \to \varepsilon \left\{\{1\}D_2\right\}_2 \tag{4.3}$$

Moreover

$$\langle 1(11)\rangle \to \varepsilon \left\{\{11\}D_2\right\}_2 \tag{4.4}$$

We need also the formula (8.18) of [60], that give the tensor product of two harmonic representations of $Sp(2n,\mathbb{R})$

$$\langle \frac{1}{2}k(\mu)\rangle \times \langle \frac{1}{2}l(\lambda)\rangle = \langle \frac{1}{2}(k+l)\left(\left(\{\mu_s\}^k \cdot \{\lambda_s\}^l \cdot D\right)\right)_{k+l,n}\rangle \tag{4.5}$$

where $((\lambda))_{k+l,n}$ is defined equal to zero for $\tilde{\lambda_1} \leq n$ and $\tilde{\lambda_1} + \tilde{\lambda_2} \leq k+l$ else (λ) is retained. For example

$$\langle \frac{1}{2}(0)\rangle \times \langle \frac{1}{2}(0)\rangle = \sum_{i\geq 0}\langle 1(2i)\rangle \tag{4.6}$$

$$\langle \frac{1}{2}(0)\rangle \times \langle \frac{1}{2}(1)\rangle = \sum_{i\geq 0}\langle 1(2i+1)\rangle \tag{4.7}$$

$$\langle \frac{1}{2}(1)\rangle \times \langle \frac{1}{2}(1)\rangle = \langle 1(11)\rangle + \sum_{i\geq 1}\langle 1(2i)\rangle \tag{4.8}$$

Finally, the branching rule $Sp(2nk,\mathbb{R}) \downarrow Sp(2n,\mathbb{R}) \times O(k)$ ([60] Eq.(4.3)) is given by

$$\tilde{\Delta} \to \sum_{\lambda}\langle \frac{1}{2}k(\lambda)\rangle \times [\lambda] \tag{4.9}$$

The harmonic representation $\tilde{\Delta}_{2nk}$ of $Sp(2nk,\mathbb{R})$ has the following restrictions, with respect with the group subgroup chain $Sp(2nk,\mathbb{R}) \supset Sp(2n,\mathbb{R}) \times O(k) \supset Sp(2n,\mathbb{R}) \times 1$

$$\tilde{\Delta}_{2nk} \to \sum_{\lambda}\langle \frac{1}{2}k(\lambda)\rangle \times [\lambda] \to \tilde{\Delta}_{2n}^{\times k}$$

where the partitions λ verify $\lambda'_1 + \lambda'_2 \leq k$ et $\lambda'_1 \leq n$.

4.3 The $U(p,q)$ and $SO^*(2n)$ groups

The techniques we explained in these chapter for the symplectic groups can be applied in the same manner for the other two groups. Essentially because we will use the same kind of arguments for the metaplectic representation of $SO^\star(2n)$ and we will use an embedding of $U(p,q)$ in $Sp(2p+2q,\mathbb{R})$ to resolve the Kronecker square of the fundamental representations of $U(p,q)$. Complements of notations and explanations will be given in time.

Chapter 5

Representations of $Sp(2n, \mathbb{R})$

This chapter is the matter of a common work realised with J. Y. Thibon and B. G. Wybourne, published in *Journal of Physics A: Mathematical and General.* [102].

5.1 Introduction

In the previous chapters, we have given an account of symmetric functions and their role in representation theory. At the same time we have explained the importance of non-compact groups. The non-compact symplectic group $Sp(2n, \mathbb{R})$, well-known as the dynamical group of the isotropic $n-$dimensional harmonic oscillator [106], also finds significant applications in symplectic models of nuclei [15][92] and in the mesoscopic properties of quantum dots [38][39][107]. A central problem in making applications is the resolution of Kronecker powers of the two fundamental irreducible unitary representations (unirreps) which, following King and Wybourne [60] and the previous chapter we shall designate as $\langle\frac{1}{2}(0)\rangle$ and $\langle\frac{1}{2}(1)\rangle$, into their various symmetry types. This problem has recently been studied by Grudzinski and Wybourne [35] whom, on the basis of numerical calculations, conjectured the explicit decompositions

$$\langle\tfrac{1}{2}(0)\rangle \otimes \{2\} = \sum_{i\geq 0}\langle 1\,(4i)\rangle \tag{5.1}$$

$$\langle\tfrac{1}{2}(0)\rangle \otimes \{1^2\} = \sum_{i\geq 0}\langle 1\,(2+4i)\rangle \tag{5.2}$$

$$\langle\tfrac{1}{2}(1)\rangle \otimes \{2\} = \sum_{i\geq 0}\langle 1\,(2+4i)\rangle \tag{5.3}$$

$$\langle\tfrac{1}{2}(1)\rangle \otimes \{1^2\} = \langle 1\,(1^2)\rangle + \sum_{i\geq 1}\langle 1\,(4i)\rangle\ . \tag{5.4}$$

It was shown that the validity of such conjectures implies some unusual $S-$function identities which they duly established but without establishing a rigorous proof of the conjectured equivalences. Herein we supply a formal proof of (5.1) to (5.4). Moreover, the methods introduced lead to the establishment of a complete resolution of the Kronecker cube of the fundamental irreps of $Sp(2n,\mathbb{R})$. These results are then used to demonstrate the existence of a symmetry relationship linking plethysms of the $\langle\frac{1}{2}(0)\rangle$ unirrep to those of $\langle\frac{1}{2}(1)\rangle$.

5.2 Decomposition of symmetrized squares

In this section we give a proof of Eqs. (5.1)-(5.4).

Let us first look at the $U(n)$ content of the left-hand-side (l.h.s.) of (5.1) and (5.3). We know that under $Sp(2n,\mathbb{R}) \downarrow U(n)$ [35],

$$\langle\tfrac{1}{2}(0)\rangle \otimes \{2\} \to \left(\epsilon^{\frac{1}{2}} \sum_{m\geq 0}\{2m\}\right) \otimes \{2\} = \epsilon h_2 \left[\sum_{m\geq 0} h_{2m}\right]$$

and

$$\langle\tfrac{1}{2}(1)\rangle \otimes \{2\} \to \left(\epsilon^{\frac{1}{2}} \sum_{m\geq 0}\{2m+1\}\right) \otimes \{2\} = \epsilon h_2 \left[\sum_{m\geq 0} h_{2m+1}\right]$$

where for later convenience the r.h.s. has been rewritten in terms of symmetric functions following the notation of [73]. Hence, the sum of the left-hand sides correspond to the terms of even degree in the series

$$h_2[\sigma_1] = h_2\left[\sum_{m\geq 0} h_{2m} + \sum_{m\geq 0} h_{2m+1}\right]$$

$$= h_2\left[\sum_{m\geq 0} h_{2m}\right] + \left(\sum_{m\geq 0} h_{2m}\right)\cdot\left(\sum_{m\geq 0} h_{2m+1}\right) + h_2\left[\sum_{m\geq 0} h_{2m+1}\right] \ ,$$

which (up to the factor ϵ), describes the $U(n)$ content of $\tilde{\Delta}\otimes\{2\}$. Therefore, to compute the sum

$$\langle\tfrac{1}{2}(0)\rangle \otimes \{2\} + \langle\tfrac{1}{2}(1)\rangle \otimes \{2\}$$

it is enough to evaluate

$$\tilde{\Delta}\otimes\{2\} = \langle\tfrac{1}{2}(0)\rangle \otimes \{2\} + \langle\tfrac{1}{2}(1)\rangle \otimes \{2\} + \langle\tfrac{1}{2}(0)\rangle \times \langle\tfrac{1}{2}(1)\rangle \ . \tag{5.5}$$

Now, we can resolve $\tilde{\Delta}^2_{2n}$ into its symmetric and antisymmetric part by regarding it as the restriction of $\tilde{\Delta}_{4n}$ under $Sp(4n,\mathbb{R}) \downarrow Sp(2n,\mathbb{R})$ and introducing the chain of groups

$$Sp(4n,\mathbb{R}) \supset Sp(2n,\mathbb{R}) \times O(2) \supset Sp(2n,\mathbb{R}) \times S_2 \ .$$

Indeed, under $Sp(4n,\mathbb{R}) \downarrow Sp(2n,\mathbb{R}) \times S_2$,we have

$$\tilde{\Delta}_{4n} \to \left(\tilde{\Delta}_{2n}\otimes\{2\}\right) \times (2) + \left(\tilde{\Delta}_{2n}\otimes\{1^2\}\right) \times (11) \ .$$

Under $Sp(4n,\mathbb{R}) \downarrow Sp(2n,\mathbb{R}) \times O(2)$, the branching rule (4.9) gives

$$\tilde{\Delta}_{2n} \rightarrow \langle 1(1^2)\rangle \times [1^2] + \sum_{i\geq 0} \langle 1(i)\rangle \times [i]$$

The restriction $O(k) \downarrow S_k$ is given by [55, 97]

$$[\lambda] \rightarrow \langle 1\rangle \otimes \{\lambda/G\} \tag{5.6}$$

where $G = \{0\} + \{1\} - \{21\} \ldots$ involves only self conjugate partitions, so that

$$[1^2] \rightarrow \langle 1\rangle \otimes \big(\{1^2\} + \{1\}\big) = \langle 11\rangle + \langle 1\rangle$$

and for $k = 2$, $[1^2] \rightarrow (\langle 11\rangle + \langle 1\rangle)_{S_2} = (11) + (01) = (11)$ (since according to the modification rules, (01) is null) reduces to the alternating representation of S_2. For $i \geq 1$,

$$[i] \downarrow \langle 1\rangle \otimes (\{i\} + \{i-1\})$$

where for S_2, $\langle 1\rangle \otimes \{j\}$ reduces to (2) for j even and to (11) for j odd. Therefore, the restriction

$$\tilde{\Delta}_{2n} \downarrow Sp(2n,\mathbb{R}) \times S_2$$

is equal to

$$\langle 1(1^2)\rangle \times (11) + \langle 1(0)\rangle \times (2) + \sum_{i\geq 1} \langle 1(i)\rangle \times [(2) + (11)]$$

so that

$$\tilde{\Delta} \otimes \{2\} \;=\; \sum_{i\geq 0} \langle 1(i)\rangle \tag{5.7}$$

$$\tilde{\Delta} \otimes \{1^2\} \;=\; \langle 1(1^2)\rangle + \sum_{i\geq 1} \langle 1(i)\rangle \tag{5.8}$$

If we rewrite (5.5) where under restriction to $U(n)$, $\langle \frac{1}{2}(0)\rangle \otimes \{2\}$ yields symmetric functions of degree $d \equiv 0 \bmod [4]$, $\langle \frac{1}{2}(1)\rangle \otimes \{2\}$ give terms of degree $d \equiv 2 \bmod [4]$ and $\langle \frac{1}{2}(0)\rangle \times \langle \frac{1}{2}(1)\rangle$ the terms of odd degree.
From (5.7) and (4.7) we have

$$\langle \tfrac{1}{2}(0)\rangle \otimes \{2\} + \langle \tfrac{1}{2}(1)\rangle \otimes \{2\} = \sum_{i\geq 0} \langle 1(2i)\rangle \;. \tag{5.9}$$

Comparing (5.9) and (4.6), we obtain

$$\langle \tfrac{1}{2}(0)\rangle \otimes \{1^2\} = \langle \tfrac{1}{2}(1)\rangle \otimes \{2\} \;, \tag{5.10}$$

(since $\langle \frac{1}{2}(0)\rangle^2 = \langle \frac{1}{2}(0)\rangle \otimes \{2\} + \langle \frac{1}{2}(0)\rangle \otimes \{1^2\}$.)

To conclude, consider the $U(n)$ content of the difference

$$\langle \tfrac{1}{2}(0)\rangle \otimes \{2\} - \langle \tfrac{1}{2}(0)\rangle \otimes \{1^2\} = \langle \tfrac{1}{2}(0)\rangle \otimes p_2 \ , \tag{5.11}$$

where p_2 is the second power sum. Under restriction to $U(n)$, the r.h.s reads in terms of symmetric functions

$$p_2\left[\epsilon^{\frac{1}{2}} \sum_{n\geq 0} h_{2n}\right] = \epsilon \sum_{n\geq 0} p_2\left[h_{2n}\right] \ .$$

This expression can be expanded in the basis of Schur functions

$$\sum_{n\geq 0} p_2\left[h_{2n}\right] = \sum_{n\geq 0}\sum_{k=0}^{2n} (-1)^k s_{4n-k,k} \tag{5.12}$$

Under restriction to $U(n)$, the left-hand-side of (5.11) corresponds to the even terms in the series obtained by taking the difference of the two expressions

$$\sum_{k\geq 0} \langle 1(4k)\rangle \downarrow \left(\epsilon\left(\sum_{k\geq 0} h_{4k}\right)\sigma_1[h_2] - \epsilon\left(\sum_{k\geq 1} s_{4k,2}\right)\sigma_1[h_2]\right)_{d\equiv 0[4]}$$

and

$$\sum_{k\geq 0} \langle 1(4k)\rangle \downarrow \left(\epsilon\left(\sum_{k\geq 0} h_{4k}\right)\sigma_1[h_2] - \epsilon\left(\sum_{k\geq 0} s_{4k+2,2}\right)\sigma_1[h_2]\right)_{d\equiv 2[4]}$$

We can extract these terms by taking the real part of

$$\sigma_1[h_2]\left[\sum_{n\geq 0} (\mathrm{i}t)^n h_n - \sum_{n\geq 2} (\mathrm{i}t)^n s_{n,2}\right]$$

which is equal for $t=1$ to

$$\sigma_1[h_2]\left[\sigma_{\mathrm{i}} - \sigma_{\mathrm{i}}\left(s_2 - \frac{1}{\mathrm{i}} s_1\right) + \frac{1}{\mathrm{i}} s_1 + s_{1^2}\right] \tag{5.13}$$

We want to show that in two variables $\{x, y\}$, the symmetric functions coincide with the r.h.s of (5.12).
Indeed,

$$\begin{aligned} p_2\left[\sum_{n\geq 0} h_{2n}\right] &= \frac{p_2}{2}\left[\sigma_1 + \sigma_{-1}\right] \\ &= \frac{p_2}{2}\left[\frac{1}{(1-x)(1-y)} + \frac{1}{(1+x)(1+y)}\right] \\ &= \frac{1}{2}\left[\frac{1}{(1-x^2)(1-y^2)} + \frac{1}{(1+x^2)(1+y^2)}\right] \end{aligned}$$

and using the specialization of

$$\sigma_1\left[h_2(x,y)\right] = \sigma_1\left[x^2+xy+y^2\right] = \frac{1}{(1-x^2)(1-xy)(1-y^2)} ,$$

$$\sigma_{\mathrm{i}}(x,y) = \frac{1}{(1-\mathrm{i}x)(1-\mathrm{i}y)}$$

and so on, we see that the specialization S of (5.13) is equal to

$$\begin{aligned} S &= \frac{1}{2}\left[\frac{1}{(1-x^2)(1-y^2)} + \frac{1}{(1+x^2)(1+y^2)}\right] \\ &\quad +\mathrm{i}\frac{y+x+x^3y^2+x^2y^3}{(1+x^2)(1+y^2)(1-x^2)(1-xy)(1-y^2)} \end{aligned}$$

and we have now the result that

$$\langle \tfrac{1}{2}(0)\rangle \otimes \{2\} - \langle \tfrac{1}{2}(0)\rangle \otimes \{1^2\} = \sum_{i\geq 0}\langle 1(4i)\rangle - \sum_{i\geq 0}\langle 1(4i+2)\rangle \ . \tag{5.14}$$

From (5.14) and (4.6) we obtain (5.1) and (5.2), and taking into account (5.10) and (4.8), we obtain (5.3) and (5.4).

5.3 Symmetrized cubes

We shall now establish similar decompositions for the symmetrized cubes of the fundamental representations $\langle \frac{1}{2}(0)\rangle$ and $\langle \frac{1}{2}(1)\rangle$. As above, we obtain $\tilde{\Delta}_{2n}^{\times 3}$ by restriction of $\tilde{\Delta}_{6n}$ through the chain

$$Sp(6n,\mathbb{R}) \supset Sp(2n,\mathbb{R}) \times O(3) \supset Sp(2n,\mathbb{R}) \times S_3 \supset Sp(2n,\mathbb{R}) \ .$$

To obtain the plethysm $\tilde{\Delta}_{2n} \otimes \{\mu\}$, we need to determine the restriction from $O(3)$ to S_3. We remark first that in the restriction $O(3) \downarrow S_3$ only the partitions $[1^3]$, $[m]$ and $[m,1]$ are retained. Since $[m,1] = [m]^*$ and $[m]^* \downarrow S_3 = \mathrm{sgn} \times [m] \downarrow S_3$ (see [95]), it is enough to determine $[m] \downarrow S_3$ (clearly, $[1^3] \to \mathrm{sgn}$). Applying (5.6), we find

$$[m] \downarrow \langle 1\rangle \otimes \{m/G\} = \langle 1\rangle \otimes \{m\} + \langle 1\rangle \otimes \{m-1\} \ .$$

The value of $\langle 1\rangle \otimes \{m\}$ for S_k is equal to the coefficient of z^m in the series

$$(1-z)h_n\left(\frac{X}{1-z}\right) = \prod_{j=2}^{n}\frac{1}{1-z^j}\cdot\sum_{\lambda\vdash n}\tilde{K}_{\lambda,1^n}(z)s_\lambda(X)$$

(cf. [97]) where $\tilde{K}_{\lambda,1^k}(z)$ are the Kostka-Foulkes polynomials. In particular, for $n=3$, $[m] \downarrow S_3$ is the sum of coefficient of z^m and z^{m-1} in

$$F(z) = \frac{1}{(1-z^2)(1-z^3)}\left[z^3(111) + (z^2+z)(21) + (3)\right] \ . \tag{5.15}$$

Writing

$$\frac{1}{(1-z^2)(1-z^3)} = \sum_{n\geq 0} a_n z^n \tag{5.16}$$

we obtain

$$F(z) = \left(\sum_{n\geq 0} a_n z^n\right)(3) + \left(\sum_{n\geq 1} a_{n-2} z^n + a_{n-1} z^n\right)(21) + \left(\sum_{n\geq 3} a_{n-3} z^n\right)(111)\ ,$$

and after simplification, one finds that

$$[m] \downarrow S_3 = (a_m + a_{m-1})(3) + (a_{m-1} + 2a_{m-2} + a_{m-3})(21) + (a_{m-3} + a_{m-4})(111)\ ,$$

$$[m,1] \downarrow S_3 = (a_{m-3} + a_{m-4})(3) + (a_{m-1} + 2a_{m-2} + a_{m-3})(21) + (a_m + a_{m-1})(111)\ ,$$

and

$$[1^3] \downarrow S_3 = \mathrm{sgn} = (111)$$

Hence,

$$\tilde{\Delta}_{3n} \downarrow Sp(2n) \times S_3 = \sum_{m\geq 0} \langle \tfrac{3}{2}(m)\rangle E_1(m) + \sum_{m\geq 1} \langle \tfrac{3}{2}(m,1)\rangle E_2(m) + \langle \tfrac{3}{2}(1^3)\rangle (111)$$

where

$$E_1(m) = (a_m + a_{m-1})(3) + (a_{m-1} + 2a_{m-2} + a_{m-3})(21) + (a_{m-3} + a_{m-4})(111)\ ,$$

and

$$E_2(m) = (a_{m-3} + a_{m-4})(3) + (a_{m-1} + 2a_{m-2} + a_{m-3})(21) + (a_m + a_{m-1})(111)$$

On the other hand, since $\tilde{\Delta} = \langle \frac{1}{2}(0)\rangle + \langle \frac{1}{2}(1)\rangle$,

$$\begin{aligned} \tilde{\Delta} \otimes \{3\} &= \langle \tfrac{1}{2}(0)\rangle \otimes \{3\} + \langle \tfrac{1}{2}(1)\rangle \otimes \{3\} \\ &+ \langle \tfrac{1}{2}(0)\rangle \times \langle \tfrac{1}{2}(1)\rangle \otimes \{2\} + \langle \tfrac{1}{2}(1)\rangle \times \langle \tfrac{1}{2}(0)\rangle \otimes \{2\} \end{aligned} \tag{5.17}$$

$$\begin{aligned} \tilde{\Delta} \otimes \{21\} &= \langle \tfrac{1}{2}(1)\rangle \otimes \{21\} + \langle \tfrac{1}{2}(0)\rangle \otimes \{21\} \\ &+ \langle \tfrac{1}{2}(0)\rangle \times \langle \tfrac{1}{2}(1)\rangle \otimes \{2\} + \langle \tfrac{1}{2}(1)\rangle \times \langle \tfrac{1}{2}(0)\rangle \otimes \{2\} \\ &+ \langle \tfrac{1}{2}(0)\rangle \times \langle \tfrac{1}{2}(1)\rangle \otimes \{1^2\} + \langle \tfrac{1}{2}(1)\rangle \times \langle \tfrac{1}{2}(0)\rangle \otimes \{1^2\} \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} \tilde{\Delta} \otimes \{1^3\} &= \langle \tfrac{1}{2}(1)\rangle \otimes \{1^3\} + \langle \tfrac{1}{2}(0)\rangle \otimes \{1^3\} \\ &+ \langle \tfrac{1}{2}(0)\rangle \times \langle \tfrac{1}{2}(1)\rangle \otimes \{1^2\} + \langle \tfrac{1}{2}(1)\rangle \times \langle \tfrac{1}{2}(0)\rangle \otimes \{1^2\} \end{aligned} \tag{5.19}$$

To extract the plethysms $\langle \frac{1}{2}(i)\rangle \otimes \{\lambda\}$ from these expressions, we rewrite the left-hand-sides as (5.17),(5.18) and (5.19), and we manage to subtract the contribution of the crossed products $\langle \frac{1}{2}(i)\rangle \times \langle \frac{1}{2}(i-1)\rangle \otimes \{\mu\}$, ($|\mu| = 2$).
From Eqs. (5.1)-(5.4), we obtain

$$\langle \tfrac{1}{2}(1)\rangle \times \langle \tfrac{1}{2}(0)\rangle \otimes \{2\} = \langle \tfrac{1}{2}(1)\rangle \times \sum_{i\geq 0} \langle 1(4i)\rangle\ , \tag{5.20}$$

$$\langle \tfrac{1}{2}(1)\rangle \times \langle \tfrac{1}{2}(0)\rangle \otimes \{1^2\} = \langle \tfrac{1}{2}(1)\rangle \times \sum_{i\geq 0}\langle 1(4i+2)\rangle \ , \tag{5.21}$$

$$\langle \tfrac{1}{2}(0)\rangle \times \langle \tfrac{1}{2}(1)\rangle \otimes \{2\} = \langle \tfrac{1}{2}(0)\rangle \times \sum_{i\geq 0}\langle 1(4i+2)\rangle \ , \tag{5.22}$$

and

$$\langle \tfrac{1}{2}(0)\rangle \times \langle \tfrac{1}{2}(1)\rangle \otimes \{1^2\} = \langle \tfrac{1}{2}(0)\rangle \times \left(\langle 1(1^2)\rangle + \sum_{i\geq 1}\langle 1(4i)\rangle\right) \ . \tag{5.23}$$

From (4.5) we find

$$\langle \tfrac{1}{2}(1)\rangle \times \langle 1(0)\rangle = \sum_{i\geq 0}\langle \tfrac{3}{2}(2i+1)\rangle \tag{5.24}$$

and for $m \neq 0$

$$\langle \tfrac{1}{2}(1)\rangle \times \langle 1(m)\rangle = \sum_{i\geq 0}\langle \tfrac{3}{2}(m+2i,1)\rangle + \langle \tfrac{3}{2}(m+2i+1)\rangle \tag{5.25}$$

Let $p_2(m) = \lceil \frac{m+1}{2} \rceil$, the number of partitions of m into parts not greater that 2, given by the generating function

$$\sum_{m\geq 0} p_2(m) z^m = \frac{1}{(1-z)(1-z^2)}$$

Expanding the r.h.s of (5.20) by means of (5.24) and (5.25), we get

$$\begin{aligned}\langle \tfrac{1}{2}(1)\rangle \times \langle \tfrac{1}{2}(0)\rangle \otimes \{2\} &= \textstyle\sum_{i\geq 0}\langle \tfrac{3}{2}(2i+1)\rangle \\ &+ \textstyle\sum_{\substack{i\geq 1\\ j\geq 0}}\langle \tfrac{3}{2}(4i+2j,1)\rangle + \langle \tfrac{3}{2}(4i+2j+1)\rangle \\ &= \textstyle\sum_{i\geq 0}\langle \tfrac{3}{2}(2i+1)\rangle \\ &+ \textstyle\sum_{m\geq 1} p_2(m-2)\left(\langle \tfrac{3}{2}(2m,1)\rangle + \langle \tfrac{3}{2}(2m+1)\rangle\right)\end{aligned}$$

where $4i+2j = 2m$ and $p_2(m-2)$ comes from the rearrangement

$$\sum_{\substack{i\geq 1,\ j\geq 0\\ 2i+j=m}} 1 = \sum_{\substack{k\geq 0,\ j\geq 0\\ 2k+j=m-2}} 1 = p_2(m-2)$$

For equation (5.22) we have a similar expression. For $m \neq 0$

$$\langle \tfrac{1}{2}(0)\rangle \times \langle 1(m)\rangle = \sum_{j\geq 0}\langle \tfrac{3}{2}(m+2j+1,1)\rangle + \langle \tfrac{3}{2}(m+2j)\rangle$$

then

$$\begin{aligned}\langle \tfrac{1}{2}(0)\rangle \times \langle \tfrac{1}{2}(1)\rangle \otimes \{2\} &= \textstyle\sum_{i\geq 0}\sum_{j\geq 1}\langle \tfrac{3}{2}(4i+2j+1,1)\rangle + \langle \tfrac{3}{2}(4i+2j)\rangle \\ &= \textstyle\sum_{m\geq 1} p_2(m-1)\left(\langle \tfrac{3}{2}(2m+1,1)\rangle + \langle \tfrac{3}{2}(2m)\rangle\right)\end{aligned}$$

Now it is possible to give a closed expression for

$$\begin{aligned}&\langle \tfrac{1}{2}(0)\rangle \otimes \{3\} + \langle \tfrac{1}{2}(1)\rangle \otimes \{3\} \\ &= \tilde{\Delta} \otimes \{3\} - \langle \tfrac{1}{2}(0)\rangle \times \langle \tfrac{1}{2}(1)\rangle \otimes \{2\} - \langle \tfrac{1}{2}(1)\rangle \times \langle \tfrac{1}{2}(0)\rangle \otimes \{2\}\end{aligned}$$

that is,

$$\begin{array}{rcl} \langle\frac{1}{2}(0)\rangle \otimes \{3\} + \langle\frac{1}{2}(1)\rangle \otimes \{3\} & = & \sum_{m\geq 0}(a_m + a_{m-1})\langle\frac{3}{2}(m)\rangle \\ & + & \sum_{m\geq 3}(a_{m-3} + a_{m-4})\langle\frac{3}{2}(m,1)\rangle \\ & - & \sum_{i\geq 0}\langle\frac{3}{2}(2i+1)\rangle \\ & - & \sum_{k\geq 1} p_2(k-2)\left(\langle\frac{3}{2}(2k,1)\rangle + \langle\frac{3}{2}(2k+1)\rangle\right) \\ & - & \sum_{k\geq 1} p_2(k-1)\left(\langle\frac{3}{2}(2k+1,1)\rangle + \langle\frac{3}{2}(2k)\rangle\right) \end{array} \quad (5.26)$$

We can now give the coefficients of $\langle\frac{3}{2}(m)\rangle$ and $\langle\frac{3}{2}(m,1)\rangle$ in (5.26). The coefficient of $\langle\frac{3}{2}(m)\rangle$ is equal to

$$\begin{array}{rcll} a_m + a_{m-1} - p_2(2k-1) & = & a_m + a_{m-1} - a_{2k-1} - a_{2k-2} - a_{2k-3} & \text{for } m = 4k, \\ a_m + a_{m-1} - p_2(2k) & = & a_m + a_{m-1} - 1 - a_{2k-1} - a_{2k-2} - a_{2k-3} & \text{for } m = 4k+2, \\ a_m + a_{m-1} - 1 - p_2(k-2)) & = & a_m + a_{m-1} - 1 - a_{k-2} - a_{k-3} - a_{k-4} & \text{for } m = 2k+1, \end{array}$$

and the coefficient of $\langle\frac{3}{2}(m,1)\rangle$ is equal to

$$\begin{array}{rcll} a_{m-3} + a_{m-4} - p_2(k-2) & = & a_{m-3} + a_{m-4} - a_{k-2} - a_{k-3} - a_{k-4} & \text{for } m = 2k, \\ a_{m-3} + a_{m-4} - p_2(k-1)) & = & a_{m-3} + a_{m-4} - a_{k-1} - a_{k-2} - a_{k-3} & \text{for } m = 2k+1, \end{array}$$

To separate the components of $\langle\frac{1}{2}(0)\rangle \otimes \{3\}$ and $\langle\frac{1}{2}(1)\rangle \otimes \{3\}$, we consider their $U(n)$ contents. It is enough to see that the terms of $\langle\frac{1}{2}(0)\rangle \otimes \{3\}$ are those $\langle\frac{3}{2}(\lambda)\rangle$ with $|\lambda|$ even and the terms of $\langle\frac{1}{2}(1)\rangle \otimes \{3\}$ are those with $|\lambda|$ odd.

$$\begin{array}{rcl} \langle\frac{1}{2}(0)\rangle \otimes \{3\} & = & \sum_{k\geq 0}(a_{4k} + a_{4k-1} - 1 - a_{2k-1} - a_{2k-2} - a_{2k-3})\langle\frac{3}{2}(4k)\rangle \\ & + & \sum_{k\geq 0}(a_{4k+2} + a_{4k+1} - 1 - a_{2k-1} - a_{2k-2} - a_{2k-3})\langle\frac{3}{2}(4k+2)\rangle \\ & + & \sum_{k\geq 0}(a_{2k-2} + a_{2k-3} - a_{k-1} - a_{k-2} - a_{k-3})\langle\frac{3}{2}(2k+1,1)\rangle \end{array} \quad (5.27)$$

and

$$\begin{array}{rcl} \langle\frac{1}{2}(1)\rangle \otimes \{3\} & = & \sum_{k\geq 0}(a_{2k+1} + a_{2k} - 1 - a_{k-2} - a_{k-3} - a_{k-4})\langle\frac{3}{2}(2k+1)\rangle \\ & + & \sum_{k\geq 0}(a_{2k-3} + a_{2k-4} - a_{k-2} - a_{k-3} - a_{k-4})\langle\frac{3}{2}(2k,1)\rangle \end{array} \quad (5.28)$$

The derivation of the other plethysms is similar. The right-hand-side of (5.23) evaluates to

$$\begin{array}{rcl} \langle\frac{1}{2}(0)\rangle \times \langle\frac{1}{2}(1)\rangle \otimes \{1^2\} & = & \sum_{i\geq 0}\langle\frac{3}{2}(2i+1,1)\rangle + \sum_{i\geq 1}\langle\frac{3}{2}(4i)\rangle \\ & + & \sum_{i\geq 1}\sum_{j\geq 1}\langle\frac{3}{2}(4i+2j-1,1)\rangle + \langle\frac{3}{2}(4i+2j)\rangle \\ & = & \sum_{i\geq 0}\langle\frac{3}{2}(2i+1,1)\rangle + \sum_{i\geq 1}\langle\frac{3}{2}(4i)\rangle \\ & + & \sum_{m\geq 3} p_2(m-3)\left(\langle\frac{3}{2}(2m-1,1)\rangle + \langle\frac{3}{2}(2m)\rangle\right) \end{array}$$

and for (5.21) we obtain

$$\begin{array}{rcl} \langle\frac{1}{2}(1)\rangle \times \langle\frac{1}{2}(0)\rangle \otimes \{1^2\} & = & \sum_{i\geq 0}\sum_{j\geq 1}\langle\frac{3}{2}(4i+2j+1)\rangle + \langle\frac{3}{2}(4i+2j,1)\rangle \\ & = & \sum_{m\geq 1} p_2(m-1)\left(\langle\frac{3}{2}(2m,1)\rangle + \langle\frac{3}{2}(2m+1)\rangle\right) \end{array}$$

Subtracting the contributions of the crossed products in (5.19) we obtain

$$\begin{aligned}&\langle \tfrac12(1)\rangle \otimes \{1^3\} + \langle \tfrac12(0)\rangle \otimes \{1^3\}\\ &= \tilde\Delta \otimes \{1^3\} - \langle \tfrac12(0)\rangle \times \langle \tfrac12(1)\rangle \otimes \{1^2\} - \langle \tfrac12(1)\rangle \times \langle \tfrac12(0)\rangle \otimes \{1^2\}\end{aligned}$$

and this gives

$$\begin{aligned}&\langle \tfrac12(0)\rangle \otimes \{1^3\} + \langle \tfrac12(1)\rangle \otimes \{1^3\}\\ &= \textstyle\sum_{m\geq 0}(a_{m-3}+a_{m-4})\langle \tfrac32(m)\rangle + \displaystyle\sum_{m\geq 4}(a_m+a_{m-1})\langle \tfrac32(m,1)\rangle\\ &+\langle \tfrac32(1^3)\rangle\\ &-\textstyle\sum_{i\geq 0}\langle \tfrac32(2i+1,1)\rangle - \displaystyle\sum_{i\geq 1}\langle \tfrac32(4i)\rangle\\ &-\textstyle\sum_{m\geq 3} p_2(m-3)\left(\langle \tfrac32(2m-1,1)\rangle + \langle \tfrac32(2m)\rangle\right)\\ &-\textstyle\sum_{m\geq 1} p_2(m-1)\left(\langle \tfrac32(2m,1)\rangle + \langle \tfrac32(2m+1)\rangle\right)\end{aligned}$$

We can now write down the coefficients of $\langle \tfrac32(m)\rangle$ and $\langle \tfrac32(m,1)\rangle$. We give first the coefficient of $\langle \tfrac32(m)\rangle$

$$\begin{array}{ll} a_{m-3}+a_{m-4}-1-p_2(2k-3) & \\ = a_{m-3}+a_{m-4}-1-a_{2k-3}-a_{2k-4}-a_{2k-5} & \text{for } m=4k,\\ a_{m-3}+a_{m-4}-p_2(2k-2) & \\ = a_{m-3}+a_{m-4}-a_{2k-2}-a_{2k-3}-a_{2k-4} & \text{for } m=4k+2,\\ a_{m-3}+a_{m-4}-p_2(k-1) & \\ = a_{m-3}+a_{m-4}-a_{k-1}-a_{k-2}-a_{k-3} & \text{for } m=2k+1,\end{array}$$

and then the coefficient of $\langle \tfrac32(m,1)\rangle$

$$\begin{array}{ll} a_m+a_{m-1}-p_2(k-1) & \\ = a_m+a_{m-1}-a_{k-1}-a_{k-2}-a_{k-3} & \text{for } m=2k,\\ a_m+a_{m-1}-1-p_2(k-2) & \\ = a_m+a_{m-1}-1-a_{k-2}-a_{k-3}-a_{k-4} & \text{for } m=2k+1.\end{array}$$

Separating the terms as above, we obtain

$$\begin{aligned}\langle \tfrac12(0)\rangle \otimes \{1^3\} &= \textstyle\sum_{k\geq 0}(a_{4k-3}+a_{4k-4}-1-a_{2k-3}-a_{2k-4}-a_{2k-5})\langle \tfrac32(4k)\rangle\\ &+ \textstyle\sum_{k\geq 0}(a_{4k-1}+a_{4k-2}-a_{2k-2}-a_{2k-3}-a_{2k-4})\langle \tfrac32(4k+2)\rangle\\ &+ \textstyle\sum_{k\geq 0}(a_{2k+1}+a_{2k}-1-a_{k-2}-a_{k-3}-a_{k-4})\langle \tfrac32(2k+1,1)\rangle\end{aligned} \tag{5.29}$$

and

$$\begin{aligned}\langle \tfrac12(1)\rangle \otimes \{1^3\} &= \textstyle\sum_{k\geq 0}(a_{2k-2}+a_{2k-3}-a_{k-1}-a_{k-2}-a_{k-3})\langle \tfrac32(2k+1)\rangle\\ &+ \textstyle\sum_{k\geq 1}(a_{2k}+a_{2k-1}-a_{k-1}-a_{k-2}-a_{k-3})\langle \tfrac32(2k,1)\rangle\\ &+ \langle \tfrac32(1^3)\rangle\end{aligned} \tag{5.30}$$

Finally

$$\begin{aligned}&\langle \tfrac12(1)\rangle \otimes \{21\} + \langle \tfrac12(0)\rangle \otimes \{21\}\\ &= \tilde\Delta \otimes \{21\}\\ &-\langle \tfrac12(0)\rangle \times \langle \tfrac12(1)\rangle \otimes \{2\} - \langle \tfrac12(1)\rangle \times \langle \tfrac12(0)\rangle \otimes \{2\}\\ &-\langle \tfrac12(0)\rangle \times \langle \tfrac12(1)\rangle \otimes \{1^2\} - \langle \tfrac12(1)\rangle \times \langle \tfrac12(0)\rangle \otimes \{1^2\}\end{aligned}$$

which yields

$$\begin{array}{rcl}\langle\frac{1}{2}(0)\rangle\otimes\{21\}+\langle\frac{1}{2}(1)\rangle\otimes\{21\} & = & \sum_{m\geq 0}\langle\frac{3}{2}(m)\rangle(a_{m-1}+2a_{m-2}+a_{m-3})\\ & + & \sum_{m\geq 1}\langle\frac{3}{2}(m,1)\rangle(a_{m-1}+2a_{m-2}+a_{m-3})\\ & - & \sum_{i\geq 0}\langle\frac{3}{2}(2i+1)\rangle\\ & - & \sum_{k\geq 1}p_2(k-2)\left(\langle\frac{3}{2}(2k,1)\rangle+\langle\frac{3}{2}(2k+1)\rangle\right)\\ & - & \sum_{k\geq 1}p_2(k-1)\left(\langle\frac{3}{2}(4i)\rangle\right)\\ & - & \sum_{m\geq 3}p_2(m-3)\left(\langle\frac{3}{2}(2m-1,1)\rangle+\langle\frac{3}{2}(2m)\rangle\right)\\ & - & \sum_{m\geq 1}p_2(m-1)\left(\langle\frac{3}{2}(2m,1)\rangle+\langle\frac{3}{2}(2m+1)\rangle\right)\end{array}$$

and hence the coefficient of $\langle\frac{3}{2}(m)\rangle$ is
for $m=4k$,

$$\begin{array}{l}a_{m-1}+2a_{m-2}+a_{m-3}-1-p_2(2k-1)-p_2(2k-3)\\ =a_{m-1}+2a_{m-2}+a_{m-3}-1-a_{2k-1}-a_{2k-2}-2a_{2k-3}-a_{2k-4}-a_{2k-5}\end{array} \tag{5.31}$$

for $m=4k+2$,

$$\begin{array}{l}a_{m-1}+2a_{m-2}+a_{m-3}-p_2(2k)-p_2(2k-2)\\ =a_{m-1}+2a_{m-2}+a_{m-3}-a_{2k-1}-2a_{2k-2}-2a_{2k-3}-a_{2k-4}\end{array} \tag{5.32}$$

for $m=2k+1$,

$$\begin{array}{l}a_{m-1}+2a_{m-2}+a_{m-3}-p_2(k-1)-p_2(k-2)\\ =a_{m-1}+2a_{m-2}+a_{m-3}-a_{k-1}-2a_{k-2}-2a_{k-3}-a_{k-4}\end{array} \tag{5.33}$$

and for the coefficient of $\langle\frac{3}{2}(m,1)\rangle$, for $m=2k$,

$$\begin{array}{l}a_{m-1}+2a_{m-2}+a_{m-3}-p_2(k-1)-p_2(k-2)\\ =a_{m-1}+2a_{m-2}+a_{m-3}-a_{k-1}-2a_{k-2}-2a_{k-3}-a_{k-4}\end{array} \tag{5.34}$$

for $m=2k+1$,

$$\begin{array}{l}a_{m-1}+2a_{m-2}+a_{m-3}-1-p_2(k-1)-p_2(k-2)\\ =a_{m-1}+2a_{m-2}+a_{m-3}-1-a_{k-1}-2a_{k-2}-2a_{k-3}-a_{k-4}\end{array} \tag{5.35}$$

To separate the terms of $\langle\frac{1}{2}(0)\rangle\otimes\{21\}$ and $\langle\frac{1}{2}(1)\rangle\otimes\{21\}$ we again make use of the $U(n)$ content of $\langle\frac{1}{2}(0)\rangle\otimes\{21\}+\langle\frac{1}{2}(1)\rangle\otimes\{21\}$, which yields

$$\begin{array}{l}\langle\frac{1}{2}(0)\rangle\otimes\{21\}\\ =\sum_{k\geq 0}(a_{4k-1}+2a_{4k-2}+a_{4k-3}-1-a_{2k-1}-a_{2k-2}-2a_{2k-3}-a_{2k-4}-a_{2k-5})\langle\frac{3}{2}(4k)\rangle\\ +\sum_{k\geq 0}(a_{4k+1}+2a_{4k}+a_{4k-1}-a_{2k-1}-2a_{2k-2}-2a_{2k-3}-a_{2k-4})\langle\frac{3}{2}(4k+2)\rangle\\ +\sum_{k\geq 0}(a_{2k}+2a_{2k-1}+a_{2k-2}-1-a_{k-1}-2a_{k-2}-2a_{k-3}-a_{k-4})\langle\frac{3}{2}(2k+1,1)\rangle\end{array} \tag{5.36}$$

and

$$\begin{array}{l}\langle\frac{1}{2}(1)\rangle\otimes\{21\}\\ =\sum_{k\geq 0}(a_{2k}+2a_{2k-1}+a_{2k-2}-a_{k-1}-2a_{k-2}-2a_{k-3}-a_{k-4})\langle\frac{3}{2}(2k+1)\rangle\\ +\sum_{k\geq 0}(a_{2k-1}+2a_{2k-2}+a_{2k-3}-a_{k-1}-2a_{k-2}-2a_{k-3}-a_{k-4})\langle\frac{3}{2}(2k,1)\rangle\end{array} \tag{5.37}$$

5.4 Star equivalent $Sp(2n,\mathbb{R})$ unirreps and plethysm

Recall the modification rule for $O(k)$

$$[\lambda] = (-1)^{c-1}[\lambda - h]^*, \qquad h = 2\lambda_1{}' - k \geq 0 \tag{5.38}$$

where a continuous strip of boxes of length h, starting at the foot of the first column and working up along the right edge and ending in column c is removed from the Ferrers graph of (λ). The constraints on partitions (λ) corresponding to unirreps $\langle\frac{k}{2}(\lambda)\rangle$ of $Sp(2n,\mathbb{R})$ allows us to restrict our attention to those cases corresponding to removing boxes from just the first column. Thus for the unirreps $\langle\frac{k}{2}(\lambda)\rangle$ of $Sp(2n,\mathbb{R})$, we have

$$(\lambda) \to (\lambda - h)^*, \qquad h = 2\lambda_1{}' - k \geq 0 \tag{5.39}$$

and conversely we may add boxes to the first column of a standard partition (λ) for $O(k)$ to produce a $(\lambda + h)$ that is non-standard for $O(k)$ but standard for $Sp(2n,\mathbb{R})$

$$(\lambda + h) \to (\lambda)^*, \qquad h = k - 2\lambda_1{}' \geq 0 \tag{5.40}$$

Unirreps of $Sp(2n,\mathbb{R})$ will be said to be *star equivalent* if

$$\langle\tfrac{k}{2}(\lambda)\rangle^* = \langle\tfrac{k}{2}(\lambda)^*\rangle = \langle\tfrac{k}{2}(\lambda \pm h)\rangle \tag{5.41}$$

which for brevity, we shall write as

$$\langle\tfrac{k}{2}(\lambda)\rangle \overset{*}{\equiv} \langle\tfrac{k}{2}(\lambda \pm h)\rangle \tag{5.42}$$

where it is assumed that $k \leq n$. Thus $\langle\frac{1}{2}(0)\rangle \overset{*}{\equiv} \langle\frac{1}{2}(1)\rangle$ is a pair of star equivalent unirreps of $Sp(2n,\mathbb{R})$.

Bearing in mind the preceding remarks and definitions, consider the leading terms in the plethysm $\langle\frac{1}{2}(0)\rangle \otimes \{3\}$

$$\begin{array}{lllll}
\langle\frac{3}{2}(0)\rangle & +\langle\frac{3}{2}(4)\rangle & +\langle\frac{3}{2}(6)\rangle & +\langle\frac{3}{2}(8)\rangle & +\langle\frac{3}{2}(91)\rangle \\
+\langle\frac{3}{2}(10\ )\rangle & +2\langle\frac{3}{2}(12\ )\rangle & +\langle\frac{3}{2}(13\ 1)\rangle & +\langle\frac{3}{2}(14\ )\rangle & +\langle\frac{3}{2}(15\ 1)\rangle \\
+2\langle\frac{3}{2}(16\ )\rangle & +\langle\frac{3}{2}(17\ 1)\rangle & +2\langle\frac{3}{2}(18\ )\rangle & +\langle\frac{3}{2}(19\ 1)\rangle & +2\langle\frac{3}{2}(20\ )\rangle \\
+2\langle\frac{3}{2}(21\ 1)\rangle & +2\langle\frac{3}{2}(22\ )\rangle & +\langle\frac{3}{2}(23\ 1)\rangle & &
\end{array}$$

Replacing each unirrep by its star equivalent gives the leading terms of $(\langle\frac{1}{2}(0)\rangle \otimes \{3\})^*$ as

$$\begin{array}{lllll}
\langle\frac{3}{2}(1^3)\rangle & +\langle\frac{3}{2}(41)\rangle & +\langle\frac{3}{2}(61)\rangle & +\langle\frac{3}{2}(81)\rangle & +\langle\frac{3}{2}(9)\rangle \\
+\langle\frac{3}{2}(10\ 1)\rangle & +2\langle\frac{3}{2}(12\ 1)\rangle & +\langle\frac{3}{2}(13\ )\rangle & +\langle\frac{3}{2}(14\ 1)\rangle & +\langle\frac{3}{2}(15\ )\rangle \\
+2\langle\frac{3}{2}(16\ 1)\rangle & +\langle\frac{3}{2}(17\ )\rangle & +2\langle\frac{3}{2}(18\ 1)\rangle & +\langle\frac{3}{2}(19\ )\rangle & +2\langle\frac{3}{2}(20\ 1)\rangle \\
+2\langle\frac{3}{2}(21\ )\rangle & +2\langle\frac{3}{2}(22\ 1)\rangle & +\langle\frac{3}{2}(23\ )\rangle & &
\end{array}$$

which are precisely the leading terms in $\langle\frac{1}{2}(0)\rangle \otimes \{1^3\}$

$$\begin{array}{lllll}
\langle\frac{3}{2}(1^3)\rangle & +\langle\frac{3}{2}(41)\rangle & +\langle\frac{3}{2}(61)\rangle & +\langle\frac{3}{2}(81)\rangle & +\langle\frac{3}{2}(9)\rangle \\
+\langle\frac{3}{2}(10\ 1)\rangle & +2\langle\frac{3}{2}(12\ 1)\rangle & +\langle\frac{3}{2}(13\ )\rangle & +\langle\frac{3}{2}(14\ 1)\rangle & +\langle\frac{3}{2}(15\ )\rangle \\
+2\langle\frac{3}{2}(16\ 1)\rangle & +\langle\frac{3}{2}(17\ )\rangle & +2\langle\frac{3}{2}(18\ 1)\rangle & +\langle\frac{3}{2}(19\ )\rangle & +2\langle\frac{3}{2}(20\ 1)\rangle \\
+2\langle\frac{3}{2}(21\ )\rangle & +2\langle\frac{3}{2}(22\ 1)\rangle & +\langle\frac{3}{2}(23\ )\rangle & &
\end{array}$$

Similarly, the leading terms in $(\langle\frac{1}{2}(0)\rangle\otimes\{21\})^*$ are

$$\begin{array}{lllll}
\langle\frac{3}{2}(2)\rangle & +\langle\frac{3}{2}(4)\rangle & +\langle\frac{3}{2}(51)\rangle & +\langle\frac{3}{2}(6)\rangle & +\langle\frac{3}{2}(71)\rangle \\
+\,2\langle\frac{3}{2}(8)\rangle & +\langle\frac{3}{2}(91)\rangle & +\,2\langle\frac{3}{2}(10\ )\rangle & +\,2\langle\frac{3}{2}(11\ 1)\rangle & +\,2\langle\frac{3}{2}(12\ )\rangle \\
+\,2\langle\frac{3}{2}(13\ 1)\rangle & +\,3\langle\frac{3}{2}(14\ )\rangle & +\,2\langle\frac{3}{2}(15\ 1)\rangle & +\,3\langle\frac{3}{2}(16\ )\rangle & +\,3\langle\frac{3}{2}(17\ 1)\rangle \\
+\,3\langle\frac{3}{2}(18\ )\rangle & +\,3\langle\frac{3}{2}(19\ 1)\rangle & +\,4\langle\frac{3}{2}(20\ )\rangle & +\,3\langle\frac{3}{2}(21\ 1)\rangle & +\,4\langle\frac{3}{2}(22\ )\rangle \\
+\,4\langle\frac{3}{2}(23\ 1)\rangle & & & &
\end{array}$$

which are identical to the leading terms in $\langle\frac{1}{2}(0)\rangle\otimes\{21\}$

$$\begin{array}{lllll}
\langle\frac{3}{2}(2)\rangle & +\langle\frac{3}{2}(4)\rangle & +\langle\frac{3}{2}(51)\rangle & +\langle\frac{3}{2}(6)\rangle & +\langle\frac{3}{2}(71)\rangle \\
+\,2\langle\frac{3}{2}(8)\rangle & +\langle\frac{3}{2}(91)\rangle & +\,2\langle\frac{3}{2}(10\ )\rangle & +\,2\langle\frac{3}{2}(11\ 1)\rangle & +\,2\langle\frac{3}{2}(12\ )\rangle \\
+\,2\langle\frac{3}{2}(13\ 1)\rangle & +\,3\langle\frac{3}{2}(14\ )\rangle & +\,2\langle\frac{3}{2}(15\ 1)\rangle & +\,3\langle\frac{3}{2}(16\ )\rangle & +\,3\langle\frac{3}{2}(17\ 1)\rangle \\
+\,3\langle\frac{3}{2}(18\ )\rangle & +\,3\langle\frac{3}{2}(19\ 1)\rangle & +\,4\langle\frac{3}{2}(20\ )\rangle & +\,3\langle\frac{3}{2}(21\ 1)\rangle & +\,4\langle\frac{3}{2}(22\ )\rangle \\
+\,4\langle\frac{3}{2}(23\ 1)\rangle & & & &
\end{array}$$

The above results lead us to conjecture that

$$\langle\tfrac{1}{2}(0)\rangle\otimes\{\lambda\}\overset{*}{\equiv}\langle\tfrac{1}{2}(1)\rangle\otimes\{\lambda'\} \tag{5.43}$$

That the conjecture holds for $\lambda\vdash 2$ is evident from (5.1) to (5.4).That the conjecture holds for $\lambda\vdash 3$ follows by noting that

$$\langle\tfrac{3}{2}(m)\rangle\overset{*}{\equiv}\langle\tfrac{3}{2}(m,1)\rangle$$

and

$$\langle\tfrac{3}{2}(0)\rangle\overset{*}{\equiv}\langle\tfrac{3}{2}(1^3)\rangle$$

Application of these star equivalences to the results obtained for the symmetrization of the cubes of the star equivalent pair $\langle\frac{1}{2}(0)\rangle\overset{*}{\equiv}\langle\frac{1}{2}(1)\rangle$ together with inspection of the given coefficients, noting in particular (5.31) to (5.35), completes the verification of (5.43) for all $(\lambda)\vdash 2,3$. That proof the conjecture (5.43) holds for all (λ) and is given in a recent paper from King and Wybourne [61] together with a proof of a more general star equivalence

$$\begin{aligned}
\langle\tfrac{k}{2}(\mu)\rangle &\overset{*}{\equiv} \langle\tfrac{k}{2}(\mu\pm h)\rangle\otimes\{\lambda'\} \quad (\ k\ \ \text{odd}) \\
&\overset{*}{\equiv} \langle\tfrac{k}{2}(\mu\pm h)\rangle\otimes\{\lambda\} \quad (\ k\ \ \text{even})
\end{aligned} \tag{5.44}$$

5.5 Concluding remarks

It is now possible to establish results for higher powers by applying the same methods to obtain the plethysms. But because longer partitions are involved the calculations rapidly become cumbersome. For example, the branching rule $O(4)\downarrow S_4$ is obtained by examination of partitions of the form $[1^4]$, $[m,1,1]$, $[m,2]$, $[m,1]$ and $[m]$. One remarks that $[m,1,1]=[m,1]^*$ and $[1^4]=[0]^*$ and one needs to examine only the partitions of the form $[m]$, $[m,1]$, $[m,2]$. One applies the formula to give $[m]\downarrow S_4$, $[m,1]\downarrow S_4$, $[m,2]\downarrow S_4$ which is equivalent to

$$[m]\downarrow\langle 1\rangle\otimes\{m/G\}$$

$$[m,1] \downarrow \langle 1 \rangle \otimes \{m1/G\}$$

and

$$[m,2] \downarrow \langle 1 \rangle \otimes \{m2/G\}$$

Now, to obtain the result, one writes that $\{m,j\} = \{m\}\{j\} - \{m+1\}\{j-1\}$ and finally

$$\langle 1 \rangle \otimes \{m,j\} = (\langle 1 \rangle \otimes \{m\}) * (\langle 1 \rangle \otimes \{j\}) - (\langle 1 \rangle \otimes \{m+1\}) * (\langle 1 \rangle \otimes \{j-1\})$$

These remarks are sufficient to give a method to compute the fourth power. The existence of the star equivalence operation for unirreps of $Sp(2n,\mathbb{R})$ implies certain one-to-one mappings between the states of N-particles in even parity orbitals and those of N-particles in odd parity orbitals.

A practical application of this results is the possibility of a very simple way to calculate the coefficients of the plethysms. We give here some examples of such calculations for $\langle \frac{1}{2}(0) \rangle \otimes \{3\}$, $\langle \frac{1}{2}(0) \rangle \otimes \{1^3\}$, $\langle \frac{1}{2}(1) \rangle \otimes \{3\}$ and $\langle \frac{1}{2}(1) \rangle \otimes \{1^3\}$.

5.6 Calculations

Once obtained the closed formulas giving the coefficients of the irreducible representations in the plethysms given in this document, it is possible to calculate these coefficients in a simple and automatic way for example with MAPLE.

Here is a sample program giving the coefficients for the plethysms $\langle \frac{1}{2}(0) \rangle \otimes \{3\}$ and $\langle \frac{1}{2}(1) \rangle \otimes \{3\}$. Of course, it is possible to give similar calculations for the other plethysms formulae.

This very simple calculation goes as follows. First of all, one calculates the coefficients to a sufficiently large order in Taylor expansion of the function $\frac{1}{(1-z^2)(1-z^3)}$ which corresponds to the coefficients Kostka polynomials. Then, one adds up these various coefficients by applying our formulas, using the fact that exists a simple relation between these coefficients and the number of partitions of n into parts lower or equal to two.

```
###########################################################
# Taylor expansion of                                     #
# 1/(1-z**2)/(1-z**3)                                     #
###########################################################
TableCoeff := proc (ordre)
local i,j,k,f,degmax,d;
global TCoeffs;

f := sort(convert ( taylor(1/(1-z**2)/(1-z**3), z=0, ordre), polyn\^om));
degmax := degree(op(1,f),z);
```

```
TCoeffs := array(sparse,1 .. degmax+1);

for i from 1 to nops(f) do
d := degree(op(i,f),z);
k :=  coeff(f,z,d);
TCoeffs[d+1] := k;
od;
end;

#############################################################
# Calculation of plethysms  <s, 01> \otimes {3}             #
#  and  <s,1> times {3}                                     #
#############################################################
# Used dunctions :                                          #
# C3m4kfor3 gives the coeff of <s1,m>  for m=4k             #
# C3m4k2for3  gives the coeff of <s1,m>  for  m=4k+2        #
# C3m2k1for3  gives the coeff of <s1,m>  for m=2k+1         #
# C3m12kfor3  gives the coeff of <s1,m1> for m=2k           #
# C3m12k1for3 gives the coeff of <s1,m1> for m=2k+1         #
#############################################################
# From this functions :                                     #
#    ---> pls03   gives  <s,0> \otimes {3}                  #
#    ---> pls13   gives   <s,1> \otimes {3}                 #
#############################################################

### Coeff of <s1,m>, m=4k in pl s;0,3 + pl s;1,3 #################

C3m4kfor3 := proc(i)
local c;
  c:= TCoeffs[i+1]+TCoeffs[i]-TCoeffs[i/2]
-TCoeffs[i/2-1]-TCoeffs[i/2-2];
RETURN(c);
end;

###Coeff of <s1,m>, m= 2k+1 in pl s;0,3 + pl s;1,3 ####

C3m2k1for3 := proc(i)
local c;
c := TCoeffs[i+1]+TCoeffs[i]-1-TCoeffs[(i-1)/2-1]
-TCoeffs[(i-1)/2-2]-TCoeffs[(i-1)/2-3];
RETURN(c);
end;

### Coeff of <s1,m>, m=4k+2 in pl s;0,3 + pl s;1,3 ###

C3m4k2for3 := proc(i)
local c;
c := TCoeffs[i+1]+TCoeffs[i]-TCoeffs[(i-2)/2+1]
-TCoeffs[(i-2)/2]-TCoeffs[(i-2)/2-1];
RETURN(c);
```

```
end;

### Coeff of <s1,m1>, m=2k  in pl s;0,3 + pl s;1,3 ####

C3m12kfor3 := proc(i)
local c;
c := TCoeffs[i-2]+TCoeffs[i-3]-TCoeffs[i/2-1]
-TCoeffs[i/2-2]-TCoeffs[i/2-3];
RETURN(c);
end;

### Coeff of <s1,m1>, m=2k+1  in pl s;0,3 + pl s;1,3 ###

C3m12k1for3 := proc(i)
local c;
c := TCoeffs[i-2]+TCoeffs[i-3]-TCoeffs[(i-1)/2]
-TCoeffs[(i-1)/2-1]-TCoeffs[(i-1)/2-2];
RETURN(c);
end;

############ Calculation of pl s;0,3 ######################

pls03 := proc(ordre)
local i,l1;

l1:= [[1,0,0],[1,4,0],[1,6,0]];

TableCoeff(ordre+4);
for i from 8 to ordre do
if ( i mod 4 = 0 ) then
l1:=[op(l1),[C3m4kfor3(i),i,0]];
elif ( i mod 2 = 0 ) then
l1:=[op(l1),[C3m4k2for3(i),i,0]];
else
  l1 := [op(l1),[C3m12k1for3(i),i,1]];
fi;
od;
affiche(l1);
#RETURN(l1);
end;

############ Calculation of pl s;1,3 ######################

pls13 := proc(ordre)
local i,l1;

l1 := [[1,3,0],[1,6,1],[1,7,0]];

TableCoeff(ordre+4);
for i from 9 to ordre do
if ( i mod 2 = 0 ) then
```

```
l1:=[op(l1),[C3m12kfor3(i),i,1]];
else
l1 := [op(l1),[C3m2k1for3(i),i,0]];
fi;
od;
affiche(l1);
#RETURN(l1);
end;

####################################################
# Procedure output  (in symplectic form ... )       #
####################################################

affiche := proc(l)
local i,j,n,a,b, chaine, cpt;

n := nops(l);
chaine := ``;
a := `<s1,`;
b := `> + `;
cpt := 0;

for i from 1 to n do
   if ( op(1,op(i,l)) <> 0 ) then
chaine:=cat(chaine,op(1,op(i,l)),a,op(2,op(i,l)));
if ( op(3,op(i,l)) <> 0 ) then
chaine:=cat(chaine,` `,op(3,op(i,l)),b);
else
chaine:=cat(chaine,b);
fi;
   cpt := cpt+1;
   fi;

   if (cpt mod 6 = 0 ) then
print(chaine);
chaine :=``;
   fi;
od;
print ( chaine );
end;
```

With this program we obtain for the 25 first terms of $\langle\frac{1}{2}(0)\rangle \otimes \{3\}\rangle$

```
> pls03(25);

    1<s1,0> + 1<s1,4> + 1<s1,6> + 1<s1,8> + 1<s1,9 1> + 1<s1,10> +
    2<s1,12> + 1<s1,13 1> + 1<s1,14> + 1<s1,15 1> + 2<s1,16> + 1<s1,17 1> +
    2<s1,18> + 1<s1,19 1> + 2<s1,20> + 2<s1,21 1> + 2<s1,22> + 1<s1,23 1> +
    3<s1,24> + 2<s1,25 1>
```

and for the 25 first terms of $\langle\frac{1}{2}(1)\rangle \otimes \{3\}\rangle$

```
> pls13(25);

    1<s1,3> + 1<s1,6 1> + 1<s1,7> + 1<s1,9> + 1<s1,10 1> + 1<s1,11> +
    1<s1,12 1> + 1<s1,13> + 1<s1,14 1> + 2<s1,15> + 1<s1,16 1> + 1<s1,17> +
    2<s1,18 1> + 2<s1,19> + 1<s1,20 1> + 2<s1,21> + 2<s1,22 1> + 2<s1,23> +
    2<s1,24 1> + 2<s1,25>
```

These results are coherent with those obtained with SCHUR.

Chapter 6

Representations of $U(p,q)$

This chapter is the subject of a joint work realised with J. Y. Thibon and B. G. Wybourne and published in *Journal of Physics A: Mathematical and General* [101].

6.1 Introduction

Bohr, in his very first paper [9], on what has become known as "The Bohr-model" of the atom, made the surprising discovery that the energies of levels of the non-relativistic hydrogen atom could be expressed (in appropriate units) as simply

$$E_n = -\frac{1}{n^2} \quad \text{with} \quad n = 1, 2, \ldots$$

With the advent of the Schrödinger equation for the H-atom it became apparent that each value of n could be associated with orbital angular momenta of

$$\ell = 0, 1, 2, \ldots, n-1$$

and associated with each value of ℓ there were $(2\ell+1)$ values of the angular momentum projection eigenvalues m_ℓ leading to each energy level E_n being associated with $(n-1)^2$ eigenfunctions. Initially such a high degeneracy appeared surprising. Pauli [86] noted that in a purely Coulombic central field there was an additional constant of the motion associated with the Runge-Lenz vector and from there was led to the realisation that the observed degeneracies were precisely the dimensions of certain of the irreducible representations of the group $SO(4) \sim SU(2) \times SU(2)$, in particular those commonly designated as $[n-1, 0] \sim \{n-1\} \times \{n-1\}$.

Much later, Barut and Kleinert [5] observed that all the discrete levels of a H-atom spanned a single infinite dimensional irreducible representation of the non-compact group $SO(4,2) \sim SU(2,2)$ with the group being referred to as the *dynamical group* of the H-atom [5, 106]. The Runge-Lenz vector ceases to be a constant of the motion for two or more electrons in a central Coulomb field [5, 106, 109, 10] and the $SO(4)$ symmetry is broken. Nevertheless, it can be useful to consider the $n-$electron states starting with the single irreducible representation of $SU(2,2)$, or more simply $U(2,2)$, and then forming symmetrized $n-$fold tensor products which will be the central problem considered here. For greater generality we shall initially consider the group $U(p,q)$ as previously studied [60] by King and Wybourne. After a brief sketch of the relevant properties of $U(p,q)$ we tackle the problem of resolving the Kronecker powers of the relevant irreducible representation into its relevant symmetrized powers, namely the problem of plethysms in $U(p,q)$. In the process we are able to give closed results for the second powers of the fundamental harmonic series irreducible representations of $U(p,q)$ which thus yields, in the case of two electrons, the appropriate spin triplet and singlet states.

6.2 The fundamental harmonic series irreducible representations of $U(p,q)$

Following [60], we may embed the non-compact group $U(p,q)$ in $Sp(2p+2q,\mathbb{R})$ whose harmonic representation $\tilde{\Delta}$ decomposes as

$$\tilde{\Delta} \to H = H_0 + \sum_{m=1}^{\infty}(H_m + H_{-m}) \tag{6.1}$$

where

$$\begin{aligned} H_0 &= \{1(\bar{0};0)\} && (6.2)\\ H_m &= \{1(\bar{0};m)\} \quad m=1,2,\ldots && (6.3)\\ H_{-m} &= \{1(\bar{m};0)\} \quad m=1,2,\ldots && (6.4)\end{aligned}$$

Upon restriction to the maximal compact subgroup $U(q)\times U(p)$ we have

$$H_0 = \{1(\bar{0};0\} \ \to \ (0\times\varepsilon)\cdot(\sum_{j=0}^{\infty}\{\bar{j}\}\times\{j\}) \tag{6.5}$$

$$H_m = \{1(\bar{0};m)\} \ \to \ (0\times\varepsilon)\cdot(\sum_{j=0}^{\infty}\{\bar{j}\}\times\{m+j\}) \tag{6.6}$$

$$H_{-m} = \{1(\bar{m};0)\} \ \to \ (0\times\varepsilon)\cdot(\sum_{j=0}^{\infty}\{\overline{m+j}\}\times\{j\}) \tag{6.7}$$

The harmonic series unitary irreducible representations (unirreps for short) $\{k(\bar{\nu};\mu)\}$ of $U(p,q)$ are generated by considering powers [46] H^k of H. Under restriction from $U(pk,qk)$ to $U(p,q)\times U(k)$

$$H \to \sum_{\nu,\mu}\{k(\bar{\nu};\mu)\}\times\{\bar{\nu};\mu\} \tag{6.8}$$

where the conjugate partitions $(\tilde{\nu})$ and $(\tilde{\mu})$ satisfy the constraints [46]

$$\begin{aligned} \tilde{\mu}_1+\tilde{\nu}_1 &\le k && (6.9)\\ \tilde{\mu}_1 &\le p \quad \text{and} \quad \tilde{\nu}_1 \le q\ . && (6.10)\end{aligned}$$

6.3 Kronecker products for all of the fundamental harmonic series unirreps

The Kronecker product of two arbitrary unirreps of $U(p,q)$ may be evaluated following [60] to give

$$\{k(\bar{\nu};\mu)\} \times \{\ell(\bar{\tau};\sigma)\} = \sum_{\zeta} \{k+\ell((\{\bar{\nu}_s\}^k \cdot \{\bar{\tau}_s\}^\ell \cdot \{\bar{\zeta}\}; \{\mu_s\}^k \cdot \{\sigma_s\}^\ell \cdot \{\zeta\}))\} \tag{6.11}$$

where the notation is as in [60] and it is understood that

$$((\bar{\rho};\lambda))_{k+\ell,p,q} = \begin{cases} (\bar{\rho};\lambda) & \text{if } \tilde{\lambda}_1 \le p, \tilde{\rho}_1 \le q \text{ and } \tilde{\lambda}_1 + \tilde{\rho}_1 \le k+\ell \\ 0 & \text{otherwise} \end{cases} \tag{6.12}$$

Specialization of (6.11) to the fundamental harmonic series of $U(p,q)$ yields the following cases: for $m, r, s > 0$,

$$\begin{aligned}
H_0^2 &= \sum_{n=0}^{\infty} \{2(\bar{n};n)\} && (6.13)\\
H_m^2 &= \sum_{n=0}^{\infty} \{2(\bar{n};n+2m)\} + \sum_{p=1}^{m} \{2(\bar{0};2m-p,p)\} && (6.14)\\
H_{-m}^2 &= \sum_{n=0}^{\infty} \{2(\overline{n+2m};n)\} + \sum_{p=1}^{m} \{2(\overline{2m-p,p};0)\} && (6.15)\\
H_m \times H_{-m} &= \sum_{k=0}^{\infty} \{2(\overline{m+k};m+k)\} && (6.16)\\
H_r \times H_s &= \sum_{x=0}^{\min(r,s)} \{2(\bar{0};r+s-x,x)\} + \sum_{k=1}^{\infty} \{2(\bar{k};r+s+k\} && (6.17)\\
H_{-r} \times H_{-s} &= \sum_{x=0}^{\min(r,s)} \{2(\overline{r+s-x,x};0)\} + \sum_{k=1}^{\infty} \{2(\overline{r+s+k};k)\} && (6.18)\\
H_{-r} \times H_s &= \{2(\bar{r};s)\} + \sum_{k=1}^{\infty} \{2(\overline{r+k};s+k\} && (6.19)\\
H_0 \times H_m &= \{2(\bar{0};m)\} + \sum_{k=1}^{\infty} \{2(\bar{k};m+k)\} && (6.20)\\
H_0 \times H_{-m} &= \{2(\bar{m};0)\} + \sum_{k=1}^{\infty} \{2(\overline{m+k};k)\} && (6.21)
\end{aligned}$$

6.4 Symmetrized squares of the fundamental harmonic representations

To separate the Kronecker squares of the representations H_m of $U(p,q)$ into its symmetric and antisymmetric parts, we first solve the corresponding problem for the complete harmonic representation H. This is done by restricting the H of $U(2p,2q)$ through the chain

$$U(2p,2q) \supset U(p,q) \times U(2) \supset U(p,q) \times S_2 \supset U(p,q) \ . \tag{6.22}$$

Under $U(2p,2q) \downarrow U(p,q) \times U(2)$, Eq. (6.8), and the constraints (6.9)-(6.10) with $k = 2$ yield

$$H \to \sum_{\tilde{\nu_1}+\tilde{\mu_1} \le 2} \{2(\bar{\nu};\mu)\} \times \{\bar{\nu};\mu\} \ . \tag{6.23}$$

Therefore, we just have to determine the restriction to S_2 of the $U(2)$ representations $\{\bar{\nu};\mu\}$.

It is known ([1], see also [97]) that the Frobenius characteristic of the decomposition of $\{m\}$ under $U(k) \downarrow S_k$ is the coefficient of z^m in the series

$$h_k\left(\frac{X}{1-z}\right) = \prod_{j=1}^{k} \frac{1}{1-z^j} \cdot \sum_{\lambda \vdash k} \tilde{K}_{\lambda,1^k}(z) s_\lambda \tag{6.24}$$

where $\tilde{K}_{\lambda,1^k}(z)$ are the (cocharge) Kostka-Foulkes polynomials. In particular for $k = 2$, $\{m\} \downarrow S_2$ is the coefficient of z^m in

$$\frac{1}{(1-z)(1-z^2)} [(2) + z(11)] \tag{6.25}$$

so that

$$\{m\} \to p_2(m)(2) + p_2(m-1)(11) \tag{6.26}$$

where $p_2(m)$ is the number of partitions of m into parts not greater that 2, that is, $p_2(m) = \lceil \frac{m+1}{2} \rceil$.

Taking into account the $U(2)$ equivalences

$$\begin{aligned}
\{\bar{0};\mu_1\mu_2\} &\equiv \epsilon^{\mu_2}\{\mu_1 - \mu_2\} \ , \\
\{\bar{m};n\} &\equiv \epsilon^{-m}\{n+m\} \\
\{\overline{\nu_1\nu_2};0\} &\equiv \epsilon^{-\nu_1}\{\nu_1 - \nu_2\} \ ,
\end{aligned}$$

we obtain

$$\{\bar{0};\mu_1\mu_2\} \quad \rightarrow \quad \begin{cases} p_2(\mu_1-\mu_2)(2)+p_2(\mu_1-\mu_2-1)(11) & \text{for} \quad \mu_2 \text{ even} \\ p_2(\mu_1-\mu_2-1)(2)+p_2(\mu_1-\mu_2)(11) & \text{for} \quad \mu_2 \text{ odd} \end{cases} \tag{6.27}$$

$$\{\bar{m};n\} \quad \rightarrow \quad \begin{cases} p_2(m+n)(2)+p_2(m+n-1)(11) & \text{for} \quad m \text{ even} \\ p_2(m+n-1)(2)+p_2(m+n)(11) & \text{for} \quad m \text{ odd} \end{cases} \tag{6.28}$$

$$\{\overline{\nu_1\nu_2};0\} \quad \rightarrow \quad \begin{cases} p_2(\nu_1-\nu_2)(2)+p_2(\nu_1-\nu_2-1)(11) & \text{for} \quad \nu_1 \text{ even} \\ p_2(\nu_1-\nu_2-1)(2)+p_2(\nu_1-\nu_2)(11) & \text{for} \quad \nu_1 \text{ odd} \end{cases} \tag{6.29}$$

Now, we have

$$\begin{aligned} H\otimes\{2\} &= \left(H_0+\sum_{m=1}^{\infty}(H_m+H_{-m})\right)\otimes\{2\} \\ &= H_0\otimes\{2\}+H_0\times\sum_{m=1}^{\infty}(H_m+H_{-m})+\left(\sum_{m=1}^{\infty}H_m\right)\otimes\{2\} \\ &+ \sum_{r,s=1}^{\infty}H_r\times H_{-s}+\left(\sum_{m=1}^{\infty}H_{-m}\right)\otimes\{2\} \\ &= H_0\otimes\{2\}+\sum_{m=1}^{\infty}H_m\times H_{-m}+R\ . \end{aligned}$$

To extract $H_0\otimes\{2\}$ from $H\otimes\{2\}$, we remark that since under $U(p,q)\downarrow U(p)\times U(q)$

$$H_0 \rightarrow (0\times\epsilon)\sum_{m=0}^{\infty}\{\bar{m}\}\times\{m\}$$

the Kronecker square of H_0 can only contain terms whose restriction to $U(p)\times U(q)$ is a sum of representations $(0\times\epsilon^2)\{\bar{\nu};\mu\}$ such that $|\nu|=|\mu|$. Clearly, the terms in R are not of this form, and to obtain $H_0\otimes\{2\}$, we just need to compute the terms of the form $\{2(\bar{m};m)\}$ in $H\otimes\{2\}$ and to remove the contribution of $\sum_{m=1}^{\infty}H_m\times H_{-m}$.

We know from the above discussion that the multiplicity of $\{2(\bar{m};m)\}$ in $H\otimes\{2\}$ is equal to $p_2(m+m)=m+1$ for m even, and to $p_2(m+m-1)=m$ for m odd. On the other hand,

$$H_m\times H_{-m}=\sum_{k=0}^{\infty}\{2(\overline{m+k};m+k)\}$$

so that a given $\{2(\bar{m};m)\}$ occurs exactly m times in $\sum_{k=1}^{\infty}H_k\times H_{-k}$. Removing this contribution, we are left with

$$H_0\otimes\{2\}=\sum_{k=0}^{\infty}\{2(\overline{2k};2k)\}\ . \tag{6.30}$$

Since $H_0^2=\sum_{m=0}^{\infty}\{2(\bar{m};m)\}$, we also have

$$H_0\otimes\{1^2\}=\sum_{k=0}^{\infty}\{2(\overline{2k+1};2k+1)\}\ . \tag{6.31}$$

To split the square of H_m $(m\geq 1)$, we first observe that under restriction to $U(p)\times U(q)$, it yields a sum of representations of the form $(0\times\epsilon^2)\{\bar{\nu}\}\times\{\mu\}$ such that $|\mu|=|\nu|+2m$. Next, we proceed as above to extract it from $H\otimes\{2\}$. We have

$$\begin{aligned} H\otimes\{2\} &= \left(H_m+\sum_{j\neq m}H_j\right)\otimes\{2\} \\ &= H_m\otimes\{2\}+H_m\times\sum_{j\neq m}H_j+\left(\sum_{j=1}^{\infty}H_{m-j}\right)\otimes\{2\} \\ &+ \left(\sum_{j=1}^{\infty}H_{m-j}\right)\times\left(\sum_{k=1}^{\infty}H_{m+k}\right)+\left(\sum_{j=1}^{\infty}H_{m+j}\right)\otimes\{2\} \end{aligned}$$

Therefore, to extract $H_m\otimes\{2\}$, we just have to select from $H\otimes\{2\}$ the terms having the correct restriction property to $U(p)\times U(q)$ and to subtract the contribution of the crossed products $H_{m-j}\times H_{m+j}$ $(j\geq 1)$. Suppose first that $m\geq 1$. Then,

$$\sum_{j=1}^{\infty}H_{m-j}\times H_{m+j}=H_0\times H_{2m}+\sum_{r=1}^{m-1}H_r\times H_{2m-r}+\sum_{r=1}^{\infty}H_{-r}\times H_{2m+r}\ . \tag{6.32}$$

The terms of this sum are

$$H_0\times H_{2m} = \sum_{k=0}^{\infty}\{2(\bar{k};2m+k)\}\ , \tag{6.33}$$

$$H_r\times H_{2m-r} = \sum_{i=1}^{r}\{2(\bar{0};2m-i,i)\}+\sum_{k=0}^{\infty}\{2(\bar{k};2m+k)\}\ , \tag{6.34}$$

$$H_{-r}\times H_{2m+r} = \sum_{k=0}^{\infty}\{2(\overline{r+k},2m+r+k)\} \tag{6.35}$$

so that

$$\sum_{j=1}^{\infty}H_{m-j}\times H_{m+j}=\sum_{i=1}^{m-1}(m-i)\{2(\bar{0};2m-i,i)\}+\sum_{k=0}^{\infty}(m+k)\{2(\bar{k};2m+k)\}\ . \tag{6.36}$$

Now, the multiplicity of $\{2(\bar{0};2m-i,i)\}$ in $H\otimes\{2\}$ is $p_2(2m-2i)=m-i+1$ for i even, and $p_2(2m-2i-1)=m-i$ for i odd. Similarly, the multiplicity of $\{2(\bar{k};2m+k)\}$ in

$H \otimes \{2\}$ is equal to $p_2(2m+2k) = m+k+1$ for k even, and to $p_2(2m+2k-1) = m+k$ for k odd. Finally, we are left with

$$H_m \otimes \{2\} = \sum_{i=1}^{\lfloor m/2 \rfloor} \{2(\bar{0}; 2m-2i, 2i)\} + \sum_{k=0}^{\infty} \{2(\overline{2k}; 2m+2k)\} \ . \tag{6.37}$$

Similarly, we obtain

$$H_m \otimes \{1^2\} = \sum_{i=0}^{\lfloor (m-1)/2 \rfloor} \{2(\bar{0}; 2m-2i-1, 2i+1)\} + \sum_{k=0}^{\infty} \{2(\overline{2k+1}; 2m+2k+1)\} \tag{6.38}$$

Likewise,

$$H_{-m} \otimes \{2\} = \sum_{i=1}^{\lfloor m/2 \rfloor} \{2(\overline{2m-2i, 2i}; 0)\} + \sum_{k=0}^{\infty} \{2(\overline{2m+2k}; 2k)\} \ , \tag{6.39}$$

$$H_{-m} \otimes \{1^2\} = \sum_{i=0}^{\lfloor (m-1)/2 \rfloor} \{2(\overline{2m-2i-1, 2i+1}; 0)\} + \sum_{k=0}^{\infty} \{2(\overline{2m+2k+1}; 2k+1)\} \tag{6.40}$$

6.5 Conclusion

We have been able to obtain complete results for all the Kronecker products, and their symmetrized squares, for all the fundamental harmonic unirreps of $U(p,q)$ expressing them in a compact closed form. The plethysms of the square of the unirrep H_0 for $U(2,2)$ give the complete set of $U(2,2)$ unirreps that arise in a two-electron hydrogenic-like atom with the symmetric part describing the spin singlets ($S=0$) and the antisymmetric part the spin triplets ($S=1$). The groundstate $1s^2(^1S)$ is the first level of an infinite tower of states associated with the $\{2(\bar{0};0)\}$ unirrep while the lowest ^3SP level is the first level of an infinite tower associated with the $\{2(\bar{1};1)\}$ unirrep. A complete account of the two-electron hydrogen like states remains to be considered but knowing the relevant $U(2,2)$ unirepps is a significant beginning. For an $n-$electron hydrogen-like atom ($n>2$) the resolution of plethysms of the type $H_0 \otimes \{\lambda\} \quad (\lambda \vdash n)$ is a formidable task and complete results of the type considered herein cannot be expected.

6.6 Calculations with SCHUR

6.6.1 Standardisation and signed sequences

The problem of computing Kronecker products for the non compact group $U(p,q)$ was considered by King and Wybourne [60]. We have given here more precise results that we can exploit to give explicit calculations with SCHUR.

Using the standard notation developed here we are looking for a method to calculate the following product

$$\{k(\bar{\nu};\mu)\}\times\{l(\bar{\tau};\sigma)\}=\sum_{\zeta}\{k+l\left(\left(\{\bar{\nu}_s\}^k\{\bar{\tau}_s\}^l\{\bar{\zeta}\};\{\mu_s\}^k\{\sigma_s\}^l\{\zeta\}\right)\right)\} \qquad (6.41)$$

The first important step in the implementation of (6.41) is to determine the the required terms in the signed sequences, used and defined in this chapter.

We will illustrate the various steps of the calculation of Kronecker product by examining the product

$$\{2(\bar{3};1)\}\times\{2(\bar{1};2)\}$$

in $U(2,2)$ and $U(4,4)$ and first we recall how to calculate the signed sequence.

We first note that an irreducible representation (irrep) $\{\bar{\nu};\mu\}$ is standard in $U(k)$ if and only if

$$\ell_\nu+\ell_\mu\leq k.$$

All others irreps are either null or reducible to a signed standard irrep. A non-standard irrep of $U(k)$ may be modified by applying the modification rule

$$\{\bar{\nu};\mu\}=(-1)^{c+d-1}\{\overline{\nu-h};\mu-h\},\ h=\ell_\nu+\ell_\mu-k-1\geq 0$$

In applying the rule one removes from the partition (ν) a continuous strip of h boxes starting at the top of the first column and ending in the c-th column and similarly for the partition (μ) ending with the d-th column. This operation is continued until either ordered partitions or a null result is obtained. This is done automatically in SCHUR using the command "std".

Thus for example we have the signed sequences for

$$\{\bar{3};1\}_s^2=(\bar{3};1)-(\overline{31};1^2)+(\overline{32};1^3)+(\overline{31^3};2^2)\ \cdots$$

and

$$\{\bar{1};2\}_s^2=(\bar{1};2)-(\overline{1^2};21)+(\overline{1^3};2^2)+(\overline{2^2};21^3)\ \cdots$$

This signed sequences can be verified in SCHUR as in the following fragment

```
DPrep Mode (with function)
DP>
rep
REP mode
REP>
gr u2
Group is U(2)
```

```
REP>
std31;11
       - {3;1}

REP>
std32;111
      {3;1}

REP>
std11;21
       - {1;2}

REP>
std111;22
      {1;2}

REP>
```

6.6.2 The product of the signed sequences

The next step is to compute the product of the two signed sequences for the group $U(p) \times U(q)$ while at the same time satisfying the constraints imposed by (6.12). In our case this restricts each of the signed sequences to just the first two terms. Thus for $U(4,4)$ we have with SCHUR

```
DP>
gr2u4u4
Groups are   U(4) * U(4)
DP>
p[3*1]-[31*11],[1*2]-[11*21]
      {42}{32} + {42}{31^2 } + {42}{2^2 1} + {42}{21^3 }
       + {41^2 }{32} + {41^2 }{31^2 } + {41^2 }{2^2 1}
       + {41^2 }{21^3 } - 2{41}{31} - {41}{2^2 }
       - 2{41}{21^2 } + {4}{3} + {4}{21} + {321}{32}
       + {321}{31^2 } + {321}{2^2 1} + {321}{21^3 }
       - {32}{31} - {32}{21^2 } + {31^3 }{32}
       + {31^3 }{31^2 } + {31^3 }{2^2 1} + {31^3 }{21^3 }
       - 2{31^2 }{31} - {31^2 }{2^2 } - 2{31^2 }{21^2 }
       + {31}{3} + {31}{21}
DP>
```

The constraint (6.12) eliminate all $U(4)$ irreps involving partitions with three non-zero parts. That is done in SCHUR by

```
DP>
len1,2last
      {42}{32} + {42}{31^2 } + {42}{2^2 1} + {42}{21^3 }
       - 2{41}{31} - {41}{2^2 } - 2{41}{21^2 } + {4}{3}
       + {4}{21} - {32}{31} - {32}{21^2 } + {31}{3}
       + {31}{21}
DP>
len2,2last
      {42}{32} - 2{41}{31} - {41}{2^2 } + {4}{3}
       + {4}{21} - {32}{31} + {31}{3} + {31}{21}
DP>
setp1last
DP>
```

They are all the terms required for the product of the two signed sequences. Note that one has saved the result in v1 for later use.

6.6.3 The sum $\sum_\zeta\{\bar{\zeta}\}\times\{\zeta\}$

The next step is to prepare the terms for the sum$\sum_\zeta\{\bar{\zeta}\}\times\{\zeta\}$. One notices that equation (6.12) restricts the partitions (ζ) to at most two non-zero parts. The series is infinite and a suitable cutoff must be chosen We shall restrict ourselves to terms of weight lower or equal to 6. The relevant (ζ) may be determined by the following command

```
SFN>
  len2wt6ser6,f       {6} + {51} + {5} + {42} + {41} + {4} + {3^2 } + {32}
       + {31} + {3} + {2^2 } + {21} + {2} + {1^2 } + {1}
       + {0}

SFN>
```

We now return to the "DPmode" of SCHUR and prepare a second variable v2 as the appropriate list

```
DP>
setp2[0*0]+[1*1]+[2*2]+[11*11]+[3*3]+[21*21]+[4*4]+[31*31]+[22*22]+[5*5]
DP>
setp2add v2,[41*41]+[32*32]+[6*6]+[51*51]+[42*42]+[33*33]
DP>
v2
      {6}{6} + {51}{51} + {5}{5} + {42}{42} + {41}{41}
       + {4}{4} + {3^2 }{3^2 } + {32}{32} + {31}{31}
       + {3}{3} + {2^2 }{2^2 } + {21}{21} + {2}{2}
```

```
       + {1^2 }{1^2 } + {1}{1} + {0}{0}
DP>
```

6.6.4 The product obtained

We now form the product of v1 with v2 in $U(4) \times U(4)$, keeping terms of weight up to 6 via the following SCHUR commands

```
DP>
set_pwt6
DP>
wt1,6p v1,v2
      {6}{5} + 2{6}{41} + 2{6}{32} + {6}{31^2 }
       + {6}{2^2 1} + 2{51}{5} + 3{51}{41} + 2{51}{32}
       + 2{51}{31^2 } + 2{51}{2^2 1} + {51}{21^3 } + {5}{4}
       + 2{5}{31} + {5}{2^2 } + {5}{21^2 } + 2{42}{5}
       + 2{42}{41} + 2{42}{32} + {42}{31^2 } + 2{42}{2^2 1}
       + {42}{21^3 } + {41^2 }{5} + 2{41^2 }{41}
       + {41^2 }{32} + 3{41^2 }{31^2 } + 2{41^2 }{2^2 1}
       + 2{41^2 }{21^3 } + 2{41}{4} + 2{41}{31}
       + {41}{2^2 } + 2{41}{21^2 } + {4}{3} + {4}{21}
       + {3^2 }{5} + {3^2 }{41} + {3^2 }{32}
       + {3^2 }{2^2 1} + {321}{5} + 2{321}{41} + 2{321}{32}
       + 2{321}{31^2 } + 2{321}{2^2 1} + {321}{21^3 }
       + {32}{4} + {32}{31} + {32}{2^2 } + {32}{21^2 }
       + {31^3 }{41} + {31^3 }{32} + 2{31^3 }{31^2 }
       + {31^3 }{2^2 1} + {31^3 }{21^3 } + {31^2 }{4}
       + 2{31^2 }{31} + {31^2 }{2^2 } + {31^2 }{21^2 }
       + {31}{3} + {31}{21}
DP>
setp3last
```

where we have saved the result as v3. Again we must consider (6.12) which allows us to eliminate all cases where both partitions have three non-zero parts, those where one has three non-zero parts and the other two non-zero parts and those where one partition has four non-zero parts with the other also having non-zero parts. These latter are eliminated in this SCHUR fragment

```
DP>
sub v3,len1,-4v3
      {6}{5} + 2{6}{41} + 2{6}{32} + {6}{31^2 }
       + {6}{2^2 1} + 2{51}{5} + 3{51}{41} + 2{51}{32}
       + 2{51}{31^2 } + 2{51}{2^2 1} + {51}{21^3 } + {5}{4}
       + 2{5}{31} + {5}{2^2 } + {5}{21^2 } + 2{42}{5}
```

```
       + 2{42}{41} + 2{42}{32} + {42}{31^2 } + 2{42}{2^2 1}
       + {42}{21^3 } + {41^2 }{5} + 2{41^2 }{41}
       + {41^2 }{32} + 3{41^2 }{31^2 } + 2{41^2 }{2^2 1}
       + 2{41^2 }{21^3 } + 2{41}{4} + 2{41}{31}
       + {41}{2^2 } + 2{41}{21^2 } + {4}{3} + {4}{21}
       + {3^2 }{5} + {3^2 }{41} + {3^2 }{32}
       + {3^2 }{2^2 1} + {321}{5} + 2{321}{41} + 2{321}{32}
       + 2{321}{31^2 } + 2{321}{2^2 1} + {321}{21^3 }
       + {32}{4} + {32}{31} + {32}{2^2 } + {32}{21^2 }
       + {31^2 }{4} + 2{31^2 }{31} + {31^2 }{2^2 }
       + {31^2 }{21^2 } + {31}{3} + {31}{21}
DP>
sub last,len2,-4last
      {6}{5} + 2{6}{41} + 2{6}{32} + {6}{31^2 }
       + {6}{2^2 1} + 2{51}{5} + 3{51}{41} + 2{51}{32}
       + 2{51}{31^2 } + 2{51}{2^2 1} + {5}{4} + 2{5}{31}
       + {5}{2^2 } + {5}{21^2 } + 2{42}{5} + 2{42}{41}
       + 2{42}{32} + {42}{31^2 } + 2{42}{2^2 1}
       + {41^2 }{5} + 2{41^2 }{41} + {41^2 }{32}
       + 3{41^2 }{31^2 } + 2{41^2 }{2^2 1} + 2{41}{4}
       + 2{41}{31} + {41}{2^2 } + 2{41}{21^2 } + {4}{3}
       + {4}{21} + {3^2 }{5} + {3^2 }{41} + {3^2 }{32}
       + {3^2 }{2^2 1} + {321}{5} + 2{321}{41} + 2{321}{32}
       + 2{321}{31^2 } + 2{321}{2^2 1} + {32}{4} + {32}{31}
       + {32}{2^2 } + {32}{21^2 } + {31^2 }{4}
       + 2{31^2 }{31} + {31^2 }{2^2 } + {31^2 }{21^2 }
       + {31}{3} + {31}{21}
DP>
setp3last
DP>
```

The next step is to keep the cases where one partition has three non-zero parts and the other one non-zero part. This may be done by first putting all these cases into a single variable v4 as indicated in the SCHUR sequence

```
DP>
len1,1v3
      {6}{5} + 2{6}{41} + 2{6}{32} + {6}{31^2 }
       + {6}{2^2 1} + {5}{4} + 2{5}{31} + {5}{2^2 }
       + {5}{21^2 } + {4}{3} + {4}{21}
DP>
add last,len2,1v3
      2{6}{5} + 2{6}{41} + 2{6}{32} + {6}{31^2 }
       + {6}{2^2 1} + 2{51}{5} + 2{5}{4} + 2{5}{31}
```

```
       + {5}{2^2 } + {5}{21^2 } + 2{42}{5} + {41^2 }{5}
       + 2{41}{4} + 2{4}{3} + {4}{21} + {3^2 }{5}
       + {321}{5} + {32}{4} + {31^2 }{4} + {31}{3}
DP>
add len1,-3last,len2,-3last
      {6}{31^2 } + {6}{2^2 1} + {5}{21^2 } + {41^2 }{5}
       + {321}{5} + {31^2 }{4}
DP>
setp4last
DP>
```

Now we remove from v3 all cases involving at least one partition into three non-zero parts and then adding back the cases saved as shown below

```
DP>
len1,2v3
      {6}{5} + 2{6}{41} + 2{6}{32} + {6}{31^2 }
       + {6}{2^2 1} + 2{51}{5} + 3{51}{41} + 2{51}{32}
       + 2{51}{31^2 } + 2{51}{2^2 1} + {5}{4} + 2{5}{31}
       + {5}{2^2 } + {5}{21^2 } + 2{42}{5} + 2{42}{41}
       + 2{42}{32} + {42}{31^2 } + 2{42}{2^2 1} + 2{41}{4}
       + 2{41}{31} + {41}{2^2 } + 2{41}{21^2 } + {4}{3}
       + {4}{21} + {3^2 }{5} + {3^2 }{41} + {3^2 }{32}
       + {3^2 }{2^2 1} + {32}{4} + {32}{31} + {32}{2^2 }
       + {32}{21^2 } + {31}{3} + {31}{21}
DP>
len2,2last
      {6}{5} + 2{6}{41} + 2{6}{32} + 2{51}{5}
       + 3{51}{41} + 2{51}{32} + {5}{4} + 2{5}{31}
       + {5}{2^2 } + 2{42}{5} + 2{42}{41} + 2{42}{32}
       + 2{41}{4} + 2{41}{31} + {41}{2^2 } + {4}{3}
       + {4}{21} + {3^2 }{5} + {3^2 }{41} + {3^2 }{32}
       + {32}{4} + {32}{31} + {32}{2^2 } + {31}{3}
       + {31}{21}
DP>
add last,v4
      {6}{5} + 2{6}{41} + 2{6}{32} + {6}{31^2 }
       + {6}{2^2 1} + 2{51}{5} + 3{51}{41} + 2{51}{32}
       + {5}{4} + 2{5}{31} + {5}{2^2 } + {5}{21^2 }
       + 2{42}{5} + 2{42}{41} + 2{42}{32} + {41^2 }{5}
       + 2{41}{4} + 2{41}{31} + {41}{2^2 } + {4}{3}
       + {4}{21} + {3^2 }{5} + {3^2 }{41} + {3^2 }{32}
       + {321}{5} + {32}{4} + {32}{31} + {32}{2^2 }
```

```
       + {31^2 }{4} + {31}{3} + {31}{21}
DP>
setp4last
DP>
```

which is the result saved as v4. In the case of $U(2,2)$, we simply eliminate all terms with three non-zero parts and store the result as v5.

```
DP>
sb_rev true
DP>
columns 5
DP>
len1,2last
      {31}{21} + {31}{3} + {32}{2^2 } + {32}{31}
       + {32}{4} + {3^2 }{32} + {3^2 }{41} + {3^2 }{5}
       + {4}{21} + {4}{3} + {41}{2^2 } + 2{41}{31}
       + 2{41}{4} + 2{42}{32} + 2{42}{41} + 2{42}{5}
       + {5}{21^2 } + {5}{2^2 } + 2{5}{31} + {5}{4}
       + 2{51}{32} + 3{51}{41} + 2{51}{5} + {6}{2^2 1}
       + {6}{31^2 } + 2{6}{32} + 2{6}{41} + {6}{5}
DP>
len2,2last
      {31}{21} + {31}{3} + {32}{2^2 } + {32}{31}
       + {32}{4} + {3^2 }{32} + {3^2 }{41} + {3^2 }{5}
       + {4}{21} + {4}{3} + {41}{2^2 } + 2{41}{31}
       + 2{41}{4} + 2{42}{32} + 2{42}{41} + 2{42}{5}
       + {5}{2^2 } + 2{5}{31} + {5}{4} + 2{51}{32}
       + 3{51}{41} + 2{51}{5} + 2{6}{32} + 2{6}{41}
       + {6}{5}
DP>
setp5last
DP>
```

Now if we transform the results using the TEXoutput from SCHUR one obtains $\{2(\bar{3};1)\}\times\{2(\bar{1};2)\}$, in weight 6 for $U(4,4)$

$$
\begin{array}{lllll}
\{4(\overline{31};21)\} & +\ \{4(\overline{31};3)\} & +\ \{4(\overline{31^2};4)\} & +\ \{4(\overline{32};2^2)\} & +\ \{4(\overline{32};31)\} \\
+\ \{4(\overline{32};4)\} & +\ \{4(\overline{321};5)\} & +\ \{4(\overline{3^2};32)\} & +\ \{4(\overline{3^2};41)\} & +\ \{4(\overline{3^2};5)\} \\
+\ \{4(\overline{4};21)\} & +\ \{4(\overline{4};3)\} & +\ \{4(\overline{41};2^2)\} & +\ 2\{4(\overline{41};31)\} & +\ 2\{4(\overline{41};4)\} \\
+\ \{4(\overline{41^2};5)\} & +\ 2\{4(\overline{42};32)\} & +\ 2\{4(\overline{42};41)\} & +\ 2\{4(\overline{42};5)\} & +\ \{4(\overline{5};21^2)\} \\
+\ \{4(\overline{5};2^2)\} & +\ 2\{4(\overline{5};31)\} & +\ \{4(\overline{5};4)\} & +\ 2\{4(\overline{51};32)\} & +\ 3\{4(\overline{51};41)\} \\
+\ 2\{4(\overline{51};5)\} & +\ \{4(\overline{6};2^21)\} & +\ \{4(\overline{6};31^2)\} & +\ 2\{4(\overline{6};32)\} & +\ 2\{4(\overline{6};41)\} \\
+\ \{4(\overline{6};5)\} & & & &
\end{array}
$$

and for $U(2,2)$

$$\begin{array}{lllll}
\{4(\overline{31};21)\} & +\{4(\overline{31};3)\} & +\{4(\overline{32};2^2)\} & +\{4(\overline{32};31)\} & +\{4(\overline{32};4)\} \\
+\{4(\overline{3^2};32)\} & +\{4(\overline{3^2};41)\} & +\{4(\overline{3^2};5)\} & +\{4(\overline{4};21)\} & +\{4(\overline{4};3)\} \\
+\{4(\overline{41};2^2)\} & +2\{4(\overline{41};31)\} & +2\{4(\overline{41};4)\} & +2\{4(\overline{42};32)\} & +2\{4(\overline{42};41)\} \\
+2\{4(\overline{42};5)\} & +\{4(\overline{5};2^2)\} & +2\{4(\overline{5};31)\} & +\{4(\overline{5};4)\} & +2\{4(\overline{51};32)\} \\
+3\{4(\overline{51};41)\} & +2\{4(\overline{51};5)\} & +2\{4(\overline{6};32)\} & +2\{4(\overline{6};41)\} & +\{4(\overline{6};5)\}
\end{array}$$

6.6.5 Writing a function with SCHUR

The previous example illustrate the general problems involved in the practical determination of Kronecker product for a air of harmonic series irreps of $U(p,q)$. Much of the preceding work can be avoided by defining a function that can be executed in SCHUR and run for any pair of irreps. Such a function can be written as

```
gr4u4u4u4u4
sb_rev true
setlimit6
set_pwt6
enter v1
enter v2
setp3[0*0*0*0]+[0*0*1*1]+[0*0*2*2]+[0*0*11*11]+[0*0*3*3]+[0*0*21*21]
setp3 add v3,[0*0*4*4]+[0*0*31*31]+[0*0*22*22]+[0*0*5*5]+[0*0*41*41]
setp3 add v3,[0*0*32*32]+[0*0*6*6]+[0*0*51*51]+[0*0*42*42]+[0*0*33*33]
wt1,6p v1,v2
len1,2last
len2,2last
wt1,6p last,v3
wt1,6contract1,3last
wt1,6contract2,3last
len1,3 last
len2,3 last
setp1 last
len1,1 last
add len2,1 v1,last
setp2 last
len1,-3 last
add len2,-3v2 last
setp3 last
len1,2 v1
len2,2 last
add last, v3
setp3last
columns5
```

```
last
stop
```

This function is run by the following command

```
readfn1'upq.fn'
```

and the execution gives

```
DP>
fn1
Groups are   U(4) * U(4) * U(4) * U(4)

enter v1
[3*1*0*0]-[31*11*0*0]

enter v2
[1*2*0*0]-[11*21*0*0]
Groups are   U(4) * U(4) * U(4)
Groups are   U(4) * U(4)
      {31}{21} + {31}{3} + {31^2 }{4} + {32}{2^2 }
       + {32}{31} + {32}{4} + {321}{5} + {3^2 }{32}
       + {3^2 }{41} + {3^2 }{5} + {4}{21} + {4}{3}
       + {41}{2^2 } + 2{41}{31} + 2{41}{4} + {41^2 }{5}
       + 2{42}{32} + 2{42}{41} + 2{42}{5} + {5}{21^2 }
       + {5}{2^2 } + 2{5}{31} + {5}{4} + 2{51}{32}
       + 3{51}{41} + 2{51}{5} + {6}{2^2 1} + {6}{31^2 }
       + 2{6}{32} + 2{6}{41} + {6}{5}
DP>
```

to produce the same final results. One may also change the groups set to obtain results for various $U(p,q)$.

Chapter 7

Representations of $SO^*(2n)$

7.1 Introduction

This chapter is the subject of a joint work realised with R. C. King and B. G. Wybourne and published in *Journal of Physics A: Mathematical and General* [56].

7.2 $SO^*(2n) \to U(n)$ branching rule

It should first be noted that the non-compact group $SO^*(2n)$ and the compact group $Sp(2k)$ form a dual pair in $Mp(4nk)$ in the sense that each is a maximal centraliser of the other in the metaplectic group $Mp(4nk)$. As a direct consequence of this, the metaplectic representation of $Sp(4nk,\mathbb{R})$ decomposes under restriction to $SO^*(2n) \times Sp(2k)$ in accordance with the rule:

$$\tilde{\Delta} \to \sum_{\lambda} [k(\lambda)] \times \langle \lambda \rangle, \tag{7.1}$$

where the summation is over all λ such that

$$\lambda_1' \leq k \quad \text{and} \quad \lambda_1' \leq n. \tag{7.2}$$

The procedure necessary to determine the $U(n)$ content of each harmonic series irreducible representation$[k(\lambda)]$ of $SO^*(2n)$ is closely related to that used for $Sp(2n,\mathbb{R})$ in Section 5 of [60]. Consider the two group-subgroup chains:

$$Sp(4nk,\mathbb{R}) \to SO^*(2n) \times Sp(2k) \to U(n) \times Sp(2k) \tag{7.3}$$

$$Sp(4nk,\mathbb{R}) \to U(2nk) \to U(n) \times U(2k) \to U(n) \times Sp(2k). \tag{7.4}$$

$$\tag{7.5}$$

The first of these gives

$$\tilde{\Delta} \to \sum_{\lambda} [k(\lambda)] \times \langle \lambda \rangle \to \sum_{\lambda,\mu} \varepsilon^k B_\lambda^\mu \{\mu\} \times \langle \lambda \rangle, \tag{7.6}$$

while the second leads to

$$\tilde{\Delta} \to \sum_m \varepsilon^{1/2}\{m\} \to \sum_\mu \varepsilon^k\{\mu\} \times \varepsilon^{n/2}\{\mu\} \to \sum_{\mu,\lambda} \varepsilon^k R^\mu_\lambda\{\mu\} \times \langle\lambda\rangle, \tag{7.7}$$

or, equivalently,

$$\begin{aligned}\tilde{\Delta} \to \textstyle\sum_m \varepsilon^{1/2}\{m\} \to \sum_\mu \varepsilon^k\{\mu\} \times \varepsilon^{n/2}\{\mu\} &\to \textstyle\sum_\mu \varepsilon^k\{\mu\} \times \langle\mu/B\rangle \\ &= \textstyle\sum_\nu \varepsilon^k\{\nu \cdot B\} \times \langle\nu\rangle\end{aligned} \tag{7.8}$$

where [49, 55, 8]

$$B = \sum_\beta \{\beta\} = \{0\} + \{1^2\} + \{2^2\} + \{1^4\} + \{3^2\} + \{2^2 1^2\} + \{1^6\} + \cdots, \tag{7.9}$$

with the summation taken over all partitions β such that each distinct part is repeated an even number of times. In (7.6) the coefficients B^μ_λ are the required branching rule coefficients for $SO^*(2n) \to U(n)$, while in (7.7) the coefficients R^μ_λ are the known branching rule coefficients for $U(2k) \to Sp(2k)$. These are defined implicitly by (7.8). We thus arrive at the following:

Proposition 7.2.1 *Let λ be such that $\lambda'_1 \le \min(k,n)$. Then on restriction from $SO^*(2n)$ to $U(n)$ the irreducible representation $[k(\lambda)]$ of $SO^*(2n)$ decomposes in accordance with the branching rule:*

$$[k(\lambda)] \to \sum_\mu \varepsilon^k R^\mu_\lambda\{\mu\} = \varepsilon^k \cdot \{\lambda_s\}^{\langle 2k\rangle} \cdot B, \tag{7.10}$$

where $\{\lambda_s\}^{\langle 2k\rangle}$ is the signed sequence([60, 35, 61, 95])

$$\{\lambda_s\}^{\langle 2k\rangle} = \sum_\nu \xi^\lambda_\nu\{\nu\} \tag{7.11}$$

with the summation extending over all ν with $\nu'_1 \le 2k$ such $\langle\nu\rangle = \xi^\lambda_\mu\langle\lambda\rangle$ under the modification rules of $Sp(2k)$.

The superscript $\langle 2k\rangle$ has been used as a notational device to emphasise that the signed sequences are constructed from a knowledge of the modification rules of $Sp(2k)$. These rules [49, 55, 8] are such that the non-vanishing coefficients ξ^λ_ν are all ± 1.

It follows from further consideration of the limitations imposed by branching via $U(n) \times U(2k)$ in (7.7) and (7.8) that (7.10) can be re-written in the computational simpler form:

$$[k(\lambda)] \to \varepsilon^k \cdot \{\{\lambda_s\}^{\langle 2k\rangle}_N \cdot B_N\}_N. \tag{7.12}$$

where $N = min(2k,n)$. The first $\cdot$ indicates a product in $U(n)$ and the second $\cdot$ a product in $U(N)$ as implied by the various subscripts N which limit all terms to those labelled by partitions into no more than N parts.Irreducible representations$[k(\lambda)]$ of $SO^*(2n)$ satisfying (7.2) will be said to be *standard*. The signed sequence $\{\lambda_s\}^{\langle 2k\rangle}_N$ associated with modifications in $Sp(2k)$ is rendered finite by the constraint implied by the subscript N. Thus for $k=2$ and $n=4$ we have $N=4$ and, for example,

$$\{31\}_4^{\langle 4\rangle} = \{31\} - \{31^3\} \tag{7.13}$$

whereas for $k=2$ and $n=3$ we have $N=3$ and only the first term survives:

$$\{31\}_3^{\langle 4\rangle} = \{31\} \tag{7.14}$$

In general the modification rules [49, 55] for $Sp(2k)$ are such that for each standard irreducible representation $[k(\lambda)]$ of $SO^*(2n)$ the corresponding signed sequence takes the form:

$$\{\lambda_s\}^{\langle 2k\rangle} = \{\lambda\} - \{\nu\} \pm \{\rho\} + \cdots \quad \text{with} \quad \lambda'_1 < \nu'_1 \le \rho'_1 \le \cdots . \tag{7.15}$$

where

$$\{\nu'\} = \{2k+2-\lambda'_1, \lambda'_2, \lambda'_3, \ldots\}. \tag{7.16}$$

Standard irreducible representations$[k(\lambda)]$ associated with signed sequences $\{\lambda_s\}_N^{2k}$ involving just one term are said to be *highly standard*. From (7.15) it can be seen that this will be the case whenever

$$2k+2-\lambda'_1 > N = \min(n, 2k). \tag{7.17}$$

In particular this condition is automatically satisfied if $\lambda'_1 \le 1$. Hence all the irreducible representations$[k(m)]$ are highly standard, including $H_m = [1(m)]$ for all m.

More generally, for all highly standard irreducible representationsof $SO^*(2n)$ (7.12) simplifies to just

$$[k(\lambda)] \to \varepsilon^k \cdot \{\{\lambda\} \cdot B_N\}_N \tag{7.18}$$

For example, the highly standard irreducible representation[2(31)] of $SO^*(6)$ branches under $SO^*(6) \to U(3)$ as

$$\begin{aligned}
[2(31)] \to \varepsilon^2 \cdot \{\{31\} \cdot B_3\}_3 & \\
&= \varepsilon^2 \cdot \{\{31\} \cdot (\{0\} + \{1^2\} + \{2^2\} + \{3^2\} + \{4^2\} + \cdots)\}_3 \\
&= \varepsilon^2 \cdot (\{31\} + \{321\} + \{3^22\} + \{41^2\} + \{42\} + \{42^2\} \\
&\quad + \{431\} + \{521\} + \{53\} + \cdots) \\
&= \{532\} + \{543\} + \{5^24\} + \{63^2\} + \{642\} + \{64^2\} \\
&\quad + \{653\} + \{743\} + \{752\} + \cdots .
\end{aligned} \tag{7.19}$$

In the case of the standard, but not highly-standard irreducible representation[2(31)] of $SO^*(8)$ we have from (7.12) and (7.13), for $SO^*(8) \to U(4)$

$$\begin{aligned}
[2(31)] &\to \varepsilon^2 \cdot \{(\{31\} - \{31^3\}) \cdot B_4\}_4 \\
&= \varepsilon^2 \cdot \{(\{31\} - \{31^3\}) \cdot (\{0\} + \{1^2\} + \{1^4\} + \{2^2\} + \{2^21^2\} \\
&\quad + \{2^4\} + \{3^2\} + \{3^21^2\} + \{4^2\} + \cdots)\}_4 \\
&= \varepsilon^2 \cdot (\{31\} + \{321\} + \{3^22\} + \{41^2\} + \{42\} + \{421^2\} \\
&\quad + \{42^2\} + \{431\} + \{521\} + \{53\} + \cdots) \\
&= \{532^2\} + \{5432\} + \{5^242\} + \{63^22\} + \{642^2\} + \{643^2\} \\
&\quad + \{64^22\} + \{6532\} + \{7432\} + \{752^2\} + \cdots .
\end{aligned} \tag{7.20}$$

In the particular case of the fundamental irreducible representationsH_m of $SO^*(2n)$ we have

$$H_m = [1(m)] \to \varepsilon \cdot \{\{m\} \cdot B_2\}_2 = \sum_{r=0}^{\infty} \varepsilon \cdot \{m+r, r\} = \sum_{r=1}^{\infty} \{m+r, r\}, \tag{7.21}$$

since $B_2 = \sum_{r=0}^{\infty} \{r^2\}$ and the relevant products are taken in $U(2)$ with $\varepsilon = \{1^2\}$. It follows that under $SO^*(2n) \to U(n)$ the basic harmonic representation decomposes as

$$H = \sum_{m=0}^{\infty} [1(m)] \to \sum_{m,r=0}^{\infty} \varepsilon \cdot \{m+r, r\}. \tag{7.22}$$

Equivalently, but more formally, we have

$$H = \sum_{m=0}^{\infty} [1(m)] \to \sum_{m=0}^{\infty} \varepsilon \cdot \{\{m\} \cdot B_2\}_2 = \varepsilon \cdot \{MB_2\}_2 = \varepsilon \cdot F_2, \tag{7.23}$$

where quite generally [49, 55, 8]

$$M = \sum_{m} \{m\} \quad \text{and} \quad MB = F = \sum_{\zeta} \{\zeta\}, \tag{7.24}$$

with the latter sum being over taken over all partitions ζ, although in (7.23) the subscript on F_2 indicates that the series is to be restricted to partitions involving at most two parts.

7.3 Tensor products for harmonic unirreps of $SO^*(2n)$

The case of tensor products for the holomorphic discrete series of $SO^*(2n)$ was considered in KW1 who gave the result as ([60] (7.12))

$$[\{\mu\}] \times [\{\nu\}] = [\{\mu \cdot \nu \cdot B\}] \tag{7.25}$$

The corresponding results for the harmonic irreducible representations follow in a very similar fashion to [60] (8.10 - 8.15) by consideration of the two group-subgroup chains

$$\begin{aligned} Sp(4nk+4n\ell,\mathbb{R}) &\to Sp(4nk,\mathbb{R}) \times Sp(4n\ell,\mathbb{R}) \\ &\to SO^*(2n) \times Sp(2k) \times SO^*(2n) \times Sp(2\ell) \\ &\to SO^*(2n) \times Sp(2k) \times Sp(2\ell) \end{aligned} \tag{7.26}$$

and

$$\begin{aligned} Sp(4nk+4n\ell,\mathbb{R}) &\to SO^*(2n) \times Sp(2k+2\ell) \\ &\to SO^*(2n) \times Sp(2k) \times Sp(2\ell). \end{aligned} \tag{7.27}$$

Under (7.26) we have

$$\begin{aligned} \tilde{\Delta} \to \tilde{\Delta} \times \tilde{\Delta} &\to \textstyle\sum_{\mu,\nu} [k(\mu)] \times \langle\mu\rangle \times [\ell(\nu)] \times \langle\nu\rangle \\ &\to \textstyle\sum_{\mu,\nu,\lambda} K_{\lambda}^{\mu\nu} [k+\ell(\lambda)] \times \langle\mu\rangle \times \langle\nu\rangle, \end{aligned} \tag{7.28}$$

where $K_\lambda^{\mu\nu}$ are the required tensor product coefficients for $SO^*(2n)$. Alternatively, under (7.27) we have

$$\begin{aligned}\tilde{\Delta} \ &\to \textstyle\sum_\lambda \ [k+\ell(\lambda)] \times \langle\lambda\rangle \\ &\to \textstyle\sum_\lambda \ [k+\ell(\lambda)] \times \sum_{\mu,\nu} \ R_\lambda^{\mu\nu}\langle\mu\rangle \times \langle\nu\rangle,\end{aligned} \tag{7.29}$$

where the coefficients $R_\lambda^{\mu\nu}$ are the branching rule coefficients for the restriction $Sp(2k+2\ell) \to Sp(2k) \times Sp(2\ell)$. Comparison of (3.4) and (3.5) shows that $K_\lambda^{\mu\nu} = R_\lambda^{\mu\nu}$, thereby yielding:

Proposition 7.3.1 *The tensor product of a pair of unitary harmonic irreducible representations $[k(\mu)]$ and $[\ell(\nu)]$ of $S)^*(2n)$ decomposes in accordance with the rule*

$$[k(\mu)] \times [\ell(\nu)] = \sum_\lambda \ R_\lambda^{\mu\nu} \ [k+\ell(\lambda)]. \tag{7.30}$$

To implement (3.6) it is convenient to note that under the restriction

$$Sp(2k+2\ell) \to Sp(2k) \times Sp(2\ell)$$

we have

$$\langle\lambda\rangle \to \sum_{\mu,\nu} \ R_\lambda^{\mu\nu} \ \langle\mu\rangle \times \langle\nu\rangle = \sum_\kappa \ \langle\lambda/\kappa\rangle \times \langle\kappa/B\rangle. \tag{7.31}$$

This may be derived through the use of standard S-function method [49].

An alternative formula may be derived from a consideration of the group-subgroup chains:

$$SO^*(2n) \times SO^*(2n) \ \to \ U(n) \times U(n) \to U(n) \tag{7.32}$$
$$SO^*(2n) \times SO^*(2n) \ \to \ SO^*(2n) \to U(n). \tag{7.33}$$

Using (7.10) the first of these gives

$$\begin{aligned}[k(\mu)] \times [\ell(\nu)] \ &\to \left(\varepsilon^k \cdot \{\mu_s\}^{\langle 2k\rangle} \cdot B\right) \times \left(\varepsilon^\ell \cdot \{\nu_s\}^{\langle 2\ell\rangle} \cdot B\right) \\ &\to \varepsilon^{k+\ell} \cdot \left(\{\mu_s\}^{\langle 2k\rangle} \cdot \{\nu_s\}^{\langle 2\ell\rangle} \cdot B\right) \cdot B,\end{aligned} \tag{7.34}$$

while from (7.30) and the use once more of (7.10) we obtain

$$\begin{aligned}[k(\mu] \times [\ell(\nu)] \ &= \textstyle\sum_\lambda \ R_\lambda^{\mu\nu}[k+\ell(\lambda)] \\ &\to \textstyle\sum_\lambda \ R_\lambda^{\mu\nu}\varepsilon^{k+\ell} \cdot \{\lambda_s\}^{\langle 2(k+\ell)\rangle} \cdot B,\end{aligned} \tag{7.35}$$

with $\lambda_1' \leq N = \min(k+\ell, n)$. Comparison of (7.34) and (7.35) then yields:

Proposition 7.3.2 *The tensor product of a pair of unitary harmonic irreducible representationsof $SO^*(2n)$ decomposes in accordance with the rule*

$$[k(\mu)] \times [\ell(\nu)] = [k + \ell((\{\mu_s\}^{\langle 2k\rangle} \cdot \{\nu_s\}^{\langle 2\ell\rangle} \cdot B)_N)]. \tag{7.36}$$

where $N = \min(k+\ell, n)$ *and*

$$(\lambda)_N = \begin{cases} (\lambda) & \text{if } \lambda_1' \leq N; \\ 0 & \text{otherwise} \end{cases} \tag{7.37}$$

Either (7.30) or (7.36) may be used to evaluate tensor products. Equation (7.30) has advantages when a single coefficient $R_\lambda^{\mu\nu}$ is required. In that case signed sequences are not needed. However, equation (7.36) is particularly useful in evaluating complete products.

By way of example, consider the evaluation of the terms, to weight eight, in the tensor product $[2(21)] \times [3(1^2)]$ for the group $SO^*(8)$. Since $N = 4$ all products appearing in (7.36) may be evaluated within $U(4)$ and the signed sequences restricted to the terms

$$\{21_s\}^{\langle 4\rangle} = \{21\} - \{21^3\}, \tag{7.38}$$

$$\{1_s^2\}^{\langle 6\rangle} = \{1^2\}. \tag{7.39}$$

$$\tag{7.40}$$

Their product in $U(4)$ yields the terms

$$\{21^3\} + \{2^2 1\} - \{2^3 1\} + \{31^2\} + \{32\} - \{321^2\}. \tag{7.41}$$

Since we are evaluating terms to weight eight only terms in the B-series to weight three are relevant, that is the two terms

$$\{0\} + \{1^2\} \tag{7.42}$$

and forming the product we obtain the terms, to weight eight, as

$$\begin{array}{l} \{21^3\} + \{2^2 1\} + \{2^3 1\} + \{31^2\} + \{32\} + 3\{321^2\} \\ +2\{32^2\} + 2\{3^2 1\} + \{41^3\} + 2\{421\} + \{43\}. \end{array} \tag{7.43}$$

Changing the notation to that for $SO^*(8)$ and inserting the integer 5 in front of each partition, we finally obtain the result

$$\begin{array}{rl} [2(21)] \times [3(1^2)] = & [5(21^3)] + [5(2^2 1)] + [5(2^3 1)] + [5(31^2)] + [5(32)] \\ & +3[5(321^2)] + 2[5(32^2)] + 2[5(3^2 1)] + [5(41^3)] \\ & +2[5(421)] + [5(43)] + \cdots . \end{array} \tag{7.44}$$

7.4 The explicit decomposition of $H \times H$

Equation (7.36) is also useful in deriving explicit complete fomulae for tensor products. In particular the use of (7.36) immediately leads to the result

$$H_m \times H_{m'} = [1(m)] \times [1(m')] = \sum_{p=0}^{\infty} \sum_{x=0}^{min(m,m')} [2(m + m' + p - x, p + x)]. \tag{7.45}$$

This can be seen by noting that successive multiplication in $U(2)$ of a term $\{p^2\}$ in B by $\{m\}$ and then $\{m'\}$ can be carried out diagrammatically to give:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline a & a & a & c & c & c & c & c & c & d & d & d \\ \hline b & b & b & d & d \\ \cline{1-5}\end{array} \tag{7.46}$$

where there are precisely p columns containing the pair (a, b), m entries c in the first row and m' entries d, x of which are in the second row and the remainder in the first, with no identical entries d allowed in the same column.

Extending this analysis to the case of the square of the basic harmonic representation we have

Proposition 7.4.1 *For H as defined in (1.33)*

$$H^2 = H \times H = \sum_{r=0}^{\infty} \sum_{s=0}^{r} (r - s + 1)(s + 1)[2(r, s)]. \tag{7.47}$$

Proof We have from (1.33) and (7.45)

$$\begin{aligned} H^2 = H \times H &= \textstyle\sum_{m,m'=0}^{\infty} [1(m)] \times [1(m')] \\ &= \textstyle\sum_{p,m,m'=0}^{\infty} \sum_{x=0}^{min(m,m')} [2(m + m' + p - x, p + x)] \\ &= \textstyle\sum_{r=0}^{\infty} \sum_{s=0}^{r} C_{r,s} [2(r, s)] \end{aligned} \tag{7.48}$$

where $C_{r,s}$ is the number of diagrams of type (7.46) having rows of length r and s for any p, m, m' and x. For fixed r and s it is clear that the distribution of the letters is such that the number of d's in the second row of length s can vary from 0 to s, while the number in the first row of length r can, independently, vary from 0 to $r - s$. Thus $C_{r,s} = (s+1)(r-s+1)$, as required.

It is useful for later work to split H into its even and odd parts, H_+ and H_-, respectively, which are defined in (1.2). The coefficients $C_{rs}^{\eta\zeta}$ of the terms $[2(r, s)]$ in the various products $H_\eta \times H_\zeta$ are given by the following proposition

Proposition 7.4.2 *Let $C_{rs} = (r - s + 1)(s + 1)$. Then for $\eta, \zeta \in \{+, -\}$ we have*

$$H_\eta \times H_\zeta = \sum_{r=0}^{\infty} \sum_{s=0}^{r} C_{rs}^{\eta\zeta} [2(r, s)] \tag{7.49}$$

with

$$C_{rs}^{++} = \begin{cases} \frac{1}{2}(C_{rs} + 1) & \textit{if } r \textit{ and } s \textit{ are both even;} \\ \frac{1}{2}C_{rs} & \textit{if } r \textit{ and } s \textit{ are both odd;} \\ 0 & \textit{otherwise,} \end{cases} \tag{7.50}$$

$$C_{rs}^{--} = \begin{cases} \frac{1}{2}(C_{rs} - 1) & \textit{if } r \textit{ and } s \textit{ are both even;} \\ \frac{1}{2}C_{rs} & \textit{if } r \textit{ and } s \textit{ are both odd;} \\ 0 & \textit{otherwise,} \end{cases} \tag{7.51}$$

$$C_{rs}^{+-} = C_{rs}^{-+} = \begin{cases} \frac{1}{2}C_{rs} & \textit{if } r \textit{ is even and } s \textit{ is odd ;} \\ \frac{1}{2}C_{rs} & \textit{if } r \textit{ is odd and } s \textit{ is even ;} \\ 0 & \textit{otherwise.} \end{cases} \tag{7.52}$$

Proof First it should be noted that

$$H_\eta \times H_\zeta = [2(M_\eta \cdot M_\zeta \cdot B)_2], \tag{7.53}$$

where $M_+ = \sum_{m\text{even}}\{m\}$ and $M_- = \sum_{m\text{odd}}\{m\}$. Since all the term of B are of even weight it follows that all terms $[2(r,s)]$ of both H_+^2 and H_-^2 must be of even weight, so that r and s are either both even or both odd. Similarly, all the terms of $H_+ \times H_-$ must have r and s of opposite parity. This accounts for all the 0's appearing in (4.6a-c).

Separating $H^2 = (H_+ + H_-)^2 = (H_+^2 + H_-^2) + 2(H_+ \times H_-)$, as given by (7.47), into terms of even and odd weight, then immediately gives (7.52). Moreover to separate the terms of $H_+^2 + H_-^2$ into those of (7.50) and (7.51) it is merely necessary to show that $H_+^2 - H_-^2 = \sum_{r=0}^{\infty}\sum_{s=0}^{r}[2(r,s)]$ with the summation restricted to r and s both even. This may be established, by using (7.53) and various S-function series identities [35, 8] which imply that

$$H_+^2 - H_-^2 = [2((M_+^2 - M_-^2) \cdot B)_2] = [2(W \cdot B)_2] = [2(D_2)], \tag{7.54}$$

where the restriction of the S-function series D to two-part partitions gives $D_2 = \sum_r^{\infty}\sum_{s=0}^{r}\{r,s\}$ with r and s both even, as required.

7.5 Symmetrised powers of irreducible representations of $SO^*(2n)$

Following the techniques of Section 6 of [61] it is not difficult to derive the following general formula for symmetrised powers or plethysms of arbitrary irreducible representationsof $SO^*(2n)$:

Proposition 7.5.1 *Let the partition λ be such that $\lambda_1' \leq \min(k,n)$ and let ρ be an arbitrary partition of r, then*

$$[k(\lambda)] \otimes \{\rho\} = \sum_\mu y_{\lambda\rho}^\mu \; [kr(\mu)] \tag{7.55}$$

where the summation is over all partitions μ such that $\mu_1' \leq \min(kr,n)$ and the coefficients $y_{\lambda\rho}^\mu$ are determined by the expansion

$$\left(\left(\{\lambda_s\}^{\langle 2k\rangle} \cdot B\right) \otimes \{\rho\}\right) \cdot A = \sum_\mu y_{\lambda\rho}^\mu \; \{\mu_s\}^{\langle 2kr\rangle} \tag{7.56}$$

where $A = B^{-1}$.

Furthermore, just as for ease of calculation (7.10) can be replaced by (7.12), so (7.56) can be replaced by

$$\left(\left(\left(\{\lambda_s\}_N^{\langle 2k\rangle}\cdot B_N\right)_N\otimes\{\rho\}\right)_M\cdot A_M\right)_M=\sum_{\mu} y^{\mu}_{\lambda\rho}\ \{\mu_s\}_M^{\langle 2kr\rangle} \tag{7.57}$$

where $N=\min(2k,n)$ and $M=\min(2kr,n)$. Finally in order to read off the required plethysm coefficients in (7.55) from (7.57) it is only necessary to retain the leading term $\{\mu\}$ in each signed sequence $\{\mu_s\}_M^{\langle 2kr\rangle}$, since it is only the leading term of each signed sequence that satisfies the required $Sp(2kr)$-standardness condition $\mu_1'\leq kr$. This implies that in using (7.57) in (7.55) we may effectively replace $M=\min(2kr,n)$ by $\min(kr,n)$, a considerable simplification which leads

Corollaire 7.5.2 *With the notation of Proposition 5.1 the $SO^*(2n)$ plethysm coefficients $y^{\mu}_{\lambda\rho}$ are determined by*

$$\left(\left(\left(\{\lambda_s\}_N^{\langle 2k\rangle}\cdot B_N\right)_N\otimes\{\rho\}\right)_K\cdot A_K\right)_K=\sum_{\mu} y^{\mu}_{\lambda\rho}\ \{\mu\} \tag{7.58}$$

The significance of (7.57) and the subsequent remarks leading finally to (7.58) can be seen in the evaluation of the terms in the plethysm $[2(21)]\otimes\{21\}$ of $SO^*(24)$. In such a case we have $k=2$, $r=3$ and $n=12$ so that $N=4$, $M=12$ and $K=6$. We show how to calculate all terms $[6(\mu)]$ of $[2(21)]\otimes\{21\}$ up to weight 16 and of width $\mu_1\leq 3$ using (7.57). Such terms will necessarily have length $\mu_1'\leq 6$.

Since $k=2$ the signed sequence is evaluated in $Sp(4)$ giving

$$\{21_s\}_4^{\langle 4\rangle}=\{21\}-\{21^3\} \tag{7.59}$$

Next the terms in the B-series up to weight 16, width 3 and length 4 are:

$$\begin{array}{lllll}\{0\} & +\{1^2\} & +\{1^4\} & +\{2^2\} & +\{2^21^2\}\\ +\{2^4\} & +\{3^2\} & +\{3^21^2\} & +\{3^22^2\} & +\{3^4\}\end{array} \tag{7.60}$$

The tensor product of the terms in (7.59) with those of (7.60) is to be carried out in $U(N)$ with $N=4$. Again up to weight 16, width 3 and length 4 this gives

$$\begin{array}{lllll}\{21\} & +\{2^21\} & +\{31^2\} & +\{32\} & +\{321^2\}\\ +\{32^2\} & +\{3^21\} & +\{3^221\} & & \end{array} \tag{7.61}$$

Now we calculate the mixed symmetry third order plethysm signified by $\{21\}$ of this sum of terms in $U(M)$ with $M = 12$ to give up to weight 16, width 3 and now length 12:

$$\begin{array}{lllll}
\{2^41\} & + \{2^41^3\} & + 2\{2^51\} & + \{2^51^3\} & + 2\{2^61\} \\
+ \{2^71\} & + \{321^4\} & + 2\{32^21^2\} & + 3\{32^21^4\} & + \{32^3\} \\
+ 9\{32^31^2\} & + 5\{32^31^4\} & + 6\{32^4\} & + 15\{32^41^2\} & + 4\{32^41^4\} \\
+ 10\{32^5\} & + 11\{32^51^2\} & + 7\{32^6\} & + \{3^21^3\} & + 2\{3^21^5\} \\
+ 3\{3^221\} & + 12\{3^221^3\} & + 7\{3^221^5\} & + 18\{3^22^21\} & + 33\{3^22^21^3\} \\
+ 9\{3^22^21^5\} & + 45\{3^22^31\} & + 40\{3^22^31^3\} & + 54\{3^22^41\} & + 12\{3^31^2\} \\
+ 20\{3^31^4\} & + 5\{3^31^6\} & + 10\{3^32\} & + 60\{3^321^2\} & + 51\{3^321^4\} \\
+ 40\{3^32^2\} & + 117\{3^32^21^2\} & + 71\{3^32^3\} & + 32\{3^41\} & + 70\{3^41^3\} \\
+ 120\{3^421\} & + 28\{3^5\} & & &
\end{array} \tag{7.62}$$

Then the terms in the A-series up to weight 16, width 3 and length 12 are found to be

$$\begin{array}{lllll}
\{0\} & - \{1^2\} & + \{21^2\} & - \{2^3\} & - \{31^3\} \\
\{32^21\} & - \{3^22^2\} & + \{3^4\} & &
\end{array} \tag{7.63}$$

and their tensor product with the terms in (7.62) calculated in $U(12)$ gives

$$\begin{array}{lllll}
\{2^41\} & - \{2^41^5\} & + \{2^51\} & - \{2^51^3\} & + \{321^4\} \\
- \{321^6\} & + 2\{32^21^2\} & - 2\{32^21^6\} & + \{32^3\} & + 4\{32^31^2\} \\
- 4\{32^31^4\} & - \{32^31^6\} & + 3\{32^4\} & - 3\{32^41^4\} & + 2\{32^5\} \\
- 2\{32^51^2\} & + \{3^21^3\} & - \{3^21^7\} & + 3\{3^221\} & + 5\{3^221^3\} \\
- 5\{3^221^5\} & - 3\{3^221^7\} & + 10\{3^22^21\} & - 10\{3^22^21^5\} & + 10\{3^22^31\} \\
- 10\{3^22^31^3\} & + 7\{3^31^2\} & - 7\{3^31^6\} & + 6\{3^32\} & + 13\{3^321^2\} \\
- 13\{3^321^4\} & + 12\{3^32^2\} & + 6\{3^32^3\} & + 10\{3^41\} & + 14\{3^421\} \\
+ 4\{3^5\} & & & &
\end{array} \tag{7.64}$$

Now the labelling is changed to that of irreducible representationsof $SO^*(24)$ with $kr = 6$ inserted before each partition to yield

$$\begin{array}{lllll}
[6(2^41)] & - [6(2^41^5)] & + [6(2^51)] & - [6(2^51^3)] & + [6(321^4)] \\
- [6(321^6)] & + 2[6(32^21^2)] & - 2[6(32^21^6)] & + [6(32^3)] & + 4[6(32^31^2)] \\
- 4[6(32^31^4)] & - [6(32^31^6)] & + 3[6(32^4)] & - 3[6(32^41^4)] & + 2[6(32^5)] \\
- 2[6(32^51^2)] & + [6(3^21^3)] & - [6(3^21^7)] & + 3[6(3^221)] & + 5[6(3^221^3)] \\
- 5[6(3^221^5)] & - 3[6(3^221^7)] & + 10[6(3^22^21)] & - 10[6(3^22^21^5)] & + 10[6(3^22^31)] \\
- 10[6(3^22^31^3)] & + 7[6(3^31^2)] & - 7[6(3^31^6)] & + 6[6(3^32)] & + 13[6(3^321^2)] \\
- 13[6(3^321^4)] & + 12[6(3^32^2)] & + 6[6(3^32^3)] & + 10[6(3^41)] & + 14[6(3^421)] \\
+ 4[6(3^5)] & & & &
\end{array} \tag{7.65}$$

At first the appearance of negative terms seems disconcerting until it is realised that they correspond to non-standard terms in the signed sequences $\{\mu_s\}^{\langle 12\rangle}$ of (7.57). Restricting attention, as required, to $SO^*(24)$-standard terms in accordance with (7.58), finally yields the result

$[2(21)] \otimes \{21\} =$

$$\begin{array}{lllll} [6(2^41)] & +[6(2^51)] & +[6(321^4)] & +2[6(32^21^2)] & +[6(32^3)] \\ +4[6(32^31^2)] & +3[6(32^4)] & +2[6(32^5)] & +[6(3^21^3)] & +3[6(3^221)] \\ +5[6(3^221^3)] & +10[6(3^22^21)] & +10[6(3^22^31)] & +7[6(3^31^2)] & +6[6(3^32)] \\ +13[6(3^321^2)] & +12[6(3^32^2)] & +6[6(3^32^3)] & +10[6(3^41)] & +14[6(3^421)] \\ +4[6(3^5)] & +\cdots & & & \end{array} \tag{7.66}$$

up to weight 16 and width 3, where it is to be noted that, as promised, the surviving terms all have length $\mu_1' \leq 6$. This is because $K = \min(kr, n) = 6$. It would clearly have been simpler to use (7.58) at an earlier stage and discard all terms of length greater than 6 in (7.62)-(7.65) rather than to use (7.57) and keep terms up to length 12. At each step the calculation would have involved fewer terms and the final signed sequence problems would have been circumvented. This example, while exhibiting the fact that signed sequences do emerge in a natural way, serves to illustrate the computational merits of Corollary 4.2.

7.6 Resolution of $H^2 = H \otimes \{2\} + H \otimes \{1^2\}$

In the special case for which ρ is a partition of 2, so that $r = 2$, we have $K = N = \min(2k, n)$ in (7.58). Consequently for symmetrised squares of irreducible representationsof $SO^*(2n)$ we have

$$[k(\lambda)] \otimes \{\rho\} = [\big(\big(\{\lambda_s\}^{\langle 2k\rangle} \cdot B\big) \otimes \{\rho\}\big) \cdot A\big)_N], \tag{7.67}$$

where all products and plethysms are to be carried out in $U(N)$. Setting $\{\rho\} = \{2\}$ and $\{1^2\}$, using the algebra of plethysms and the fact that[5] $(B \otimes \{2\}) \cdot A = B_+BA = B_+$ and $(B \otimes \{1^2\}) \cdot A = B_-BA = B_-$ it follows that

$$\begin{aligned} [k(\lambda)] \otimes \{2\} &= [2k((\{\lambda_s\}^{\langle 2k\rangle} \otimes \{2\}) \cdot B_+)_N] + [2k((\{\lambda_s\}^{\langle 2k\rangle} \otimes \{1^2\}) \cdot B_-)_N], && (7.68) \\ [k(\lambda)] \otimes \{1^2\} &= [2k((\{\lambda_s\}^{\langle 2k\rangle} \otimes \{1^2\}) \cdot B_+)_N] + [2k((\{\lambda_s\}^{\langle 2k\rangle} \otimes \{2\}) \cdot B_-)_N], && (7.69) \\ & && (7.70) \end{aligned}$$

where $N = min(2k, n)$.

Further specialisation of the above result leads to

$$\begin{aligned} H_m \otimes \{2\} &= [1(m)] \otimes \{2\} = \sum_{p=0}^{\infty}\sum_{x=0}^{m} [2(2m+p-x, p+x)] \quad \text{with } p+x \text{ even}; && (7.71) \\ H_m \otimes \{1^2\} &= [1(m)] \otimes \{1^2\} = \sum_{p=0}^{\infty}\sum_{x=0}^{m} [2(2m+p-x, p+x)] \quad \text{with } p+x \text{ odd}. && (7.72) \\ & && (7.73) \end{aligned}$$

This can be seen by noting that in $U(2)$ each term $\{p^2\}$ of B belongs to B_+ or B_- according as p is even or odd, respectively, while $\{m\} \otimes \{2\}$ and $\{m\} \otimes \{1^2\}$ contain terms of the form $\{2m-x,x\}$ with x even and odd, respectively. Typical terms contributing to (7.71) are represented diagrammatically by:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline a&a&a&c&c&c&c&c&c&d&d&d\\ \hline b&b&b&d&d\\ \cline{1-5}\end{array} \tag{7.74}$$

where there are precisely p columns containing the pair (a,b), x columns containing the pair (c,d), $m-x$ columns containing just c and the same number containing just d, with p and x either both even or both odd.

Summing over all m we obtain

$$\begin{aligned}
\sum_{m=0}^{\infty} H_m \otimes \{2\} &= \sum_{r=0}^{\infty}\sum_{s=0}^{r} (s+1)[2(r,s)] \text{ with } r \text{ and } s \text{ both even;} && (7.75)\\
\sum_{m=0}^{\infty} H_m \otimes \{1^2\} &= \sum_{r=0}^{\infty}\sum_{s=0}^{r} (s+1)[2(r,s)] \text{ with } r \text{ and } s \text{ both odd;} && (7.76)\\
& && (7.77)
\end{aligned}$$

The first of these follows from the fact that for fixed r and s the distribution of letters in diagrams of type (7.74) is such that the number of d's in the second row of length s can vary from 0 to s, while the number in the first row is necessarily $(r-s)/2$. The second follows in the same way. The only difference is now that instead of r and s both being even, they are both odd.

This allows us to resolve H^2 into its symmetric and antisymmetric parts $H \otimes \{\rho\}$ with $\{\rho\} = \{2\}$ and $\{1^2\}$, respectively:

Proposition 7.6.1 *Let $C_{rs} = (r-s+1)(s+1)$. Then for $\{\rho\} = \{2\}$ and $\{1^2\}$ we have*

$$H \otimes \{\rho\} = \sum_{r=0}^{\infty}\sum_{s=0}^{r} C_{rs}^{\{\rho\}}\ [2(r,s)] \tag{7.78}$$

with

$$C_{rs}^{\{2\}} = \begin{cases} \frac{1}{2}(C_{rs}+1) & \text{if } r \text{ and } s \text{ are both even;}\\ \frac{1}{2}(C_{rs}-1) & \text{if } r \text{ and } s \text{ are both odd;}\\ \frac{1}{2}C_{rs} & \text{otherwise,}\end{cases} \tag{7.79}$$

$$C_{rs}^{\{2\}} = \begin{cases} \frac{1}{2}(C_{rs}-1) & \text{if } r \text{ and } s \text{ are both even;}\\ \frac{1}{2}(C_{rs}+1) & \text{if } r \text{ and } s \text{ are both odd;}\\ \frac{1}{2}C_{rs} & \text{otherwise,}\end{cases} \tag{7.80}$$

Proof Since

$$H^2 = \left(\sum_m H_m\right)^2 = \sum_m H_m^2 + \sum_{m\neq m'} H_m H_{m'}, \tag{7.81}$$

it follows that

$$\begin{aligned} H\otimes\{2\} &= \Big(\sum\nolimits_m H_m\Big)\otimes\{2\} = \sum\nolimits_m\Big(H_m\otimes\{2\}\Big)+\tfrac{1}{2}\sum\nolimits_{m\neq m'}H_mH_{m'} \\ &= \sum\nolimits_m\Big(H_m\otimes\{2\}\Big)+\tfrac{1}{2}\Big(H^2-\sum\nolimits_m\ H_m^2\Big) \\ &= \tfrac{1}{2}\Big(H^2+\sum\nolimits_m\Big(H_m\otimes\{2\}-H_m\otimes\{1^2\}\Big)\Big) \end{aligned} \tag{7.82}$$

Similarly

$$H\otimes\{1^2\}=\frac{1}{2}\Big(H^2-\sum_m\Big(H_m\otimes\{2\}-H_m\otimes\{1^2\}\Big)\Big). \tag{7.83}$$

The results (7.79) then follow from (7.47) and (7.75).

7.7 Relations between group chains and irreducible representationsof $SO^*(2n)$ and $Sp(2n\mathbb{R})$

Starting with the metaplectic group $Mp(4nk)$ we may relate the decompositions involving the non-compact subgroups $SO^*(2n)$ and $Sp(2n,R)$ by means of the commutative diagram:

$$\begin{tikzcd} SO^*(2n)\times Sp(2k) \arrow[d] & Mp(4nk) \arrow[l] \arrow[r] & Sp(2n,R)\times O(2k) \arrow[d] \\ U(n)\times Sp(2k) \arrow[d] & & U(n)\times O(2k) \arrow[d] \\ U(n)\times SO(2k) \arrow[r] & & U(n)\times SO(2k) \arrow[l] \end{tikzcd} \tag{7.84}$$

The terminal subgroup in each case is $U(n)\times SO(2k)$. Taking into account the labels used to distinguish mutually associate pairs of irreducible representationsof $Sp(2n\mathbb{R})$, the decomposition of the metaplectic irreducible representation$\tilde{\Delta}$ of $Mp(4nk)$ proceeds as indicated below:

$$\begin{tikzcd} \sum_\kappa[k(\kappa)]\times\langle\kappa\rangle \arrow[d] & \tilde{\Delta} \arrow[l] \arrow[r] & \sum_\lambda\langle k(\lambda)\rangle\times[\lambda] \arrow[d] \\ \sum_\kappa[k(\kappa)]_{U(n)}\times\langle\kappa\rangle \arrow[d] & & \sum_\lambda\langle k(\lambda)\rangle_{U(n)}\times[\lambda] \arrow[d] \\ \sum_\kappa[k(\kappa)]_{U(n)}\times[\kappa/AD] \arrow[r] & & \sum_\lambda(\langle k(\lambda+(1-\delta_{\lambda_1'k})\lambda^*)\rangle_{U(n)}\times[\lambda] \arrow[l] \end{tikzcd} \tag{7.85}$$

where the symbols $[\cdots]_{U(n)}$ and $\langle\cdots\rangle_{U(n)}$ signify restriction from $SO^*(2n)$ and $Sp(2n,\mathbb{R})$, respectively, to $U(n)$, while the skew products of κ with A and D correspond to passing from $Sp(2k)$ up to $U(2k)$ and then down to $SO(2k)$. It should be noted that at the level of $U(n)\times SO(2k)$ the summations over both κ and λ are restricted so that these partitions have no more than P parts with $P=\min(k,n)$.

Since [55]

$$AD = W = \sum_{r=0}^{\infty}\sum_{s=0}^{r} (-1)^s\{r,s\} \quad \text{with } r-s \text{even}, \tag{7.86}$$

it follows that on comparing the terms of the form $\cdots\times[\lambda]$ we have

$$[k(\lambda\cdot W)]_{U(n)} = \langle k(\lambda)\rangle_{U(n)} + \big(1-\delta_{\lambda_1' k}\big)\langle k(\lambda^*)\rangle_{U(n)}. \tag{7.87}$$

As special cases of this with $k=1$ and $\lambda=(0)$ and (1), we obtain:

$$\big(H_+\big)_{U(n)} = [1(M_+)]_{U(n)} = \langle 1(0)\rangle_{U(n)} + \langle 1(0^*)\rangle_{U(n)}; \tag{7.88}$$

$$\big(H_-\big)_{U(n)} = [1(M_-)]_{U(n)} = \langle 1(1)\rangle_{U(n)}. \tag{7.89}$$

$$\tag{7.90}$$

It should be stressed that quite generally a knowledge of the restriction to $U(n)$ of any direct sum of harmonic series unitary irreducible representations of both $SO^*(2n)$ and $Sp(2n,\mathbb{R})$ is sufficient to determine these representations up to equivalence. This is because such representations are determined up to equivalence by their characters which are themselves evaluated on elements of the maximal compact subgroup $U(n)$.

7.8 Powers and plethysms of irreducible representations of $SO^*(2n)$ from those of $Sp(2n,\mathbb{R})$

The results of the previous section lead to an alternative method of computing powers of the basic harmonic representation H of $SO^*(2n)$ and its constituents H_+ and H_-. Since $H=H_++H_-$ it follows from (7.88) that the $U(n)$ content of the harmonic representation H of $SO^*(2n)$ coincides with that of the representation S of $Sp(2n,\mathbb{R})$, where

$$S = \langle 1(0)\rangle + \langle 1(0^*)\rangle + \langle 1(1)\rangle. \tag{7.91}$$

The same must be true of both their powers and plethysms.

Since S is a self-associate representation of $Sp(2n,\mathbb{R})$ it follows that its pth power may be written in the form:

$$S^p = \sum_{\mu:\mu_1'\le P} g^\mu\ \langle p(\mu + (1-\delta_{\mu_1' p})\mu^*\rangle, \tag{7.92}$$

with $P = \min(p, n)$, for some set of coefficients g^μ. It then follows from (7.5) that

$$H^p = \sum_{\mu:\mu_1' \leq P} g^\mu \; [p((\mu \cdot W)_P)] \tag{7.93}$$

where the subscript P on $(\mu \cdot W)_P$ indicates that the only terms $(\nu)_P$ to be retained are those for which $\nu_1' \leq= P = \min(p, n)$. Likewise, for any partition π of p, the corresponding pth-fold symmetrised power may be expanded in the form

$$S \otimes \{\pi\} = \sum_{\mu:\mu_1' \leq P} h^\mu \; \langle p(\mu + (1 - \delta_{\mu_1' p})\mu^*\rangle, \tag{7.94}$$

for some particular set of coefficients h^μ. It then follows that

$$H \otimes \{\pi\} = \sum_{\mu:\mu_1' \leq P} h^\mu \; [p((\mu \cdot W)_P)]. \tag{7.95}$$

A similar situation applies to H_+ and H_-, and indeed to plethysms and powers of any sum of $SO^*(2n)$ representations of the form $[k(\lambda \cdot W)]$ as in (7.4). Conversely, plethysms and powers of any self-associate sum of $Sp(2n,\mathbb{R})$ representations of the form $\langle k(\kappa \cdot V)\rangle$ can be evaluated from a knowledge of the powers and plethysms of $[k(\kappa)]$ in $SO^*(2n)$, where $V = W^{-1} = W'$.

As a final example we compute the terms of $H_- \otimes \{21\}$ up to weight 12 and width 4 in $SO^*(24)$ starting from the $Sp(24,\mathbb{R})$ plethysm $\langle 1(1)\rangle \otimes \{21\} =$

$$\begin{array}{lllll} \langle 3(21)\rangle & + \langle 3(21^3)\rangle & + \langle 3(2^21)\rangle & + 2\langle 3(31^2)\rangle & + 2\langle 3(32)\rangle \\ + 2\langle 3(321^2)\rangle & + 2\langle 3(32^2)\rangle & + 3\langle 3(3^21)\rangle & + 2\langle 3(41)\rangle & + 2\langle 3(41^3)\rangle \\ + 6\langle 3(421)\rangle & + 4\langle 3(43)\rangle & + 4\langle 3(431^2)\rangle & + 5\langle 3(432)\rangle & + 5\langle 3(4^21)\rangle \\ + 2\langle 3(4^23)\rangle & & & & \end{array} \tag{7.96}$$

Now we remove the prefix $p = 3$ and standardise the irreducible representationsin the group $U(3)$ to give

$$\begin{array}{lllll} \{21\} & + \{2^21\} & + 2\{31^2\} & + 2\{32\} & + 2\{32^2\} \\ + 3\{3^21\} & + 2\{41\} & + 6\{421\} & + 4\{43\} & + 5\{432\} \\ + 5\{4^21\} & + 2\{4^23\} & & & \end{array} \tag{7.97}$$

The terms in the W-series up to weight 12 are

$$\begin{array}{lllll} \{0\} & - \{1^2\} & + \{2\} & + \{2^2\} & - \{31\} \\ - \{3^2\} & + \{4\} & + \{42\} & + \{4^2\} & \end{array} \tag{7.98}$$

Forming the tensor product, in $U(3)$, of (8.7) and (8.8), and keeping terms up weight 12 and width 4 gives

$$\begin{array}{lllll} \{21\} & + \{2^21\} & + 2\{31^2\} & + 2\{32\} & + 2\{32^2\} \\ + 4\{3^21\} & + \{3^3\} & + 3\{41\} & + 7\{421\} & + 6\{43\} \\ + 9\{432\} & + 7\{4^21\} & + 6\{4^23\} & & \end{array} \tag{7.99}$$

These $U(3)$ irreducible representationsare now converted into $SO^*(24)$ irreducible representationsby inserting $p = 3$ before each partition and adopting the notation appropriate to the group $SO^*(24)$ leading to $H_- \otimes \{21\} =$

$$\begin{array}{lllll} [3(21)] & + [3(2^21)] & + 2[3(31^2)] & + 2[3(32)] & + 2[3(32^2)] \\ + 4[3(3^21)] & + [3(3^3)] & + 3[3(41)] & + 7[3(421)] & + 6[3(43)] \\ + 9[3(432)] & + 7[3(4^21)] & + 6[3(4^23)] & + \cdots . & \end{array} \tag{7.100}$$

This is the same as the result that can be found using (7.55) and (7.58). While the above calculation was carried out for $SO^*(24)$ it should be noted that the result is valid for all $SO^*(2n)$ with $n \geq 6$.

7.9 Concluding remarks

The objective in writing this chapter has been to complete the results hinted at in KW1. In the process methods for calculating tensor products and plethysms of the infinite dimensional unitary irreducible representationsof $SO^*(2n)$ have been developed. In addition explicit results have been obtained for the case of the basic infinite-dimensional harmonic representation H of $SO^*(2n)$ and its various constituents $H_\pm$ and H_m. Furthermore, the embedding of the product groups $SO^*(2n) \times Sp(2k)$ and $Sp(2n, \mathbb{R}) \times O(2k)$ in the metaplectic group $Mp(4nk)$ has been shown to yield interesting and indeed useful relationships between irreducible representations of the non-compact groups $SO^*(2n)$ and $Sp(2n, \mathbb{R})$ as well as their powers and plethysms.

Chapter 8

Generic products

In this chapter one will find new results on generic products for $Sp(2n, \mathbb{R})$ and $SO^\star(2n)$. This work is the subject of a joint paper in preparation with R. C. King and B. G. Wybourne.

8.1 Introduction

The non-compact symplectic group $Sp(2n, \mathbb{R})$ plays a pivotal role in symplectic models of nuclei ([92, 94]) and in general many-body systems involving harmonic oscillator potentials. The states associated with such systems span infinite dimensional unitary irreducible representations of $Sp(2n, \mathbb{R})$. Tensor operators ([92, 94]) that act between these infinite sets of states normally span the finite non-unitary irreducible representations of $Sp(2n, \mathbb{R})$. The determination of the matrix elements of such tensor operators requires a knowledge of the tensor products of finite non-unitary irreducible representations with infinite dimensional unitary irreducible representations.

In recent years there has been considerable progress in developing systematic methods for calculating group-subgroup decompositions and tensor products for both $Sp(2n, \mathbb{R})$ ([95, 60, 35, 102, 61]) and $SO^*(2n)$ [60, 56]. The problem of tensor products of finite non-unitary irreducible representations with infinite unitary irreducible representations for non-compact Lie groups has received some attention in the mathematics literature ([64, 110]) and in the physics literature ([93, 34]). The general problem is complicated by the appearance of representations that are reducible but not fully reducible, and by the occurrence of both unitary and non-unitary representations. Here we restrict our attention to certain generic tensor products that can be fully resolved into a finite set of unitary irreducible representations. In this paper our primary objective is to give simple, practical and efficient algorithms for computing generic tensor products of finite non-unitary irreducible representations with infinite unitary irreducible representations.

Throughout we follow the notation developed earlier ([95, 60, 35, 61, 56]) for representations of non-compact groups and certain signed sequences. In the case of the notation for partitions and symmetric functions we follow that of Macdonald ([73]), while for S-

functions series and modification rules we call where appropriate on formulae collected together in three previous articles ([49, 55, 8]).

8.2 Finite dimensional irreducible representations of $Sp(2n,\mathbb{R})$ and $SO^*(2n)$

The finite dimensional non-unitary irreducible representations of the non-compact Lie groups $Sp(2n,\mathbb{R})$ and $SO^*(2n)$ are in one-to-one correspondence with those of the tensor unitary irreducible representations of the compact Lie groups $Sp(2n)$ and $O(2n)$ and may be similarly labelled by partitions of integers into at most n non-zero parts ([49, 55, 8]). Likewise, the branching rules for the finite non-unitary irreducible representations under the reductions $Sp(2n,\mathbb{R}) \to U(n)$ and $SO^*(2n) \to U(n)$ are identical to those of the corresponding unitary irreducible representations of compact Lie groups and hence we have [49] for $Sp(2n,\mathbb{R}) \to U(n)$

$$\langle\mu\rangle \to \sum_{\zeta}\{\bar{\zeta};\mu/D\zeta\} \tag{8.1}$$

and for $SO^*(2n) \to U(n)$

$$[\lambda] \to \sum_{\zeta}\{\bar{\zeta};\lambda/B\zeta\} \tag{8.2}$$

where the B and D are two infinite series of $S-$functions ([55, 8]). As shown in Appendix B, we may adopt an alternative, though equivalent, form for these branching rules by writing for $Sp(2n,\mathbb{R}) \to U(n)$

$$\langle\mu\rangle \to \{\bar{\mu};C\}\cdot D \tag{8.3}$$

and for $SO^*(2n) \to U(n)$

$$[\mu] \to \{\bar{\mu};A\}\cdot B \tag{8.4}$$

which are useful in formal work. Either way, under $Sp(8,\mathbb{R}) \to U(4)$ we have

$$\begin{aligned}\langle 31\rangle \to\ & \{\overline{31};0\}+\{\bar{3};1\}+\{\overline{21};1\}+\{\bar{2};2\}+\{\bar{2};1^2\}+\{\bar{2};0\}+\{\overline{1^2};2\}\\ & +\{\overline{1^2};0\}+\{\bar{1};3\}+\{\bar{1};21\}+2\{\bar{1};1\}+\{31\}+\{2\}+\{1^2\}\end{aligned} \tag{8.5}$$

while for $SO^*(8) \to U(4)$ we have

$$\begin{aligned}[31] \to\ & \{\overline{31};0\}+\{\bar{3};1\}+\{\overline{21};1\}+\{\bar{2};2\}+\{\bar{2};1^2\}+\{\bar{2};0\}+\{\overline{1^2};2\}\\ & +\{\bar{1};3\}+\{\bar{1};21\}+\{\bar{1};1\}+\{31\}+\{2\}\end{aligned} \tag{8.6}$$

For any irreducible representation $\{\overline{\mu};\tau\}$ of $U(n)$, with a mixed tensor basis, there always exists an associated irreducible representation $\{\rho\}$, with a covariant basis, such that [49]:

$$\{\overline{\mu};\tau\} = \varepsilon^{\omega}\{\rho\} \tag{8.7}$$

for some integer ω. Specifically one has for $\{\overline{\mu};\tau\}$ with (μ) a partition with p non-zero parts and (τ) with q non-zero parts in $U(n)$

$$\begin{aligned}
\rho_1 &= \tau_1 + \mu_1 \\
\rho_2 &= \tau_2 + \mu_1 \\
\ldots &= \ldots \\
\rho_q &= \tau_q + \mu_1 \\
\rho_{q+1} &= \mu_1 - \mu_{n-q} \\
\rho_{q+2} &= \mu_1 - \mu_{n-q-1} \\
\ldots &= \ldots \\
\rho_n &= 0
\end{aligned} \tag{8.8}$$

with

$$\varepsilon^{\omega} = \varepsilon^{-\mu_1} \tag{8.9}$$

Thus for the group $U(4)$ we obtain the result of (8.6) in a covariant basis as

$$\begin{aligned}
[31] \to\ & \varepsilon^0(\{2\} + \{31\}) + \varepsilon^{-1}(\{21^2\} + \{321\} + \{41^2\} + \{31\}) \\
& \varepsilon^{-2}(\{2^3\} + \{3^2 2\} + \{42^2\} + \{321\}) + \varepsilon^{-3}(43^2\} + \{3^2 2\})
\end{aligned} \tag{8.10}$$

In the case of partitions into n non-zero parts we have the $U(n)$ equivalence

$$\{\tau_1, \tau_2, \ldots, \tau_n\} \equiv \varepsilon^{-\tau_n}\{\tau_1 - \tau_n, \tau_2 - \tau_n, \ldots, 0\} \tag{8.11}$$

For subsequent usage we note that the irreducible representations $\langle\mu\rangle$ and $[\mu]$ may be mapped into one another and that ([49, 59])

$$\langle\mu\rangle \to [\mu/W] \tag{8.12}$$

$$[\mu] \to \langle\mu/V\rangle \tag{8.13}$$

where W is the infinite $S-$function series [8]

$$W = \sum_{r=0}^{\infty}\sum_{s=0}^{r} (-1)^s\{r,s\} \quad \text{with } r-s \text{ even}, \tag{8.14}$$

and $V = W^{-1}$.

If $(\mu)_{sc} = (a, a-1, \ldots, 1)$ then $(\mu)_{sc}$ is said to be a *staircase partition* with the important property [59] that

$$(\mu/W)_{sc} = (\mu/V)_{sc} = (\mu)_{sc} \tag{8.15}$$

Thus the decompositions obtained for the irreducible representations $\langle 21\rangle$ and $[21]$ have the same $U(n)$ content. In the case of $n = 4$ we obtain the $U(4)$ content

$$\begin{aligned}
& \{\overline{21};0\} + \{\bar{2};1\} + \{\overline{1^2};1\} + \{\bar{1};2\} + \{\bar{1};1^2\} + \{\bar{1};0\} + \{21\} + \{1\} \\
& = \varepsilon^0(\{1\} + \{21\}) + \varepsilon^{-1}(\{1^3\} + \{2^2 1\} + \{31^2\} + \{21\}) + \varepsilon^{-2}(32^2\} + \{2^2 1\})
\end{aligned} \tag{8.16}$$

Products of $U(n)$ irreducible representations may be evaluated by noting that [8]

$$\{\bar{\mu};\lambda\} \times \{\bar{\rho};\nu\} = \sum_{\sigma,\tau}\{\overline{(\mu/\sigma)\cdot(\rho/\tau)};(\lambda/\tau)\cdot(\nu/\sigma)\} \tag{8.17}$$

Two useful specialisations are

$$\{\bar{\mu};\lambda\} \times \{\nu\} = \sum_{\sigma} \{\overline{\mu/\sigma};\lambda \cdot (\nu/\sigma)\} \tag{8.18}$$

It follows from (8.8) and (8.9) that the largest part ρ_1 of any covariant irreducible representation $\{\rho\}$ of $U(n)$ arising from (8.1) or (8.2) is given by

$$\rho_1 = \mu_1 + \mu_2 \tag{8.19}$$

with $\ell_\rho \leq n-1$.

Consider the $U(n)$ product $\{\mu\} \times \{\tau\}$ of two covariant irreducible representations with $\ell_\mu, \ell_\tau \leq n-1$. For a given $\{\mu\}$ the largest possible number of $U(n)$ irreducible representations will arise when

$$\tau_i - \tau_{i+1} \geq \mu_1 \quad i = 1, \ldots, n-1 \tag{8.20}$$

In that case the number of terms will be $dim(\mu)$. For example in $U(4)$ the product $\{321\} \times \{963\}$ contains 64 irreducible representations and indeed $dim(321) = 64$. The irreducible representation multiplicities occurring in the product are precisely those of the weight multiplicities associated with the irreducible representation $\{\mu\}$. These weight multiplicities form upper bounds for the irreducible representation multiplicities that can arise in any product in $U(n)$ involving the irreducible representation $\{\mu\}$.

8.3 The unitary irreducible representations of $Sp(2n,\mathbb{R})$ and $SO^*(2n)$

The standard discrete series unitary irreducible representations of $Sp(2n,\mathbb{R})$ may be labelled as ([95, 60]) $\langle \frac{k}{2}(\lambda)\rangle$ where

$$\lambda'_1 + \lambda'_2 \leq k \quad and \quad \lambda'_1 \leq n \tag{8.21}$$

while those of $SO^*(2n)$ may be labelled as ([60, 56]) $[k(\lambda)]$ where

$$\lambda'_1 \leq min(2k, n) \tag{8.22}$$

Under restriction to the $U(n)$ subgroup we have ([95, 60]) for $Sp(2n,\mathbb{R}) \to U(n)$

$$\langle \tfrac{k}{2}(\lambda)\rangle \to \varepsilon^{\frac{k}{2}} \cdot \{\{\lambda_s\}_N^{[k]} \cdot D_N\}_N \tag{8.23}$$

with $N = min(n,k)$. The first $\cdot$ indicates a product in $U(n)$ and the second $\cdot$ a product in $U(N)$ as implied by the final subscript N. $\{\lambda_s\}_N^k$ is a signed sequence ([95, 60]) of terms $\pm\{\rho\}$ such that $\pm[\rho]$ is equivalent to $[\lambda]$ under the modification rules ([8, 47]) of $O(k)$.

Likewise, under restriction to the $U(n)$ subgroup we have [56] for $SO^*(2n) \to U(n)$

$$[k(\lambda)] \to \varepsilon^{k} \cdot \{\{\lambda_s\}_N^{\langle 2k\rangle} \cdot B_N\}_N \tag{8.24}$$

where $N = min(2k, n)$ and the signed sequences are evaluated in $Sp(2k)$.

Equation (8.23) greatly simplifies if

$$k + 1 - \lambda_2' > N = \min(k, n) \tag{8.25}$$

implying that either

$$k \geq n + \lambda_2' \quad or \quad k < n \quad and \quad \lambda_2' = 0 \tag{8.26}$$

to give (8.23) as just

$$\langle \tfrac{k}{2}(\lambda)\rangle \to \varepsilon^{\frac{k}{2}} \cdot \{\{\lambda\} \cdot D_N\}_N \tag{8.27}$$

Irreducible representations satisfying (8.25) will be said to be *highly standard* [60]) and the products are all evaluated in $U(n)$. In such cases the signed sequence consists of the single leading term. Eq (8.26) thus implies that $\langle \frac{k}{2}(1^m)\rangle$ is highly standard for all k and n.

An irreducible representation of $SO^*(2n)$ will be highly standard if

$$2k + 2 - \lambda_1' > N = min(n, 2k) \tag{8.28}$$

implying that if $n \leq 2k$ then $2k \geq n + \lambda_1' - 1$ or if $n > 2k$ then $2k + 2 - \lambda_1' > 2k$ and thus $\lambda_1' \leq 1$ leading to the conclusion that if $\lambda_1' \leq 1$ then $[k(m)]$ is highly standard for all k and n while if $\lambda_1' \geq 2$ then $[k(\lambda)]$ is highly standard if $2k \geq n + \lambda_1' - 1$. Hence for highly standard irreducible representations of $SO^*(2n)$ (8.24) simplifies to

$$[k(\lambda)] \to \varepsilon^k \cdot \{\{\lambda\} \cdot B_N\}_N \tag{8.29}$$

Highly standard irreducible representations have the important, and useful, distinction that (8.27) and (8.29) can be inverted by forming the product, in $U(n)$, of the inverse $S-$function series ($C = D^{-1}$) and ($A = B^{-1}$) respectively. For irreducible representations that are standard but not highly standard the occurrence of signed sequences prevents such a simple inversion.

Two irreducible representations of $Sp(2n,\mathbb{R})$ (or $SO^*(2n)$) will be said to be *equivalent* if under $Sp(2n,\mathbb{R}) \to U(n)$ (or under $SO^*(2n) \to U(n)$) they yield the same $U(n)$ characters[4]. These equivalences arise in the particular case of highly standard irreducible representations. Thus in $Sp(6,\mathbb{R})$ we have the equivalences

$$\langle 6(1)\rangle \equiv \langle 5(211)\rangle \equiv \langle 4(322)\rangle \equiv \langle 3(433)\rangle \tag{8.30}$$

All these irreducible representations satisfy the constraints of (8.21) whereas the irreducible representation $\langle 2(544)\rangle$ is necessarily null.

Suppose that (8.1) or (8.2) lead to a covariant irreducible representation $\varepsilon^{-w}\{\rho\}$ of the subgroup $U(n)$. Then the property of highly standardness means that for a sufficiently large value of k we may write

$$\varepsilon^{-w}\{\rho\} \times \langle \tfrac{k}{2}(\lambda)\rangle \quad \to \langle \tfrac{k}{2} - w(\rho \cdot \lambda)\rangle \tag{8.31}$$

$$\tag{8.32}$$

$$\varepsilon^{-w}\{\rho\} \times [k(\lambda)] \quad \to [k - w(\rho \cdot \lambda)] \tag{8.33}$$
$$\tag{8.34}$$

where the $\cdot$ implies a product in $U(n)$ and on both sides of each equation the $U(n)$ content is the same. In the case of terms in the product having n non-zero parts the irreducible representations may be replaced by equivalent irreducible representations. It follows from (8.33) that in $SO^*(6)$ that

$$\begin{aligned}\varepsilon^{-2}\{21\} \times [k(42)] &= [k-2(63)] + [k-2(621)] + [k-2(54)] + 2[k-2(531)] \\ &\quad +[k-2(522)] + [k-2(441)] + [k-2(432)] \\ &\equiv [k-2(63)] + [k-1(51)] + [k-2(54)] + 2[k-1(42)] \\ &\quad +[k(3)] + [k-1(33)] + [k(21)]\end{aligned}$$

The above list of irreducible representations will be highly standard for $k \geq 4$. For $k = 4$ we have

$$\begin{aligned}\varepsilon^{-2}\{21\} \times [4(42)] &= [4(3)] + [4(21)] + [3(51)] + 2[3(42)] + [3(3^2)] \\ &\quad +[2(63)] + [2(54)]\end{aligned}$$

We note that in $U(3)$ that $dim(21) = 8$ and that the weight multiplcities are 6 of multiplicity 1 and one of multiplicity 2. In the above product there are 6 irreducible representations with multiplicity 1 and one irreducible representation with multiplicity 2.

Evaluation of the product, in $SO^*(6)$,

$$\varepsilon^{-2}\{21\} \times [4(21)] = [4(0)] + [3(3)] + 2[3(21)] + [2(42)] + [2(3^2)]$$

yields fewer terms but with the multiplicities bounded by the weight multiplicities of the $U(3)$ irreducible representation as expected from the remarks associated with (8.20).

If k is sufficiently large it is possible to use (8.1) and (8.2) with (8.17) and the highly standard (8.27) and (8.29) to give for $Sp(2n, \mathbb{R})$

$$\langle\mu\rangle \times \langle\tfrac{k}{2}(\lambda)\rangle \to \sum_{\zeta,\sigma} \langle\tfrac{k}{2}(\{\overline{\zeta/\sigma}; (\mu/D\zeta) \cdot (\lambda/\sigma)\})\rangle \tag{8.35}$$

and for $SO^*(2n)$

$$[\mu] \times [k(\lambda)] \to \sum_{\zeta,\sigma} [k(\{\overline{\zeta/\sigma}; (\mu/B\zeta) \cdot (\lambda/\sigma)\})] \tag{8.36}$$

with the $\cdot$ indicating a product in $U(n)$. These two equations are valid if k is sufficiently large so as to ensure that every irreducible representation on their right-hand-sides are highly standard. The irreducible representations appearing in (8.35) and (8.36) are given in mixed tensor notation and should be converted to covariant form as indicated by (8.7) and following equations.

Alternative expressions can be obtained using (8.3) and (8.4) to give

$$\langle\mu\rangle \times \langle \tfrac{k}{2}(\lambda)\rangle \to \langle \tfrac{k}{2}((\bar{\mu}; C)_n \cdot (\lambda \cdot D)_n)\rangle \tag{8.37}$$

$$[\mu] \times [k(\lambda)] \to [k((\bar{\mu}; A)_n \cdot (\lambda \cdot B)_n)] \tag{8.38}$$

where the various products are evaluated in $U(n)$ and in each case terms in the first series are retained to weight $2\omega_\mu$ and for the second series to weight $2\omega_\lambda$ and the final terms retained to weight $\omega_\mu + \omega_\lambda$.

Still further expressions can be obtained for highly generic products by noting that if (λ) is a partition into not more than n parts and $(\lambda + k^n)$ is the partition $(\lambda_1 + k, \lambda_2 + k, \ldots, \lambda_n + k)$ then if $\langle\mu\rangle$ and $\langle\lambda + k^n\rangle$ are two finite representations of $Sp(2n)$ and their product, for sufficiently large k, is

$$\langle\mu\rangle \times \langle\lambda + k^n\rangle = \sum_\rho c^\rho_{\mu\lambda}\langle\rho\rangle$$

then for $Sp(2n,\mathbb{R})$ we have

$$\langle\mu\rangle \times \langle k(\lambda)\rangle = \sum_\rho c^\rho_{\mu\lambda}\langle\rho_n(\rho - \rho^n_n)\rangle \tag{8.39}$$

and similarly for $O(2n)$ if for sufficiently large k

$$[\mu] \times [\lambda + k^n] = \sum_\rho c^\rho_{\mu\lambda}[\rho]$$

then for $SO^*(2n)$ we obtain

$$[\mu] \times [k(\lambda)] = \sum_\rho c^\rho_{\mu\lambda}[\rho_n(\rho - \rho^n_n)] \tag{8.40}$$

The proof of these two expressions is outlined in Appendix C. It follows from (8.15) and the above that if (μ) is a staircase partition then for a given n and k the right-hand-sides of (8.39) and (8.40) are, apart from their brackets, identical.

At this stage we can remark that the symplectic group $Sp(2n)$ only has tensor representations but the previous result can be extended to the case of infinite dimensional representations where $\frac{k}{2}$ is a half-integer. It is consequence of the proof given in Appendix C that for a sufficiently large value of k, the decomposition of the product is obtained by multiplying the character at the $U(n)$ level and factorising the extra $\frac{1}{2}$ term as a square root of the determinant. In the next section we will make precise the condition that is needed for k in the two cases of $Sp(2n,\mathbb{R})$ and $SO^*(2n)$.

We continue by illustrating a result for $SO^*(6)$

$$\begin{aligned}[21]\times[987] \quad &= [11\ 97]+[11\ 8^2]+[11\ 86]+[11\ 7^2]+[10^2\ 7]+2[10\ 98]+2[10\ 96]\\ &+4[10\ 87]+[10\ 85]+2[10\ 76]+[9^3]+3[9^2\ 7]+[9^2\ 5]+3[98^2]+4[986]\\ &+3[97^2]+[975]+[96^2]+3[8^2\ 7]+[8^2\ 5]+2[876]+[7^3]\end{aligned}$$

implying that

$$\begin{aligned}[21]\times[7(21)] \quad &= [9(0)]+[8(3)]+2[8(21)]+3[8(1)]+[7(42)]+[7(4)]+[7(3^2)]\\ &+4[7(31)]+3[7(2^2)]+3[7(2)]+3[7(1^2)]+[7(0)]+[6(52)]+2[6(43)]\\ &+2[6(41)]+4[6(32)]+[6(3)]+2[6(21)]+[5(53)]+[5(4^2)]+[5(42)]\\ &+[5(3^2)]\end{aligned}$$

as is indeed the case.

Suppose from (8.1) it is determined that

$$\langle\mu\rangle \to \sum_{\tau,\mu} g^{\tau\mu}\{\overline{\tau};\mu\} = \sum_{\tau,\mu} g^{\tau\mu}\varepsilon^{-\tau_1}\{\rho\} \tag{8.41}$$

where $\varepsilon^{-\tau_1}\{\rho\}$ is the covariant irreducible representation of $U(n)$ derived from $\{\overline{\tau};\mu\}$ via (8.7) to (8.9) and similarly from (8.2) that

$$[\mu] \to \sum_{\tau,\mu} g^{\tau\mu}\{\overline{\tau};\mu\} = \sum_{\tau,\mu} g^{\tau\mu}\varepsilon^{-\tau_1}\{\rho\} \tag{8.42}$$

where the $g^{\tau\mu}$ are non-negative integers. For sufficiently large k we have for $Sp(2n,\mathbb{R})$

$$\begin{aligned}\langle\mu\rangle\times\langle\tfrac{k}{2}(\lambda)\rangle &= \textstyle\sum_{\tau,\nu} g^{\tau\nu}\langle\tfrac{k}{2}-\tau_1(\rho\cdot\lambda)\rangle\\ &= \textstyle\sum_{\tau,\nu,\alpha} g^{\tau\nu}c^{\alpha}_{\rho\lambda}\langle\tfrac{k}{2}-\tau_1(\alpha)\rangle\end{aligned} \tag{8.43}$$

and similarly for $SO^*(2n)$

$$[\mu]\times[k(\lambda)] = \sum_{\tau,\nu\alpha} g^{\tau,\nu}c^{\alpha}_{\rho\lambda}[k-\tau_1(\alpha)] \tag{8.44}$$

where the $c^{\alpha}_{\rho\lambda}$ are non-negative Littlewood-Richardson coefficients with $\ell_\alpha \le n$

Now our problem is to determine the sufficiently large value of k occurring in (8.35) to (8.44) that ensures that every irreducible representation is highly standard.

8.4 Generic products

Here we consider the tensor product of a finite dimensional non-unitary irreducible representation with an infinite dimensional unitary irreducible representation. In general the problem of determining the content of an arbitrary tensor product of a finite dimensional non-unitary irreducible representation with an infinite dimensional unitary irreducible representation is complicated by the diversity of representations of non-compact groups. In practical applications there is a class of products that are of great importance, namely the class of generic products. A product is said to be *generic* if it is fully resolvable into just a finite series of standard unitary irreducible representations of the relevant non-compact Lie group. A product will be said to be *highly generic* if it is fully resolvable into highly standard unitary irreducible representations. In most of this paper we shall be considering the highly generic case.

We now take up the problem of determining the highly generic conditions, first for $SO^*(2n)$. Consider the $SO^*(2n)$ product $[\mu] \times [k(\lambda)]$ where $[k(\lambda)]$ is a highly standard irreducible representation and $[\mu]$ is a finite non-unitary irreducible representation. Under $SO^*(2n) \to U(n)$ the irreducible representation $[\mu]$ certainly yields a lowest weight $U(n)$ irreducible representation $\{\bar{\mu}\}$ such than in a covariant basis

$$\{\bar{\mu}\} = \varepsilon^{-\mu_1} \cdot \{\mu_1^{n-\ell_\mu}, \nu\} = \varepsilon^{-\mu_1} \cdot \{\mu_c\} \tag{8.45}$$

where ℓ_ν is the number of $\mu_i \neq \mu_1$. If the product is highly generic then certainly

$$[\mu] \times [k(\lambda)] \supset \varepsilon^{k-\mu_1} \cdot (\mu_c \cdot \lambda \cdot B)_n = [k - \mu_1(\mu_c \cdot \lambda)] \tag{8.46}$$

The product will be assuredly highly generic if every term on the right-hand-side of (8.46) is highly standard.

Let us write

$$[k - \mu_1(\mu_c \cdot \lambda)] = \sum_\rho [k - \mu_1(\rho)] \tag{8.47}$$

We seek the smallest value of k that ensures that every $[k - \mu_1(\rho)]$ is highly standard. If $\rho_1' = n$ then $[k - \mu_1(\rho)] \equiv [k - \mu_1 + \rho_n(\rho - \rho_n)]$ which is of higher weight than terms with $\rho_1' < n$. The worst case scenario is when $\rho_1' = n - 1$. In that case (8.28) leads to

$$k \geq n + \mu_1 - 1 \tag{8.48}$$

In the general case the minimum value of k depends upon the longest term in $(\mu_c \cdot \lambda)$. Setting

$$\ell = \min(n - 1, n - \ell_\mu + \ell_\nu + \ell_\lambda) \tag{8.49}$$

gives, for $2k \geq n$ the condition for highly standardness as

$$k \geq n + \mu_1 + \frac{\ell - n - 1}{2} \tag{8.50}$$

If $n > 2k$ then necessarily we need to have for each term in the product $\rho'_1 \leq 1$ and $[k(\rho)] = [k(n)]$ which is highly standard for all k and n. But the only possibility to have such terms in the expansion is if $\mu = 0$ and $\lambda = n$.

Now consider the case of $Sp(2n, \mathbb{R})$. The condition for highly standardness becomes, from (8.25),

$$k + 1 - \rho'_2 > \min(n, k) \tag{8.51}$$

Two cases arise.

Firstly, if $k \geq n$ then

$$\frac{k}{2} \geq n + \mu_1 + \frac{\ell - n - 1}{2} \tag{8.52}$$

where ℓ is as defined in (8.52).

Secondly, if $n > k$ then necessarily for each term in the product we need to have $\rho'_2 = 0$ and $\langle k(\rho) \rangle = \langle k(1^n) \rangle$. Terms of this form are highly standard for all k and n but the only possibility to obtain such terms in the product is if $\mu = 0$ and $\lambda = 1^n$.

We continue with an example. Consider the $SO^*(6)$ finite times infinite product $[1] \times [3(21)]$. Under $SO^*(6) \to U(3)$

$$[1] \to \{\bar{1}; 0\} + \{0; 1\} = \varepsilon^{-1}\{1^2\} + \varepsilon^0\{1\} \tag{8.53}$$

and since $[3(21)]$ is highly standard we have from (8.29)

$$[3(21)] \to (\varepsilon^3 \cdot \{21\} \cdot B_3)_3 \tag{8.54}$$

Thus at the $U(3)$ level we have the product of (8.53) with (8.54) leading to

$$\begin{aligned}
[1] \times [3(21)] \to\ & ((\{\bar{1}; 0\} + \{\bar{0}; 1\}) \cdot (\varepsilon^3 \cdot \{21\} \cdot B))_3 \\
&= ((\varepsilon^{-1}\{1^2\} + \varepsilon^0\{1\}) \cdot (\varepsilon^2 \cdot \{21\}) \cdot B) \\
&= ((\varepsilon^2(\{32\} + \{31^2\} + \{2^21\}) + \varepsilon^3(\{2^2\} + \{31\} + \{21^2\}) \cdot B)_3 \\
&= ((\varepsilon^4\{1\} + \varepsilon^3(\{2^2\} + \{31\} + \{2\} + \{1^2\}) + \varepsilon^2\{32\}) \cdot B)_3
\end{aligned} \tag{8.55}$$

The result of (8.55) may be inverted using the $A-$series which is the inverse of the $B-$series ([8, 55]) to give the final result

$$[1] \times [3(21)] \to [4(1)] + [3(2^2)] + [3(31)] + [3(2)] + [3(1^2)] + [2(32)]$$

Note that the irreducible representations in the result are all highly standard and hence the product is highly generic. That the result is highly generic could have been predicted from (8.48).

The above example illustrates a general algorithm for evaluating generic products for $SO^*(2n)$.

Algorithm I. To evaluate the generic product $[\mu] \times [k(\lambda)]$

1. *Perform the decomposition of the irreducible representation $[\mu]$ under $SO^*(2n) \to U(n)$ using (2.2) and convert the $U(n)$ irreducible representations into their covariant form using (2.5) and (2.6).*

2. *Perform the decomposition of the irreducible representation* $[k(\lambda)]$ *under* $SO^*(2n) \to U(n)$ *using (3.8).*

3. *Form the tensor product of the above two sets of* $U(n)$ *irreducible representations to give a list of irreducible representations each of the form* $\varepsilon^p\{\rho\}$.

4. *Use (2.8) to reduce every partition of* n *non-zero parts to give an equivalent* $U(n)$ *irreducible representation involving a partition into fewer than* n *non-zero parts.*

5. *Replace every* $\varepsilon^p\{\rho\}$ *by* $[p(\rho)]$ *to give the final* $SO^*(2n)$ *content of the fip.*

The corresponding algorithm for evaluating $Sp(2n,\mathbb{R})$ fips follows in the same manner.

Algorithm II. To evaluate the generic product $\langle\mu\rangle \times \langle\frac{1}{2}k(\lambda)\rangle$

1. *Perform the decomposition of the irreducible representation* $\langle\mu\rangle$ *under* $Sp(2n,\mathbb{R}) \to U(n)$ *using (2.1) and convert the* $U(n)$ *irreducible representations into their covariant form using (2.5) and (2.6).*

2. *Perform the decomposition of the irreducible representation* $\langle\frac{1}{2}k(\lambda)\rangle$ *under* $Sp(2n,\mathbb{R}) \to U(n)$ *using (3.6).*

3. *Form the tensor product of the above two sets of* $U(n)$ *irreducible representations to give a list of irreducible representations each of the form* $\varepsilon^p\{\rho\}$.

4. *Use (2.8) to reduce every partition of* n *non-zero parts to give an equivalent* $U(n)$ *irreducible representation involving a partition into fewer than* n *non-zero parts.*

5. *Replace every* $\varepsilon^p\{\rho\}$ *by* $\langle p(\rho)\rangle$ *to give the final* $Sp(2n,\mathbb{R})$ *content of the fip.*

If a product is highly generic then the algorithms yield the same results as the simpler and more direct methods outlined in §3. If the product is not highly generic caution must be exercised. In that case the algorithm *may* yield a complete resolution into characters of the positive discrete series that are all, at least, standard. However, situations can arise where the irreducible representations at the $U(n)$ level cannot be inverted to yield standard irreducible representations. For example, attempting to evaluate the product $[1] \times [3(1)]$ which is certainly not highly generic in $SO^*(8)$ fails since the lowest weight $U(4)$ irreducible representation $\varepsilon^2\{21^2\}$ cannot be inverted to give a standard irreducible representation of $SO^*(8)$.

Consider now $Sp(8,\mathbb{R})$ and the products $\langle 1\rangle \times \langle 3(1)\rangle$ and $\langle 1\rangle \times \langle 3(2)\rangle$ which are certainly not highly generic. Using algorithm II we obtain

$$\begin{aligned} \langle 1\rangle \times \langle 3(1)\rangle &= 2\langle 3(1^2)\rangle + \langle 3(2)\rangle + \langle 3(0)\rangle + \langle 2(21^2)\rangle \\ \langle 1\rangle \times \langle 3(2)\rangle &= 2\langle 3(21)\rangle + \langle 3(3)\rangle + \langle 3(1)\rangle + \langle 2(31^2)\rangle \end{aligned} \tag{8.56}$$

where $\langle 2(21^2)\rangle$ and $\langle 2(31^2)\rangle$ are standard but not highly standard. In this case while in each case algorithm II holds (8.50) does not. These products are generic but not highly generic.

The results of (8.56) can be given an alternative derivation that illustrates some of the problems associated with products that are generic but not highly generic. Consider the product $\langle 1\rangle \times \langle 3(1)\rangle$ in $Sp(8,\mathbb{R})$. Use of (8.1) and (8.23) leads to

$$\begin{aligned}\langle 1\rangle \times \langle 3(1)\rangle &\to (\varepsilon^0\{1\}+\varepsilon^{-1}\{1^3\})\cdot\varepsilon^3(1\cdot D)_4) \\ &= \varepsilon^3(((2\cdot D)_4+(1^2\cdot D)_4)+\varepsilon^2\left((1^4\cdot D)_4+(21^2\cdot D)_4\right) \\ &= \langle 3(2)\rangle+\langle 3(1^2)\rangle+\langle 3(0)\rangle+\varepsilon^2\{21^2\cdot D\}_4\end{aligned} \tag{8.57}$$

but from (8.23)

$$\begin{aligned}\langle 2(21^2)\rangle &\to \varepsilon^2\left\{(21^2-2^21^2)\cdot D\right\}_4 \\ &= \varepsilon^2\{21^2\cdot D\}_4-\varepsilon^3\{1^2\cdot D\}_4\end{aligned}$$

allowing us to recover the result of (8.56).

8.5 Bounded multiplicities

We saw in §2 that for a given $U(n)$ covariant irreducible representation $\{\mu\}$ the largest possible number of $U(n)$ irreducible representations arise in its product with a covariant irreducible representation $\{\tau\}$ when (8.20) is satisfied and in that case the number of terms is $dim(\mu)$ and that the irreducible representation multiplicities occurring in such a product are precisely those of the weight multiplicities associated with $\{\mu\}$. A given finite irreducible representation of $Sp(2n,\mathbb{R})$ or $SO^*(2n)$ can be expanded as a sum of covariant irreducible representations of $U(n)$. The largest part ρ_1 of any covariant irreducible representation $\{\rho\}$ arising from $\langle\mu\rangle$ or $[\mu]$ is (8.19)

$$\rho_1=\mu_1+\mu_2$$

It follows that for the highly generic products $\langle\mu\rangle\times\langle\frac{k}{2}(\lambda)\rangle$ and $[\mu]\times[k(\lambda)]$ if the partition (λ) satisfies the constraint

$$\lambda_i-\lambda_{i+1}\geq\mu_1+\mu_2 \quad i=1,2,\ldots,n-1 \tag{8.58}$$

then the sum of the irreducible representation multiplicities in the product will be $dim(\mu)$ and for any (λ) these multiplicities will be bounded by weight multiplicities of $\langle\mu\rangle$ or $[\mu]$. This is a consequence of the proof given in appendix C, where the multiplicities are explicitily written in term of the characters of $U(n)$ and $Sp(2n)$ which after modifications in formulae (C5) and (C6) coincide for a sufficiently large value of k.

Consider the highly generic product $[31]\times[6(12\ 84)]$ in $SO^*(8)$. We find $dim(31)=567$ in $SO^*(8)$ and that the [31] irreducible representation has 72 distinct weights that occur with multiplicity 1, 96 with multiplicity 2, 16 with multiplicity 3, 8 with multiplicity 6, 24 with multiplicity 8 and 1 with multiplicity 15. Thus, for example, in the $SO^*(8)$ product $[31]\times[6(12\ 84)]$ we find, the irreducible representation $[6(12\ 84)]$ occurs with a multiplicity of 15 while the 8 irreducible representations

$$\begin{aligned}&[8(10\ 62)]+[6(14\ 84)]+[6(12\ 10\ 4)]+[6(12\ 86)]\\ &+[6(12\ 82)]+[6(12\ 64)]+[6(10\ 84)]+[4(14\ 10\ 6)]\end{aligned}$$

all occur with multiplicity 8 etc. That is, the irreducible representation multiplicity has the same multiplicity distribution as that of the weight multiplicities of [31].

8.6 Stabilisation of products

The preceding work has made it possible to readily calculate highly generic tensor products for the group $SO^*(2n)$. The results exhibit certain stabilisation properties. Two particular types of stabilisation can occur. One can derive this result easily from examination of equation (C6) and we will now interpret it in terms of the Littlewood-Richardson rule.

Firstly, we note that if the product is highly generic we can obtain an infinite set of products from that productby increasing k by an integer or half-integer x. This follows by noting that in such a case the value of ε^k is incremented while the Schur function products are unchanged. Thus we can write

$$C^{[p(\rho)]}_{[\mu],[k(\lambda)]} = C^{[p+x(\rho)]}_{[\mu],[k+x(\lambda)]} \tag{8.59}$$

Secondly, we can consider a form of stabilisation, where k remains constant, as exemplified in (8.54), and the product is at least generic. Let us introduce the notation $(\lambda + h) = (\lambda_1 + h, \lambda_2, \dots)$ where h is an integer. Consider the Schur function product

$$\{\lambda\} \cdot \{\mu\} = \sum C^{\rho}_{\lambda\mu}\{\rho\}$$

where the $C^{\rho}_{\lambda\mu}$ are the usual Littlewood-Richardson coefficients. Increasing the first part of λ by h corresponds to increasing the difference $\lambda_1 - \lambda_2$ to h. But rewriting the new product in terms of Yamanouchi words is equivalent to putting h more 1's in the first row and this is only possible if $\mu_1 \geq h$. Otherwise we must add some 2's, 3's etc.. to the other rows. This has the consequence that

$$C^{[p(\rho)]}_{[\mu],[k(\lambda)]} = C^{[p(\rho+h)]}_{[\mu],[k(\lambda+h)]} \tag{8.60}$$

with the condition that $\lambda_1 - \lambda_2 \geq \rho_{1max}$, where ρ_{1max} corresponds to the maximal first part in the list of the covariant representations obtained using (8.1). For this reason the condition (8.60) is available with the condition that $\lambda_1 - \lambda_2 \geq \mu_1 + \mu_2$. To obtain stabilisation of a product we need to apply the relation (8.58) for $i = 1$. It is the only way to stabilize in the sense that we will preserve the multiplicity of the irreps arising in the product by increasing the first part of λ. We can illustrate these results by four examples obtained for $Sp(6, \mathbb{R})$.

$$\begin{array}{lllll}
\langle 21\rangle \times \langle 5(31)\rangle = & & & & \\
\langle 7(1)\rangle & +\langle 6(4)\rangle & +2\langle 6(31)\rangle & +\langle 6(2^2)\rangle & +3\langle 6(2)\rangle \\
+2\langle 6(1^2)\rangle & +\langle 6(0)\rangle & +\langle 5(52)\rangle & +\langle 5(5)\rangle & +\langle 5(43)\rangle \\
+4\langle 5(41)\rangle & +4\langle 5(32)\rangle & +3\langle 5(3)\rangle & +4\langle 5(21)\rangle & +\langle 5(1)\rangle \\
+\langle 4(62)\rangle & +\ 2\langle 4(53)\rangle & +\ 2\langle 4(51)\rangle & +\langle 4(4^2)\rangle & +4\langle 4(42)\rangle \\
+\langle 4(4)\rangle & +2\langle 4(3^2)\rangle & +2\langle 4(31)\rangle & +\langle 4(2^2)\rangle & +\langle 3(63)\rangle \\
+\langle 3(54)\rangle & +\langle 3(52)\rangle & +\langle 3(43)\rangle & &
\end{array} \tag{8.61}$$

$$\begin{array}{lllll}
\langle 21\rangle \times \langle 6(31)\rangle = & & & & \\
\langle 8(1)\rangle & +\langle 7(4)\rangle & +2\langle 7(31)\rangle & +\langle 7(2^2)\rangle & +3\langle 7(2)\rangle \\
+2\langle 7(1^2)\rangle & +\langle 7(0)\rangle & +\langle 6(52)\rangle & +\langle 6(5)\rangle & +\langle 6(43)\rangle \\
+4\langle 6(41)\rangle & +4\langle 6(32)\rangle & +3\langle 6(3)\rangle & +4\langle 6(21)\rangle & +\langle 6(1)\rangle \\
+\langle 5(62)\rangle & +2\langle 5(53)\rangle & +2\langle 5(51)\rangle & +\langle 5(4^2)\rangle & +4\langle 5(42)\rangle \\
+\langle 5(4)\rangle & +2\langle 5(3^2)\rangle & +2\langle 5(31)\rangle & +\langle 5(2^2)\rangle & +\langle 4(63)\rangle \\
+\langle 4(54)\rangle & +\langle 4(52)\rangle & +\langle 4(43)\rangle & &
\end{array} \tag{8.62}$$

$$\begin{array}{lllll}
\langle 21\rangle \times \langle 5(41)\rangle = & & & & \\
\langle 7(2)\rangle & +\langle 6(5)\rangle & +2\langle 6(41)\rangle & +\langle 6(32)\rangle & +3\langle 6(3)\rangle \\
+2\langle 6(21)\rangle & +\langle 6(1)\rangle & +\langle 5(62)\rangle & +\langle 5(6)\rangle & +\langle 5(53)\rangle \\
+4\langle 5(51)\rangle & +4\langle 5(42)\rangle & +3\langle 5(4)\rangle & +\langle 5(3^2)\rangle & +4\langle 5(31)\rangle \\
+\langle 5(2^2)\rangle & +\langle 5(2)\rangle & +\langle 4(72)\rangle & +2\langle 4(63)\rangle & +2\langle 4(61)\rangle \\
+\langle 4(54)\rangle & +4\langle 4(52)\rangle & +\langle 4(5)\rangle & +2\langle 4(43)\rangle & +2\langle 4(41)\rangle \\
+\langle 4(32)\rangle & +\langle 3(73)\rangle & +\langle 3(64)\rangle & +\langle 3(62)\rangle & +\langle 3(53)\rangle
\end{array} \tag{8.63}$$

$$\begin{array}{lllll}
\langle 21\rangle \times \langle 5(51)\rangle = & & & & \\
\langle 7(3)\rangle & +\langle 6(6)\rangle & +2\langle 6(51)\rangle & +\langle 6(42)\rangle & +3\langle 6(4)\rangle \\
+2\langle 6(31)\rangle & +\langle 6(2)\rangle & +\langle 5(72)\rangle & +\langle 5(7)\rangle & \langle 5(63)\rangle \\
+4\langle 5(61)\rangle & +4\langle 5(52)\rangle & +3\langle 5(5)\rangle & +\langle 5(43)\rangle & 4\langle 5(41)\rangle \\
+\langle 5(32)\rangle & +\langle 5(3)\rangle & +\langle 4(82)\rangle & +2\langle 4(73)\rangle & +2\langle 4(71)\rangle \\
+\langle 4(64)\rangle & +4\langle 4(62)\rangle & +\langle 4(6)\rangle & +2\langle 4(53)\rangle & +2\langle 4(51)\rangle \\
+\langle 4(42)\rangle & +\langle 3(83)\rangle & +\langle 3(74)\rangle & +\langle 3(72)\rangle & +\langle 3(63)\rangle
\end{array} \tag{8.64}$$

The examples illustrate the different stability properties. Examples (8.61) and (8.62) are an illustration of the property (8.59). Examples (8.61) and (8.62) show a situation where the rule (8.60) does not hold while (8.63) and (8.64) respect that particular condition.

8.7 Relationships between $Sp(2n,\mathbb{R})$ and $SO^*(2n)$ products

Recalling (8.12) and that $WB = D$ and $VD = B$ we can establish the product equivalences for the multiplicities

$$C^{[p(\rho)]}_{[\mu/W],[k(\lambda)]} = C^{\langle p(\rho)\rangle}_{\langle\mu\rangle,\langle k(\lambda)\rangle} \tag{8.65}$$

$$C^{\langle p(\rho)\rangle}_{\langle\mu/V\rangle,\langle k(\lambda)\rangle} = C^{[p(\rho)]}_{[\mu],[k(\lambda)]} \tag{8.66}$$

In the special case of (μ) being a staircase partition we recall (8.15).

8.8 Special products

Evaluation of (G3.12) for various special cases of interest leads, in the case of $SO^*(2n)$, to the typical results

$$\begin{aligned}[1] \times [k(m)] &= [k(m+1)] + [k(m,1)] + [k(m-1)] \\ &+ [k-1(m+1,1^{n-2})] \end{aligned} \tag{8.67}$$

$$\begin{aligned}[2] \times [k(m)] &= [k(m+2)] + [k(m+1,1)] + [k(m,2)] + [k(m)] \\ &+ [k(m-1,1)] + [k(m-2)] + [k-1(m+2,1^{n-2})] + [k-1(m,1^{n-2})] \\ &+ [k-1(m+1,21^{n-3})] + [k-2(m+2,2^{n-2})] \end{aligned} \tag{8.68}$$

$$\begin{aligned}[1^2] \times [k(m)] &= [k(m+1,1)] + [k(m,1^2)] + 2[k(m)] + [k(m-1)] \\ &+ [k(m-1,1)] + [k-1(m+2,1^{n-2})] + [k-1(m+1,21^{n-3})] \\ &+ [k-1(m+1,1^{n-2})] \end{aligned} \tag{8.69}$$

where the highly standard conditions are met in (8.67) and (8.69) with $k > n$ and for (8.68) with $k > n+1$. For small values of m non-standard partitions arise and the standard Littlewood modification rules [8] must be used.

The corresponding results for $Sp(2n,\mathbb{R})$ can be obtained directly from the above results by noting that

$$[1/W] = [1], \quad [2/W] = [2] + [0], \quad [1^2/W] = [1^2] - [0] \tag{8.70}$$

to give

$$\begin{aligned}\langle 1\rangle \times \langle k(m)\rangle &= \langle k(m+1)\rangle + \langle k(m,1)\rangle + \langle k(m-1)\rangle \\ &+ \langle k-1(m+1,1^{n-2})\rangle \end{aligned} \tag{8.71}$$

$$\begin{aligned}\langle 2\rangle \times \langle k(m)\rangle &= \langle k(m+2)\rangle + \langle k(m+1,1)\rangle + \langle k(m,2)\rangle + 2\langle k(m)\rangle \\ &+ \langle k(m-1,1)\rangle + \langle k(m-2)\rangle + \langle k-1(m+2,1^{n-2})\rangle + \langle k-1(m,1^{n-2})\rangle \\ &+ \langle k-1(m+1,21^{n-3})\rangle + \langle k-2(m+2,2^{n-2})\rangle \end{aligned} \tag{8.72}$$

$$\begin{aligned}\langle 1^2\rangle \times \langle k(m)\rangle &= \langle k(m+1,1)\rangle + \langle k(m,1^2)\rangle + \langle k(m)\rangle + \langle k(m-1)\rangle \\ &+ \langle k(m-1,1)\rangle + \langle k-1(m+2,1^{n-2})\rangle + \langle k-1(m+1,21^{n-3})\rangle \\ &+ \langle k-1(m+1,1^{n-2})\rangle \end{aligned} \tag{8.73}$$

The above results can be extended to $Sp(2n,\mathbb{R})$ irreducible representations involving half-integers $\langle k+\frac{1}{2}(\lambda)\rangle$ by simply making the replacement

$$k \to k + \tfrac{1}{2} \tag{8.74}$$

in (8.71) to (8.73).

For unitary irreducible representations of $SO^*(2n)$ involving two-part partitions we readily find that for $m > p$,

$$\begin{aligned}
[1] \times [k(m,p)] &= [k(m+1,p)] + [k(m,p+1)] + [k(m,p,1)] + [k(m,p-1)] \\
&+ [k(m-1,p)] + [k-1(m+1,p+1,1^{n-3})]
\end{aligned} \tag{8.75}$$

and with $m > p+1$,

$$\begin{aligned}
[2] \times [k(m,p)] &= [k(m+2,p)] + [k(m+1,p+1)] + [k(m+1,p,1)] + [k(m,p+2)] \\
&+ [k(m,p+1,1)] + [k(m+1,p-1)] + [k(m,p,2)] + 2[k(m,p)] \\
&+ [k(m,p-2)] + [k(m-1,p+1)] + [k(m-1,p-1)] + [k(m,p-1,1)] \\
&+ [k(m-1,p,1)] + [k(m-2,p)] + [k-1(m+2,p+1,1^{n-3})] \\
&+ [k-1(m+1,p+2,1^{n-3})] + [k-1(m+1,p+1,2,1^{n-4})] \\
&+ [k-1(m+1,p,1^{n-3})] + [k-1(m,p+1,1^{n-3})] \\
&+ [k-2(m+2,p+2,2^{n-3})]
\end{aligned} \tag{8.76}$$

$$\begin{aligned}
[1^2] \times [k(m,p)] &= [k(m+1,p+1)] + [k(m+1,p-1)] + [k(m+1,p,1)] + 3[k(m,p)] \\
&+ [k(m,p+1,1)] + [k(m,p-1,1)] + [k(m,p,1^2)] + [k(m-1,p+1)] \\
&+ [k(m-1,p-1)] + [k(m-1,p,1)] + [k-1(m+2,p+1,1^{n-3})] \\
&+ [k-1(m+1,p+1,1^{n-4})] + [k-1(m+1,p+2,1^{n-3})] \\
&+ [k-1(m+1,p+1,2,1^{n-4})] \\
&+ [k-1(m+1,p,1^{n-3})] + [k-1(m,p+1,1^{n-3})]
\end{aligned} \tag{8.77}$$

Again the corresponding results for $Sp(2n,\mathbb{R})$ follow trivially from use of (8.70) in the above results. In the case of $Sp(2n,\mathbb{R})$ irreducible representations involving half-integers we also note (8.74).

Setting $p = 0$ in (8.75) to (8.77) followed by application of the Littlewood modification rules to all non-standard partitions recovers the results of (8.67) to (8.69) respectively.

The results (8.67) to (8.77) are of particular relevance to the practical calculation of matrix elements of basic tensor operators. Recall that for $SO^*(2n)$ the group generators span the finite irreducible representation $[1^2]$ while for $Sp(2n,\mathbb{R})$ they span $\langle 2\rangle$. The irreducible representations $[1]$ and $\langle 1\rangle$ correspond respectively to their vector representations.

More generally we can establish some formulae for products involving only one part partitions, which we shall denote by u and v. In that case (8.36) undergoes a substantial simplification since now only the first term of the $B-$series, $\{0\}$, survives and thus for $SO^*(2n)$

$$[u] \times [k(v)] = \sum_{\zeta,\sigma} [k(\{\overline{\zeta/\sigma}; u/\zeta \cdot v/\sigma\})] \tag{8.78}$$

Since we are dealing only with one part partitions we have

$$\begin{aligned}
[u]\times[k(v)] &= \sum_{p=0}^{u}\sum_{q=0}^{min(v,p)}[k(\overline{p-q};u-p\cdot v-q)] \\
&= \sum_{p=0}^{u}\sum_{q=0}^{min(v,p)}\sum_{r=0}^{min(u-p,v-q)}[k(\overline{p-q};u+v-p-q-r,r)] \\
&= \sum_{p=0}^{u}\sum_{q=0}^{min(v,p)}\sum_{r=0}^{min(u-p,v-q)}[k-p+q(u+v-2q-r,p-q+r,(p-q)^{n-3})]
\end{aligned} \tag{8.79}$$

Now consider the product $\langle u\rangle\times\langle\frac{k}{2}(v)\rangle$ for $Sp(2n,\mathbb{R})$. We can obtain this result making use of equation (8.12) which for one part partitions is equivalent to

$$\langle u\rangle\to[u/W]=\sum_{p=0}^{\lfloor\frac{u}{2}\rfloor}[u-2p] \tag{8.80}$$

Using (8.79) and (8.80) we can derive the result

$$\langle u\rangle\times\langle\tfrac{k}{2}(v)\rangle=\sum_{x=0}^{\lfloor\frac{u}{2}\rfloor}\sum_{p=0}^{u}\sum_{q=0}^{min(v,p)}\sum_{r=0}^{min(u-2x-p,v-q)}\langle F(p,q,r,x)\rangle \tag{8.81}$$

where

$$F(p,q,r,x)=\langle\tfrac{k}{2}-p+q(u+v-2x-2q-r,p-q+r,(p-q)^{n-3})\rangle.$$

Now consider the case where the finite irreducible representation involves a column partition. The product $\langle 1^p\rangle\times\langle\frac{k}{2}(m)\rangle$ may be expanded as

$$\begin{aligned}
\langle 1^p\rangle\times\langle\tfrac{k}{2}(m)\rangle &= \sum_{\zeta,\sigma}\langle\tfrac{k}{2}(\overline{\zeta/\sigma};1^p/\zeta.m/\sigma)\rangle \\
&= \sum_{s,\sigma}\langle\tfrac{k}{2}(\overline{1^s/\sigma};1^{p-s}.m/\sigma)\rangle \\
&= \sum_{s}\langle\tfrac{k}{2}(\overline{1^s};1^{p-s}\cdot m)\rangle+\sum_{s}\langle\tfrac{k}{2}(\overline{1^{s-1}};1^{p-s}\cdot m-1)\rangle
\end{aligned} \tag{8.82}$$

Each term of the form $\langle\frac{k}{2}(\overline{1^s};m.1^{p-s})\rangle$ can be rewritten as

$$\langle\tfrac{k}{2}(\overline{1^s};m+1,1^{p-s-1})\rangle+\langle\tfrac{k}{2}(\overline{1^s};m,1^{p-s})\rangle$$

and using (8.8) to transform the mixed tensor irreducible representations into covariant representations gives except for $s=0,p,p-1$

$$\langle\tfrac{k}{2}-1(m+2,2^{p-s-1},1^{n-p})\rangle+\langle\tfrac{k}{2}-1(m+1,2^{p-s},1^{n-p-1})\rangle$$

For $s=0$ we obtain $\langle \frac{k}{2}(m+1,1^{p-1})\rangle + \langle \frac{k}{2}(m,1^p)\rangle$. If $s=p-1$ then we can transform $\langle \frac{k}{2}(\overline{1^s};m+1,1^{p-s-1})\rangle$ to its covariant form $\langle \frac{k}{2}-1(m+2,1^{n-p})\rangle$ and likewise if $s=p$ then $\langle \frac{k}{2}(\overline{1^s};m,1^{p-s})\rangle$ is transformed to $\langle \frac{k}{2}-1(m+1,1^{n-p-1})\rangle$.

We can now rewrite (8.82) as

$$\begin{aligned} S_{p,m} &= \langle 1^p\rangle \times \langle \tfrac{k}{2}(m)\rangle \\ &= \langle \tfrac{k}{2}(m+1,1^{p-1})\rangle + \langle \tfrac{k}{2}(m,1^p)\rangle + \langle \tfrac{k}{2}(m-1,1^{p-1})\rangle + \langle \tfrac{k}{2}(m,1^{p-2})\rangle \\ &+ \langle \tfrac{k}{2}-1(m+2,1^{n-p})\rangle + \langle \tfrac{k}{2}-1(m+1,1^{n-p+1})\rangle \\ &+ \langle \tfrac{k}{2}-1(m+1,1^{n-p-1})\rangle + \langle \tfrac{k}{2}-1(m,1^{n-p})\rangle \\ &+ \sum_{s=1}^{p-2}\langle \tfrac{k}{2}-1(m+2,2^{p-s-1},1^{n-p})\rangle + \sum_{s=2}^{p-2}\langle \tfrac{k}{2}-1(m+1,2^{p-s-1},1^{n-p+1})\rangle \\ &+ \sum_{s=2}^{p-1}\langle \tfrac{k}{2}-1(m,2^{p-s},1^{n-p})\rangle + \sum_{s=1}^{p-1}\langle \tfrac{k}{2}-1(m+1,2^{p-s},1^{n-p-1})\rangle \end{aligned} \tag{8.83}$$

We can derive from this formula the corresponding result for the group $SO^*(2n)$ using (8.13) to give

$$\begin{aligned} [1^p]\times[k(m)] &= [\langle 1^p/V\rangle \times \langle k(n)\rangle] \\ &= \sum_{r=0}^{\lfloor \frac{p}{2}\rfloor}\langle 1^{p-2r}\rangle \times \langle k(m)\rangle \end{aligned}$$

and hence

$$[1^p]\times[k(m)] = \sum_{r=0}^{\lfloor \frac{p}{2}\rfloor} S_{p,m} \tag{8.84}$$

8.9 Concluding remarks

The aim of this chapter has been to obtain a series of results for the practical evaluation of certain generic products of finite and infinite representations of $Sp(2n,\mathbb{R})$ and $SO^*(2n)$. In the case of highly generic products there is a considerable simplification since by the avoiding of irreducible representations associated with signed sequences it becomes unnecessary to consider branching rules for the infinite dimensional irreducible representations, effectively reducing the entire problem to applications of the Littlewood-Richardson rule with certain constraints. In practical applications these are the irreducible representations of greatest interest.

The results exhibit certain stabilisation properties which make it possible to deduce the form of an infinite set of generic products from a particular minimal product. The conditions for stabilisation have been specified.

The results presented herein represent a further step in the simplification of the calculation of the matrix elements of tensor operators as required in symplectic models of many-particle systems.

While specific applications of the group $SO^*(2n)$ seem to be rare it is useful to note the mappings that exist between that group and the group $Sp(2n, \mathbb{R})$.

8.10 Appendix A : Spin branching rules

Morris ([75, 76]) has derived for the basic spin irreducible representations of the orthogonal groups the following branching rules

$$\begin{aligned} O(4nm) &\downarrow O(2n) \times O(2m) \\ \Delta &\downarrow \sum_{\lambda} [m^n/\lambda'] \times [\lambda] \end{aligned} \tag{8.85}$$

$$\begin{aligned} O(4nm+2m) &\downarrow O(2n+1) \times O(2m) \\ \Delta &\downarrow \sum_{\lambda} [m^n/\lambda'] \times [\Delta;\lambda] \end{aligned} \tag{8.86}$$

$$\begin{aligned} O(4nm+2n+2m+1) &\downarrow O(2n+1) \times O(2m+1) \\ \Delta &\downarrow \sum_{\lambda} [\Delta; m^n/\lambda'] \times [\Delta;\lambda] \end{aligned} \tag{8.87}$$

$$\tag{8.88}$$

King [49] conjectured that

$$O(4nm) \downarrow Sp(2n) \times Sp(2m) \quad \Delta \downarrow \sum_{\lambda} \langle m^n/\lambda' \rangle \times \langle \lambda \rangle \tag{8.89}$$

This result can be obtained from (8.85). We first establish an $S-$function identity. Let $X = \sum_{\zeta}\{\zeta\}$ (with repetitions of ζ if necessary) be any $S-$function series and Y a second such series. Then

$$\begin{aligned} \sum_{\lambda} \{\rho/\lambda X\} \times \{\lambda'/Y\} &= \sum_{\lambda,\zeta} \{\rho/\lambda\zeta\} \times \{\lambda'/Y\} \\ &= \sum_{\lambda,\zeta,\mu} c^{\mu}_{\lambda\zeta} \{\rho/\mu\} \times \{\lambda'/Y\} \\ &= \sum_{\lambda,\zeta,\mu} \{\rho/\mu\} \times c^{\mu'}_{\lambda'\zeta'} \{\lambda'/Y\} \\ &= \sum_{\lambda,\mu} \{\rho/\mu\} \times \{(\mu'/\zeta')/Y\} \\ &= \sum_{\mu} \{\rho/\mu\} \times \{\mu'/X'Y\} \end{aligned} \tag{8.90}$$

If $X \equiv Y$ and $XX' = 1$ then (8.90) simplifies to

$$\sum_{\lambda} \{\rho/\lambda X\} \times \{\lambda'/X\} = \sum_{\lambda} \{\rho/\lambda\} \times \{\lambda'\} \tag{8.91}$$

The characters of $O(2n)$ may be expanded as sums of characters of $Sp(2n)$ and vice versa by noting that [59]

$$O(2n) \to Sp(2n) \quad [\lambda] \quad \to \quad \langle \lambda/V \rangle \tag{8.92}$$

$$Sp(2n) \to O(2n) \quad \langle \lambda \rangle \quad \to \quad [\lambda/W] \tag{8.93}$$

$$\tag{8.94}$$

Use of (8.92) in (8.85) followed by use of the identity (8.91) with $X = V$ immediately yields (8.89).

8.11 Appendix B : Alternative branching rules for $Sp(2n) \to U(n)$ and $O(2n) \to U(n)$

Here we sketch the derivation of the branching rules introduced in (2.1′) and (2.2′). The derivation closely follows those for the corresponding infinite dimensional unitary irreducible representations [60, 95].

Consider the two group chains

$$\begin{aligned} O(4nm) &\to O(2n) \times O(2m) \\ O(4nm) &\to U(2nm) \to U(n) \times U(2m) \to U(n) \times O(2m) \end{aligned}$$

For the first chain

$$\begin{aligned} \Delta &\to \sum_\lambda [\tfrac{k}{2}(\lambda)] \times [\lambda] \\ &\to \sum_\lambda [m^n/\lambda'] \times [\lambda] \end{aligned} \tag{8.95}$$

which follows from (4.5) to (4.9) of [60].

For the second chain

$$\begin{aligned} \Delta &\to \sum_p \varepsilon^{-\frac{1}{2}}\{1^p\} \\ &\to \sum_p \varepsilon^{-m}\{\rho'\} \times \varepsilon^{\frac{n}{2}}\{\rho\} \\ &\to \sum_p \varepsilon^{-m}\{\rho'\} \times [\rho/D] \\ &\to \sum_p \varepsilon^{-m}\{\rho' \cdot B\} \times [\rho] \end{aligned} \tag{8.96}$$

$$\tag{8.97}$$

Comparing (8.95) with (8.96) then leads to

$$[m^n/\lambda'] \to \varepsilon^{-m}\{\lambda_s\} \cdot B \tag{8.98}$$

which can be expressed in the equivalent form

$$[\lambda] \to \varepsilon^{-m}\{m^n/\lambda|_m A\} \cdot B \tag{8.99}$$

where we have made use of the following notation: Let μ be a partition into no more than p parts with its largest part no greater than q and let ν be any other partition. Define $F^{\mu|_q\nu}$ be the diagram formed by placing F^ν immediately to the right of the $q-$th column of

F^μ such that the columns are top-adjusted to the same horizontal line. The corresponding $S-$functions, denoted by $\{\mu|_q\nu\}$ are then defined in terms of $S-$functions labelled by partitions by reordering rows and columns in accordance with the usual modification rules [70].

We can transform (8.99) into a form closely resembling that for the infinite dimensional unitary irreducible representations [60] by noting that multiplication of any covariant $U(n)$ character by $\varepsilon^{-m} = \{\overline{m^n}\}$ converts the covariant indices associated with the first $m-$columns of the corresponding diagram to contravariant indices associated with the $U(n)-$complement of this portion. Here the $U(n)$-complement of (m^n/λ) is just λ itself. The symbol $|_m$ just serves to separate this part of the expression from the "signed sequence" part A, as indicated in (8.99), so that the corresponding indices remain covariant.

Thus we can rewrite (8.99) as

$$[\lambda] \to \{\overline{\lambda}; A\} \cdot B. \tag{8.100}$$

In a similar manner, starting with (8.89), we can derive for $Sp(2n) \to U(n)$

$$\langle\lambda\rangle \to \{\overline{\lambda}; C\} \cdot D. \tag{8.101}$$

It is possible to place bounds on the weights of the terms in the various series. In each case the decomposition necessarily yields the terms

$$\{\bar{\lambda}\} + \{\lambda\} \tag{8.102}$$

These are the highest weight terms in the decomposition. Recall that (8.17)

$$\{\bar{\lambda}\} \times \{\nu\} = \sum_\zeta \{\overline{\lambda/\zeta}; \nu/\zeta\}$$

and a covariant irrep of $U(n)$ arises for $\zeta = \lambda$. If we wish to obtain the covariant irrep $\{\lambda\}$ then $\omega_\nu = 2\omega_\lambda$.

It is readily seen that for any λ there always exists in the B or D series at least one term of weight $2\omega_\lambda$ that when skewed with λ yields λ. The A or C series generate the signed sequences required to eliminate repeated copies of λ and thus the terms in these series must also be retained to weight $2\omega_\lambda$.

Finally, after performing the $U(n)$ product we need only to retain terms up to weight ω_λ.

The consistency of the branching rules given by (8.1) and (8.2) with those of (8.3) and

(8.4) can be shown by noting, for example, that under $SO^*(2n) \to U(n)$

$$\begin{aligned}
[\lambda] \to \{\bar{\lambda}; A\} \cdot B &= \sum_{\zeta} \{\overline{\lambda/\zeta}; A \cdot (B/\zeta)\} \\
&= \sum_{\zeta} \{\overline{\zeta}; A \cdot (B/(\lambda/\zeta))\} \\
&= \sum_{\zeta} \{\overline{\zeta}; A \cdot ((\lambda/\zeta)/B) \cdot B\} \\
&= \sum_{\zeta} \{\overline{\zeta}; (\lambda/\zeta)/B\} \\
&= \sum_{\zeta} \{\overline{\zeta}; \lambda/B\zeta\}. \qquad (8.103)
\end{aligned}$$

where use has been made of the identity [50]

$$B/\mu = (\mu/B) \cdot B \qquad (B8)$$

A similar proof can be given for the equivalence of (8.1) and (8.3).

8.12 Appendix C : derivation of (8.39) and (8.40)

We sketch the main points for the derivation of (8.39). The corresponding derivation for (8.40) follows along similar lines.

Recall that a character of a finite dimensional irreducible representation of $Sp(2n)$ may be expressed in terms of weight vectors as:

$$\begin{aligned}\langle\lambda\rangle &= \sum_{x\in P} m_x^\lambda\ e^x \\ &= \sum_{w\in W} \mathrm{sign}(w)\ e^{w(\lambda+\rho)}\Big/\sum_{w\in W} \mathrm{sign}(w)\ e^{w(\rho)}\end{aligned} \tag{8.104}$$

where W is the Weyl group of $Sp(2n)$, ρ is half the sum of the positive roots of $Sp(2n)$, $\mathrm{sign}(w)$ is the parity of w, P is the weight lattice of $Sp(2n)$, v is a weight vector and m_x^λ is the multiplicity of v in the irrep of highest weight λ. It follows that

$$\begin{aligned}\langle\mu\rangle\times\langle\nu\rangle &= \left(\sum_{x\in P} m_x^\mu\ e^x\right)\left(\sum_{w\in W} \mathrm{sign}(w)\ e^{w(\nu+\rho)}\Big/S\right) \\ &= \left(\sum_{w\in W}\left(\sum_{x\in P} m_x^\mu\ e^x\right)\mathrm{sign}(w)\ e^{w(\nu+\rho)}\right)\Big/S \\ &= \left(\sum_{w\in W}\left(\sum_{x\in P} m_{wx}^\mu\ e^{wx}\right)\mathrm{sign}(w)\ e^{w(\nu+\rho)}\right)\Big/S \\ &= \left(\sum_{w\in W}\left(\sum_{x\in P} m_{wx}^\mu\ \right)\mathrm{sign}(w)\ e^{w(\nu+x+\rho)}\Big/S\right. \\ &= \left(\sum_{w\in W}\left(\sum_{x\in P} m_x^\mu\ \right)\mathrm{sign}(w)\ e^{w(\nu+x+\rho)}\right)\Big/S \\ &= \left(\sum_{x\in P} m_x^\mu\left(\sum_{w\in W} \mathrm{sign}(w)\ e^{w(\nu+x+\rho)}\right)\right)\Big/S \\ &= \sum_{x\in P} m_x^\mu\left(\sum_{w\in W} \mathrm{sign}(w)\ e^{w(\nu+x+\rho)}\Big/S\right) \\ &= \sum_{x\in P} m_x^\mu\ \langle\nu+x\rangle\end{aligned} \tag{8.105}$$

where

$$S = \sum_{w\in W} \mathrm{sign}(w)\ e^{w(\rho)} \tag{8.106}$$

and where use has been made of the fact that weight multiplicities are Weyl group invariant so that $m^\mu_{wx} = m^\mu_x$ for all $w \in W$. Hence the tensor product multiplicities coincide with the weight multiplicities provided that for all x such that $m^\mu_x > 0$ the weight $\nu + x$ is dominant. If this is not the case then it is necessary to use Weyl reflections to modify $\langle \nu + x \rangle$. Turning to the next simplest case, namely the product $\langle \mu \rangle \times \langle \frac{k}{2}(\nu) \rangle$ with k sufficiently large that $\langle \frac{k}{2}(\nu) \rangle$ is highly standard we have

$$\begin{aligned} \langle \tfrac{k}{2}(\nu) \rangle &= \epsilon^{k/2} \cdot \{\nu\} \cdot D \\ &= \epsilon^{k/2} \cdot D \cdot \sum_{w_c \in W_c} \mathrm{sign}(w_c)\ e^{w_c(\nu+\rho_c)} \Big/ \sum_{w_c \in W_c} \mathrm{sign}(w_c)\ e^{w_c(\rho_c)}, \end{aligned} \tag{8.107}$$

where the sums are now over elements w_c of the Weyl group w_c of $U(n)$ and ρ_c is half the sum of the positive roots of $U(n)$. Since the weight lattices P of $Sp(2n)$ and of $U(n)$ coincide we have

$$\begin{aligned} \langle \mu \rangle \times \langle \tfrac{k}{2}(\nu) \rangle &= \epsilon^{k/2} \cdot D \cdot \left(\sum_{x \in P} m^\mu_x\ e^x \right) \left(\sum_{w_c \in W_c} \mathrm{sign}(w_c)\ e^{w_c(\nu+\rho_c)} \Big/ S_c \right) \\ &= \epsilon^{k/2} \cdot D \cdot \left(\sum_{w_c \in W_c} \left(\sum_{x \in P} m^\mu_x\ e^x \right) \mathrm{sign}(w_c)\ e^{w_c(\nu+\rho_c)} \right) \Big/ S_c \\ &= \epsilon^{k/2} \cdot D \cdot \left(\sum_{w_c \in W_c} \left(\sum_{x \in P} m^\mu_{w_c x}\ e^{w_c x} \right) \mathrm{sign}(w_c)\ e^{w_c(\nu+\rho_c)} \right) \Big/ S_c \\ &= \epsilon^{k/2} \cdot D \cdot \left(\sum_{w_c \in W_c} \left(\sum_{x \in P} m^\mu_{w_c x} \right) \mathrm{sign}(w_c)\ e^{w_c(\nu+x+\rho_c)} \right) \Big/ S_c \\ &= \epsilon^{k/2} \cdot D \cdot \left(\sum_{w_c \in W_c} \left(\sum_{x \in P} m^\mu_x \right) \mathrm{sign}(w_c)\ e^{w_c(\nu+x+\rho_c)} \right) \Big/ S_c \\ &= \epsilon^{k/2} \cdot D \cdot \left(\sum_{x \in P} m^\mu_x \left(\sum_{w_c \in W_c} \mathrm{sign}(w_c)\ e^{w_c(\nu+x+\rho_c)} \right) \right) \Big/ S_c \\ &= \epsilon^{k/2} \cdot D \cdot \sum_{x \in P} m^\mu_x \left(\sum_{w_c \in W_c} \mathrm{sign}(w_c)\ e^{w_c(\nu+x+\rho_c)} \right) \Big/ S_c \\ &= \epsilon^{k/2} \cdot D \cdot \sum_{x \in P} m^\mu_x\ \{\nu + x\} \\ &= \sum_{x \in P} m^\mu_x\ \langle \tfrac{k}{2}(\nu + x) \rangle. \end{aligned} \tag{8.108}$$

where

$$S_c = \sum_{w_c \in W_c} \operatorname{sign}(w_c)\ e^{w_c(\rho_c)} \tag{8.109}$$

In the above use has been made of the fact that W_c is a subgroup of W. The penultimate step assumes that $\nu + x$ is dominant for all x, while the last step requires in addition that k is sufficiently large for $\langle \frac{k}{2}(\nu + x)\rangle$ to be highly standard for all x. The next stage is to consider those cases for which it may be necessary to invoke modification rules. This will be the case in both (8.105) and (8.108) if $\nu + x$ is not dominant. Let us start by replacing ν in (8.105) by $m^n + \nu$ where this symbol denotes the n-part partition obtained from ν by adding m to each part, including those that are originally zero. With this notation we have

$$\begin{aligned} \langle\mu\rangle \times \langle m^n + \nu\rangle &= \textstyle\sum_{x\in P}\ m_x^\mu \left(\sum_{w\in W}\ \operatorname{sign}(w)\ e^{w(m^n+\nu+x+\rho)} \Big/ \sum_{w\in W}\ \operatorname{sign}(w)\ e^{w(\rho)}\right) \\ &= \textstyle\sum_{x\in P}\ m_x^\mu\ \mathrm{ch}_{\mathrm{Sp(2n)}}(\mathrm{m^n} + \nu + \mathrm{x}) \\ &= \textstyle\sum_\lambda\ k^\lambda_{\mu,m^n+\nu}\ \langle\lambda\rangle \end{aligned} \tag{8.110}$$

where $\mathrm{ch}_{\mathrm{Sp(2n)}}(\mathrm{m^n} + \nu + \mathrm{x})$ is a formal character of $Sp(2n)$ which is either $\langle\lambda\rangle$ with $\lambda = m^n + \nu + x$ or null or $\pm\mathrm{ch}_{\mathrm{Sp(2n)}}(\lambda) = \pm\langle\lambda\rangle$ for some partition λ obtained from $m^n + \nu + x$ through the Weyl group action. The last step involves collecting together the results of these modifications. It is important to note that for all $m \geq \mu_1$ we must have each component of $m^n + \nu + x$ positive. This means that the only elements of the Weyl group W that are used to perform the modification of any non-standard $m^n+\nu+x$ are necessarily elements of W_c, the Weyl group of $U(n)$.

Returning to (8.108) we can write

$$\begin{aligned}
\langle\mu\rangle \times \langle\tfrac{k}{2}(\nu)\rangle &= \epsilon^{k/2} \cdot D \cdot \sum_{x\in P} m_x^\mu \\
&\times \left(\sum_{w_c\in W_c} \mathrm{sign}(w_c)\ e^{w_c(\nu+x+\rho_c)} \Big/ \sum_{w_c\in W_c} \mathrm{sign}(w_c)\ e^{w_c(\rho)} \right) \\
&= \epsilon^{(k/2)-m} e^{m^n} \cdot D \cdot \sum_{x\in P} m_x^\mu \\
&\times \left(\sum_{w_c\in W_c} \mathrm{sign}(w_c)\ e^{w_c(\nu+x+\rho_c)} \Big/ \sum_{w_c\in W_c} \mathrm{sign}(w_c)\ e^{w_c(\rho_c)} \right) \\
&= \epsilon^{(k/2)-m} \cdot D \cdot \sum_{x\in P} m_x^\mu \\
&\times \left(\sum_{w_c\in W_c} \mathrm{sign}(w_c)\ e^{m^n} e^{w_c(\nu+x+\rho_c)} \Big/ \sum_{w_c\in W_c} \mathrm{sign}(w_c)\ e^{w_c(\rho_c)} \right) \\
&= \epsilon^{(k/2)-m} \cdot D \cdot \sum_{x\in P} m_x^\mu \\
&\times \left(\sum_{w_c\in W_c} \mathrm{sign}(w_c)\ e^{w_c m^n} e^{w_c(\nu+x+\rho_c)} \Big/ \sum_{w_c\in W_c} \mathrm{sign}(w_c)\ e^{w_c(\rho_c)} \right) \\
&= \epsilon^{(k/2)-m} \cdot D \cdot \sum_{x\in P} m_x^\mu \\
&\times \left(\sum_{w_c\in W_c} \mathrm{sign}(w_c)\ e^{w_c(m^n+\nu+x+\rho_c)} \Big/ \sum_{w_c\in W_c} \mathrm{sign}(w_c)\ e^{w_c(\rho_c)} \right) \\
&= \epsilon^{(k/2)-m} \cdot D \cdot \sum_{x\in P} m_x^\mu \\
&\times \left(\sum_{w_c\in W_c} \mathrm{sign}(w_c)\ e^{\rho_n}\ e^{w_c(m^n+\nu+x+\rho_c)} \Big/ \sum_{w_c\in W_c} \mathrm{sign}(w_c)\ e^{\rho_n}\ e^{w_c(\rho_c)} \right) \\
&= \epsilon^{(k/2)-m} \cdot D \cdot \sum_{x\in P} m_x^\mu \\
&\times \left(\sum_{w_c\in W_c} \mathrm{sign}(w_c)\ e^{w_c(m^n+\nu+x+\rho_c+\rho_n)} \Big/ \sum_{w_c\in W_c} \mathrm{sign}(w_c)\ e^{w_c(\rho_c+\rho_n)} \right) \\
&= \epsilon^{(k/2)-m} \cdot D \cdot \sum_{x\in P} m_x^\mu \\
&\times \left(\sum_{w_c\in W_c} \mathrm{sign}(w_c)\ e^{w_c(m^n+\nu+x+\rho)} \Big/ \sum_{w_c\in W_c} \mathrm{sign}(w_c)\ e^{w_c(\rho)} \right) \\
&= \epsilon^{(k/2)-m} \cdot D \cdot \sum_{x\in P} m_x^\mu\ \mathrm{ch}_{\mathrm{U(n)}}(\mathrm{m^n} + \nu + \mathrm{x}) \\
&= \epsilon^{(k/2)-m} \cdot D \cdot \sum_{\lambda} k^\lambda_{\mu,m^n+\nu}\ \{\lambda\} \\
&= \sum_\lambda k^\lambda_{\mu,m^n+\nu}\ \langle\tfrac{k}{2} - m(\lambda)\rangle \qquad (8.111)
\end{aligned}$$

where use has been made of the facts that $\epsilon^m = e^{m^n}$ with $w_c m^n = m^n$ for all $w_c \in W_c$, and $\rho = \rho_c + \rho_n$ with $w_c \rho_n = \rho_n$ for all $w_c \in W_c$.

The crucial observation that the coefficients in (8.110) and (8.111) are identical is a consequence of the fact that the modifications of $\mathrm{ch}_{\mathrm{U(n)}}(\mathrm{m^n} + \nu + \mathrm{x})$ to give $\{\lambda\}$ coincide with those of $\mathrm{ch}_{\mathrm{Sp(2n)}}(\mathrm{m^n} + \nu + \mathrm{x})$ to give $\langle\lambda\rangle$, since in both cases it is only elements of the Weyl group W_c that are involved. As shown in the above manipulations the presence or absence of ρ_n in the exponents of both numerator and denominator is irrelevant since ρ_n is W_c invariant.

The final step still requires that each $\langle \frac{k}{2} - m(\lambda)\rangle$ be highly standard.

With that proviso the proof is complete.

The details about ρ_c and ρ_n are given elsewhere [60]. In particular for $Sp(2n, \mathbb{R})$

$$\begin{aligned} \rho_c &= \left(\tfrac{n-1}{2}, \tfrac{n-3}{2}, \ldots, \tfrac{-n+1}{2}\right); \\ \rho_n &= \left(\tfrac{n+1}{2}, \tfrac{n+1}{2}, \ldots, \tfrac{n+1}{2}\right); \\ \rho &= \rho_c + \rho_n = (n, n-1, \ldots, 1). \end{aligned} \tag{8.112}$$

The case of $SO^*(2n)$ is similar, involving just the replacement of D by B and noting that [60]

$$\begin{aligned} \rho_c &= \left(\tfrac{n-1}{2}, \tfrac{n-3}{2}, \ldots, \tfrac{-n+1}{2}\right); \\ \rho_n &= \left(\tfrac{n-1}{2}, \tfrac{n-1}{2}, \ldots, \tfrac{n-1}{2}\right); \\ \rho &= \rho_c + \rho_n = (n-1, n-2, \ldots, 0). \end{aligned} \tag{8.113}$$

8.13 Appendix D : special products with *MAPLE*

Recalling equations (8.79) and (8.81), it is also possible to produce special products for $SO^*(2n)$ and $Sp(2n,\mathbb{R})$ using the following *MAPLE* program, where the the parameters N, u, v, K correspond to notations of the text.

```
#########################################
# Calculations for generic product in  #
# SO^*                                  #
#########################################
SO := proc(N,u,v,K)
local i,j,k,L,l;
L:=[];

for i from 0 to u do
for j from 0 to min(i,v) do
for k from 0 to min(u-i,v-j) do
l := [u+v-j-i-k,k];
for m from 1 to n-3 do
l := [op(l),0];
od;
l:=[op(l),j-i];
if (j-i) < 0 then
l:=map(proc(x,y) x+y end,l,i-j);
l := [K-i+j,l];
else
l:= [K,l];
fi;
L := [op(L),l];
l:=[];
od;
od;
od;
L:=map(x->convert(x,string),L);
RETURN(L);
end;

#########################################
# Calculations for generic product in  #
# Sp(2n,R)                              #
#########################################
SpR := proc(N,u,v,K)
```

```
local i,j,k,L,l,x;
L:=[];

for x from 0 to floor(u/2) do
for i from 0 to u do
for j from 0 to min(i,v) do
for k from 0 to min(u-i-2*x,v-j) do
l := [u+v-j-2*x-i-k,k];
for m from 1 to N-3 do
l := [op(l),0];
od;
l:=[op(l),j-i];
if (j-i) < 0 then
l:=map(proc(x,y) x+y end,l,i-j);
l := [K-i+j,l];
else
l:= [K,l];
fi;

L := [op(L),l];
l:=[];
od;
od;
od;
od;
L:=map(x->convert(x,string),L);
RETURN(L);
end;
```

We can conclude with two examples.

For $SO^*(2n)$, the product $[3] \times [6(4)]$ gives

```
> SO(3,3,4,6);

[[6, [7, 0, 0]], [6, [6, 1, 0]], [6, [5, 2, 0]], [6, [4, 3, 0]],

  [5, [7, 1, 0]], [5, [6, 2, 0]], [5, [5, 3, 0]], [6, [5, 0, 0]],

  [6, [4, 1, 0]], [6, [3, 2, 0]], [4, [7, 2, 0]], [4, [6, 3, 0]],
```

```
  [5, [5, 1, 0]], [5, [4, 2, 0]], [6, [3, 0, 0]], [6, [2, 1, 0]],

  [3, [7, 3, 0]], [4, [5, 2, 0]], [5, [3, 1, 0]], [6, [1, 0, 0]]]

>
```

and for $Sp(2n,\mathbb{R})$ the product $\langle 3\rangle \times \langle 6(4)\rangle$ gives

```
> SpR(3,3,4,6);

[[6, [7, 0, 0]], [6, [6, 1, 0]], [6, [5, 2, 0]], [6, [4, 3, 0]],

  [5, [7, 1, 0]], [5, [6, 2, 0]], [5, [5, 3, 0]], [6, [5, 0, 0]],

  [6, [4, 1, 0]], [6, [3, 2, 0]], [4, [7, 2, 0]], [4, [6, 3, 0]],

  [5, [5, 1, 0]], [5, [4, 2, 0]], [6, [3, 0, 0]], [6, [2, 1, 0]],

  [3, [7, 3, 0]], [4, [5, 2, 0]], [5, [3, 1, 0]], [6, [1, 0, 0]],

  [6, [5, 0, 0]], [6, [4, 1, 0]], [5, [5, 1, 0]], [6, [3, 0, 0]]]
```

These results can also be obtained with the SCHUR command developped with the algorithms presented here.

Appendix A

HCl spectrum and Davidson potential

In this appendix, we provide a simple example of the use of a computer algebra system to calculate a spectrum from the group theoritical analysis of a model hamiltonian: the Davidson potential. Note that all the required group theoritical ingredients (spherical harmonics, Clebsch-Gordan coefficients etc.) are computed from scratch in our program.

Symplectic groups occur in different fields of physics. We can cite Hamiltonian mechanics or geometrical optics [33]. Another kind of application of the symplectic group is the understanding of the quantum interferometer. Typically this interferometer is understandable in terms of $SU(1,1)$, a group isomorphic to $Sp(2,\mathbb{R})$ that we will use in the sequel.

In fact most of the physical applications involve the notion of dual pairs [40] that we have used in the development of the previous chapters. Implicitly we will use this notion to illustrate our symplectic technics with the Davison potential or the nuclear configuration of the oxygen nucleus (Appendix II).

One of the important questions is the study of the perturbed harmonic oscillator with a given potential. We will see here that the use of the Davidson potential allows us to give, using technics of symplectic group, the rotation vibration spectrum for the HCl molecule. That is a direct application where the symplectic group appears very naturally ([3, 4]).

A.1 Preliminary remarks

The rotation-vibrational structure of molecules is a very large subject that comes from the first observations of Angstroem (1893) and of Angstroem and Balmer (1900) that showed that HCl can absorb infra-red rays under certain conditions.

After the introduction of analytical and algebraic models, Duham proposed a model to describe with greater accuracy the nature of the spectrum. Here is not the place to expose the details of the different models (we refer to [84]). One of the important treatments of the perturbed oscillator that is very useful and important in spectroscopy is the Calogero model ([11, 12]) for which the Davidson potential is the simplest case. In this section we will recall the basic objects of this study.

A.2 The one dimensional oscillator

In the introductory chapter we recall the definition of the symplectic group and its link with Lagrangian and Hamiltonian mechanics. The result of this presentation was that the solution of the one-dimensional quantum oscillator was to solve the operator equation

$$H|\phi\rangle = E|\phi\rangle$$

that can be rewritten as

$$[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2]\phi(x) = E\phi(x)$$

The natural question is about the generalization of this harmonic oscillator problem if we consider

1. Several particles.
2. Perturbations.
3. Several particles and perturbations.

These last cases introduce symmetry groups that are in our situation the symplectic groups $Sp(2n, \mathbb{R})$.

The roles of partitions and symmetric functions are sufficiently exposed in chapter 2. Also we have given the fundamental notions of representation theory and in particular the importance of the study of the representation of the Lie algebra $\mathfrak{sl}(2)$. Here we will use the isomorphism between $\mathfrak{sl}(2)$ and $\mathfrak{sp}(2)$ and also with the Lie algebra of another non-compact group $\mathfrak{su}(1,1)$.

A.3 Some isomorphisms

We started announcing that our aim is to calculate the rotational spectrum of HCl. Composed from two atoms it is evident that we cannot continue to use the one-dimensional method to solve this problem. In fact, because of the two-atoms problem, it is possible to consider the quantization of the reduced-mass and consider the HCl molecule as a three dimensional oscillator for which the dynamical group is $Sp(6, \mathbb{R})$.

In the chapter (4), we give the branching rules for $Sp(2nk,\mathbb{R}) \downarrow Sp(2n,\mathbb{R}) \times O(k)$. We will consider here this branching rule to branch $Sp(6,\mathbb{R})$ to $Sp(2,\mathbb{R}) \times O(3) \supset Sp(2,R) \times SO(3)$.

In fact, rather than working at the level of Lie groups, we will work here at the level of Lie algebras. Thus we can use the announced isomorphisms and see how it is possible to evaluate the perturbation in terms of the commutation relations of the algebra generators. First we recall the generators of $\mathfrak{sp}(2,\mathbb{R})$.

$sp(2,R)$ **generators**

We choose to label the generators differently than before

$$\begin{aligned} Z_1 &= K &&= \frac{d^2}{dx^2} \ , \\ Z_2 &= Q &&= x^2 \ , \\ Z_3 &= C &&= x\frac{d}{dx} + \frac{1}{2} \ , \end{aligned}$$

so that they are in a one to one correspondance with the generators of $\mathfrak{sl}(2,\mathbb{R})$. It is easy to check that the commutations relations for the basis e, f, h

$$[e,f] = h, \ [h,e] = 2e, \ [h,f] = -2f.$$

are conserved if we express this basis in terms of K, Q, C by

$$e = \frac{K}{2}, \ f = \frac{Q}{2} \text{ and } h = -C \ .$$

Now we can find another isomorphism between $\mathfrak{sl}(2,\mathbb{R})$ and the group algebra $\mathfrak{su}(1,1)$ of another non-compact group, the group $SU(1,1)$. This group is the transformation group leaving invariant the hermitian forms $|z_1|^2 - |z_2|^2$. It can be see at the group of matrices M such that

$$M = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \quad \text{with } \det(M) = 1.$$

To show the isomorphism $\mathfrak{su}(1,1) \simeq \mathfrak{sp}(2,\mathbb{R})$ we begin with the definition of the $\mathfrak{su}(1,1)$ generators

$$X_1 = \frac{1}{4\hbar}\left(aZ_1 - \frac{1}{a}Z_2\right), \ X_2 = \frac{1}{2\hbar}Z_3, \ X_3 = \frac{1}{4\hbar}\left(aZ_1 + \frac{1}{a}Z_2\right).$$

The commutation relations for the generator systems Z_i and X_i are identical. In fact for practical reasons one uses another generator system for $\mathfrak{su}(1,1)$ given by

$$X_\pm = X_1 \pm X_2, \ X_0 = X_3$$

and the commutation relations are given by

$$[X_-, X_+] = 2X_0, \quad [X_0, X_\pm] = \pm X_\pm$$

The choice of this second system of generators is clear if we remark that the Hamiltonian of the system is related to the generators by

$$X_0 = X_3 = \frac{1}{2\hbar\omega}\left[\frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2\right] = \frac{1}{2\hbar\omega}H$$

where we have replaced a by

$$a = \frac{1}{m\omega}$$

and the corresponding eigenstates are the set of kets $\{|n\lambda\rangle\}$ with energy

$$E_{n\lambda} = (2n+\lambda)\hbar\omega,\ \lambda = l + \frac{3}{2}\ .$$

Finally the wave function obtained is of the form

$$\Psi_{0lm} = N_l e^{\frac{r^2\hbar}{2m\omega}} r^l Y_{lm}(\theta,\phi)$$

where $\lambda = l + \frac{3}{2}$ for the spherical harmonic oscillator. The energy states are finally

$$E_{nl} = 2n + l + \frac{3}{2}$$

In fact the choice of the algebra $\mathfrak{su}(1,1)$ is motived by the coherent states representation of $SU(1,1)$ (see [89]) that we will quickly present in the appendix of this chapter.

A.4 Oscillator with Davidson potential

Now we consider the modified Hamiltonian

$$H \to \tilde{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2\left(r^2 + \frac{\varepsilon}{r^2}\right)$$

and to the new Hamiltonian we associated a new set of generators

$$\tilde{Z}_1 = Z_1 + \frac{\hbar^2\epsilon}{Z_2},\ \tilde{Z}_2 = Z_2 = r^2,\ \tilde{Z}_3 = Z_3$$

that satisfy the usual commutation relations. One supposes now that

$$E_{nl} = (2n + \lambda(l))\hbar\omega\ .$$

Thus the natural question is to determine the value of $\lambda(l)$.

The most natural way is to use the invariance of the Casimir operator

$$C_2 = X_3^2 - X_1^2 - X_2^2 = X_0(X_0-2) - X_- X_+$$

The Casimir operator eigenvalues have the spectrum

$$C_2|n\lambda\rangle = \frac{1}{4}\lambda(\lambda-2)|n\lambda\rangle \ .$$

If $\varepsilon = 0$ then one has

$$C_2 = (\lambda + \frac{3}{2})(l - \frac{1}{2})$$

else

$$C_2(\varepsilon) = \frac{1}{4\hbar^2}\left(\frac{1}{2}(\tilde{Z}_1\tilde{Z}_2 + \tilde{Z}_2\tilde{Z}_1\right) - \tilde{Z}_3{}^2 = C_2(0) + \frac{\varepsilon}{4} \ .$$

From the last equation one deduces that

$$\lambda(l)(\lambda(l)-2) = (\lambda + \frac{3}{2})(l - \frac{1}{2}) + \varepsilon$$

then

$$\lambda(l) = 1 + \sqrt{\left(l + \frac{1}{2}\right)^2 + \epsilon}$$

For the energy one obtains

$$E_{nl} = \left[2n + 1 + \sqrt{\left(l + \frac{1}{2})\right)^2 + \epsilon}\right]\hbar\omega$$

and the modified wave function is

$$\Psi_{0lm} = N_l e^{\frac{r^2\hbar}{2m\omega}} r^{b_l} Y_{lm}(\theta,\phi)$$

where

$$b_l = \sqrt{\left(l + \frac{1}{2}\right)^2 + \epsilon} - \frac{1}{2}, \ N_l = \left(\frac{2}{\Gamma(b_l + \frac{3}{2})}\right)^{\frac{1}{2}}$$

Now we have the possibility for large values of ε of developing the energy in inverse powers of ε to obtain

$$E_{nl} = E_0 + 2n\hbar\omega + Al(l+1) - Bl^2(l+1)^2 \cdots$$

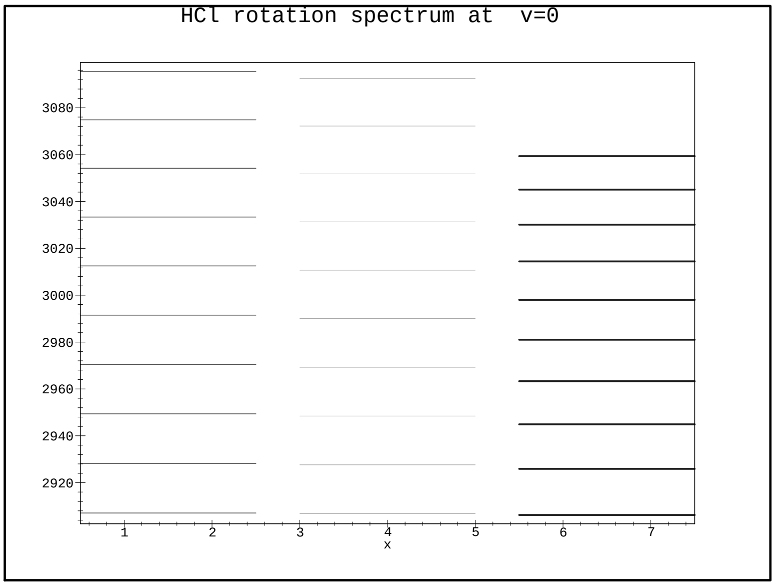
that is of the form of rotation-vibration energy. If we write

$$A = \frac{\hbar\omega}{2\sqrt{\varepsilon}}, \ B = \frac{A}{4\varepsilon}$$

and according to the usual values of physical constants [84],we can calculate and draw the rotational-vibrational spectrum of HCl (or analogous molecules). To realize the calculations we need first the dipole transition rates (see for example [43]) and practically we need to calculate the Wigner $3j$ symbols. The program presented in the next section contains all the physical values and programs necessary to obtain the following spectra for which we consider only the rotational levels given for a fixed vibrational level.

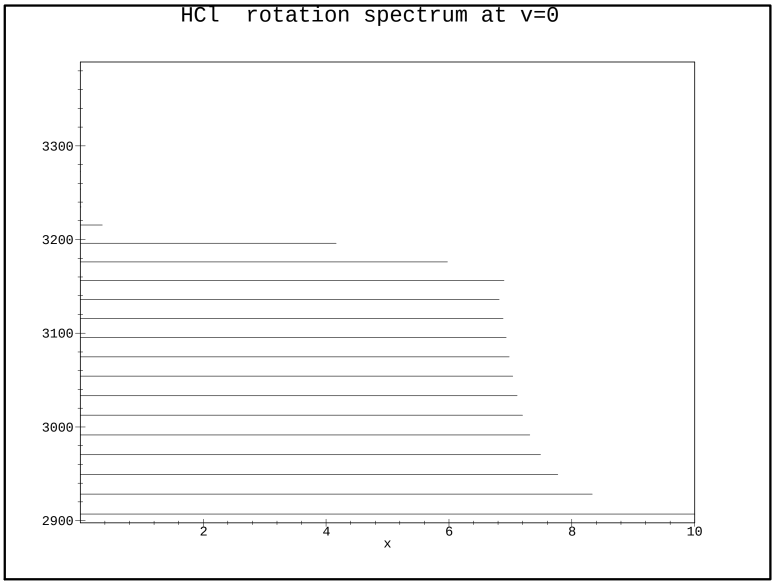
Compared rotational levels

Here we calculate the rotational energy levels with the usual formula (center of the diagram) and our expression (right part of the diagram) and we compare it to the experimental values (left of the diagram).



A computation of the spectrum

Now we can compute a spectrum that is represented below. In fact we represent only one branch of the spectrum, the other is totally symmetric. We can remark that our spectrum is not completely comparable with the experimental ones. Different reasons explain the distorsion. Certainly one is due to the fact that we have not considered the breadth of the peak and certainly we need more accuracy in the determination of the constants used in the computation.



Some remarks

Another remark can be made. The Davidson potential in a classical case can be see as a splitting of the parabolic potential into two deformed parabolas that correspond to the branch P and R of the rotational spectrum.

Another remark concerns the situation where the Poschl-Teller potential replaces the Davidson potential. In fact for this case we have to treat the perturbed system in the analogous way using the coherent states of $SU(1,1)$ [83].

A.5 Calculation of HCl spectrum

This program is very simple and allows us illustrate the commutation relations of the different generators and to calculate the HCl spectrum using the dipole operators and the Wigner $3j$ symbols that are needed to calculate the transition rate.

```
interface(verboseproc=0);
#### library for spherical harmonics ##########

read `orbitals.m`;
with(orbitals);
with(plots);
```

```
#################################################
# Data for the program                          #
#################################################
#----------------------------------------------
# Data for spectrum calculation
#----------------------------------------------

Digits := 5; # precision du calcul
Joule := 5.03396*10^22; # J en cm-1

Z := 1;
a := 1.6*10^(-19);

#--------- Spherical harmonics section
t := theta;
p := phi;
Ylm := ComplexSurfaceHarmonic;

#-------------  dipole operator ------------
D1k := Z*a*r*Ylm(1,0,t,p);


#-------------------------------------------
#---      Data for HCl          ------------
#-------------------------------------------

#---- pour Davidson ---
E0 := 2885.9/Joule;
A := 10.59/Joule;
Epsilon := (E0/4/A)^2;
B := A/4/Epsilon;

#--------- Physical data ----

A0 := 0.307/Joule;
De := 5.32*10^(-4)/Joule;
Titre := `Rotation spectrum for HCl with v=0`;
OX := 52.82/Joule;

HCl := [E0, A, B, A0, De, Titre,OX];

#------------------------------------------
#---      Data for HF          ------------
```

```
#------------------------------------------

#---- pour Davidson ---
E0 := 2998.19/Joule;
A := 11.01/Joule;
Epsilon := (E0/4/A)^2;
B := A/4/Epsilon;
OX := 45.76/Joule;


A0 := 0.302/Joule;
De := 5.94*10^(-4)/Joule;

Titre := `otation spectrum for HF with v=0`;

HF := [E0, A, B, A0, De, Titre];

#------------------------------------------
#---      Data for HBr          -----------
#------------------------------------------

#---- pour Davidson ---
E0 := 1884.75/Joule;
A := 4.24/Joule;
Epsilon := (E0/4/A)^2;
B := A/4/Epsilon;

A0 := 0.083/Joule;
De := 0.88*10^(-4)/Joule;
OX := 22.71/Joule;

Titre := `Rotation spectrum for HBr with v=0`;


HBr := [E0, A, B, A0, De, Titre];

#------------------------------------------------
# Spherical harmonic oscillator SHO
#------------------------------------------------

SHO := [0,1,0,0,0,0];

################################################
```

```
# Algebraic part
#################################################


############################
#  Sp(2,R) generators      #
############################

#------------  Z1 --------------

Z1 := proc(F)
local res;
res := -diff(diff(F,x),x);
RETURN(res);
end;

#------------  Z2 --------------

Z2 := proc(F)
local res;
res := F*x*x;
RETURN(res);
end;

#------------  Z3 --------------

Z3 := proc(F)
local res;
res := (x*diff(F,x)+F)/2;
RETURN(res);
end;


##########################
#  su(1,1) generators    #
##########################

#------------  X1 --------------
X1 := proc(F)
local res;
res := 1/4/h * (a*Z1(F)-Z2(F)/a);
RETURN(res);
end;
```

```
#------------  X2 --------------
X2 := proc(F)
local res;
res := Z3(F)/2/h;
RETURN(res);
end;

#------------  X3 --------------
X3 := proc(F)
local res;
res := 1/4/h * (a*Z1(F)+Z2(F)/a);
RETURN(res);
end;

#####################################
# su(1,1) generators modified for   #
# the  Davidson  potential          #
#####################################

#----------  W1 -------------
W1 := proc(F)
local res;
res := Z1(F)+h^2*b/Z2(F);
RETURN(res);
end;


#-------------------------------------
# Bracket and Casimir
#-------------------------------------

#------------- Bracket -------------------------
Crochet := proc(A,B,F)
local res;
res := A(B(F))-B(A(F));
RETURN(res);
end;

#------------- Casimir -------------------------
CasimirX := proc(f)
local res;
res := X3(X3(f)) - X1(X1(f)) -X2(X2(f));
```

```
RETURN(simplify(res));
end;

Casimir := proc(F)
local res;
res := 1/4/h^2*( (Z1(Z2(F))+Z2(Z1(F)))/2 - Z3(Z3(F)) );
RETURN(res);
end;

#--------- Casimir Davidson -----------
CasimirD := proc(F)
local res;
res := 1/4/h^2*( (W1(Z2(F))+Z2(W1(F)))/2 - Z3(Z3(F)) );
RETURN(simplify(res));
end;

#-------------------------
GXp := proc(F)
local res;
res := l*x*F+x^2*diff(F,x);
res := simplify(factor(res));
RETURN(res);
end;

#-------------------------
GX0 := proc(F)
local res;
res := l*F/2+x*diff(F,x);
res := simplify(factor(res));
RETURN(res);
end;

#-------------------------
GXm := proc(F)
local res;
res := diff(F,x);
res := simplify(factor(res));
RETURN(res);
end;

C := proc(n)
local p;
p := cat(K,n);
```

```
RETURN(p*x^n);
end;

############################################################
# Spectrum calculation
############################################################


#--------------------------------------------------
#  Calculation of energy levels with
# Davidson potential
#--------------------------------------------------

Enl := proc(l, MOL)
local res;

res := op(1,MOL) + op(2,MOL)*l*(l+1) - op(3,MOL)*l^2*(l+1)^2;

RETURN(res*Joule);
end;

#--------------------------------------------------
# Calcul with standard formula
#--------------------------------------------------

F := proc(v,j,MOL)
local res;
res := op(1,MOL)+(op(2,MOL)-op(4,MOL)*(v+1/2))*j*(j+1)-op(5,MOL)*j^2*(j+1)^2;
RETURN(res*Joule);
end;

#--------------------------------------------------
# Graphical part
#--------------------------------------------------

NiveauxRot := proc(n,MOL)
local res,i, En , Fn, V1,V2;
En := []; Fn := [];

for i from 1 to n do
V1 := Enl(i,MOL);
V2 := F(0,i,MOL);
En := [ op(En) , plot(V1,x=0.5..2.5,thickness=2) ];
```

```
Fn := [ op(Fn) , plot(V2,x=3..5, color=green,thickness=1) ];
print(V1,V2,V1-V2);
od;
display({op(En),op(Fn)},title=op(6,MOL),axes=boxed);
end;

#-------------------------------------------------
# HCl (or HF, HBr ) Spectrum
#-------------------------------------------------

Spectres := proc(n,MOL)
local res,i, En , Fn, V1,V2;
En := []; Fn := [];

for i from 1 to n do
V1 := Enl(0,MOL) + Enl(i,MOL)-Enl(i-1,MOL);
Amp := 1/Rl(i,MOL,4778);
#V2 := F(0,i,MOL);
En := [ op(En) , plot(V1,x=0..Amp*10,thickness=2) ];
#Fn := [ op(Fn) , plot(V2,x=3..5, color=green,thickness=1) ];
print(V1,Amp);
od;
#display({op(En),op(Fn)},title=op(6,MOL),axes=boxed);
display({op(En)},title=op(6,MOL),axes=boxed);
end;

#-------------------------------------
# Bandes
#-------------------------------------

Bandes := proc(n,MOL)
local res,i, En , Fn;
En := [];
Fn := [];
for i from 1 to n do
En := [op(En),Enl(0,MOL) + Enl(i,MOL)-Enl(i-1,MOL)];
Fn := [op(Fn),F(0,0,MOL) + F(0,i,MOL)-F(0,i-1,MOL)];
od;
res := [En,Fn];
end;


#------------------------------------------------
```

```
# Vibrationnal energy
#-----------------------------------------------

G := proc(n,MOL)
local res;
res := op(1,MOL)*(n+1/2) - op(7,MOL)*(n+1/2)^2;
RETURN(res*Joule);
end;


#------------------------------------
# branch R with experimental datas
#------------------------------------

BR := [
2906.24 ,
2925.90 ,
2944.90 ,
2963.29 ,
2981.0  ,
2998.04 ,
3014.41 ,
3030.09 ,
3045.06 ,
3059.32 ];


#################################################
# comparaison real, standard model and Davisdson
#################################################


DessineBandes := proc(n,MOL)
local res,i, En , Fn, V1,V2, V3, BRn;
En := []; Fn := []; BRn := [];

for i from 1 to n do
V1 := Enl(0,MOL) + Enl(i,MOL)-Enl(i-1,MOL);
V2 := F(0,0,MOL) + F(0,i,MOL)-F(0,i-1,MOL);
V3 := op(i,BR);
En := [ op(En) , plot(V1,x=0.5..2.5,thickness=2) ];
Fn := [ op(Fn) , plot(V2,x=3..5, color=green,thickness=1) ];
BRn := [ op(BRn), plot(V3,x=5.5..7.5, color=blue,thickness=3) ];
```

```
print(V1,V2,V1-V3,V2-V3);
od;
display({op(En),op(Fn),op(BRn)},title=op(6,MOL),axes=boxed);
end;

##########################################
#  3-j Symbols
##########################################
#
#  This Maple program is called Wigner_3j.map.
#  It calculates the 3j symbol as defined in Edmond's book equations
#  3.6.10 for the Clebsh-Gordon coefficient and 3.7.3 for the 3j symbol
#  defined in terms of the latter.
#
# Define the factorial ftn fac(n) to be GAMMA(n+1)
#

Wigner_3j:=proc(j1,j2,j3,m1,m2,m3)
        local s,ss,f,edmonds;
    if type([args],list(rational)) then
# Correction
        if abs(m1)>j1 or abs(m2)>j2 or abs(m3)>j3 then RETURN (0) fi;
        if m1+m2+m3 <> 0 then RETURN (0) fi;
        if j3 < abs(j1-j2) or j3 > j1+j2 then RETURN (0) fi;
        ss:=0:
        for s from 0 to j3+m3 do
            if j1-m1-s < 0 or j3+m3-s < 0 or j2-j3+m1+s < 0 then
            f:=0
            else
            f:=(j1+m1+s)!*(j2+j3-m1-s)!/(s!*(j1-m1-s)!*(j2-j3+m1+s)!
               *(j3+m3-s)!):
            fi;
            ss:=ss+(-1)**s*f;
        od;
        f:=(j1+j2-j3)!*(j1-m1)!*(j2-m2)!*(j3-m3)!*(j3+m3)!
        /((j1+j2+j3+1)!*(j1-j2+j3)!*(-j1+j2+j3)!*(j1+m1)!*(j2+m2)!):
        edmonds:=(-1)**(-2*j1-m1-j2-m3)*ss*sqrt(f);
        simplify(edmonds)
    else
        'Wigner_3j'(args) # return unevaluated
    fi;
end:
```

```
##################################################
### b_l calaculations
##################################################

b := proc(l,MOL,er)
local eps,res;
if r=-1 then
eps := (op(1,MOL)/4/op(2,MOL))^2;
else
eps := er;
fi;
res := ((l+1/2)^2+eps)^(1/2)-1/2;
RETURN(evalf(res));
end;



##################################################
#  Reduced matrix coeff.
##################################################

Reduit := proc(l,MOL,er)
local res,cc,G,F,I1,I2,I3,I;
F := exp(-r^2/2)*r^(b(l,MOL,er))*Ylm(l,0,t,p);
G := exp(-r^2/2)*r^(b(l-1,MOL,er))*Ylm(l-1,0,t,p);
cc := abs(Wigner_3j(l,1,l-1,0,0,0));
I := D1k*F*G*r^2*sin(t)/cc;
I1 := int(I,p=0..2*Pi);
I2 := simplify(int(I1,t=0..Pi));
I3 := int(I2,r=0..infinity);
I4 := I3/(GAMMA(b(l,MOL,er)+3/2)*GAMMA(b(l-1,MOL,er)+3/2))^(1/2);
I4 := evalf(I4^2/(2*l+1));
RETURN(I4);
end;


##################################################
### dipolar transition ratio
##################################################

Rl := proc(l,MOL,er)
local res;
res := (Reduit(l,MOL,er)/Reduit(1,MOL,er));
```

```
RETURN(res);
end;

save ‘rotation.m‘;
quit;
```

Appendix B

Nuclear configuration of oxygen

As a final example of the application of the non-compact symplectic groups in physics we consider briefly the enumeration of the states of the oxygen ${}^{16}_{8}O$ nucleus. Here we are interested in those irreducible representations of $O(n)$ that under the decomposition $O(n) \to S_n$ that yield irreducible representations of $S(n)$ involving no parts greater than 4, only those, under conjugation, will yield $U(4)$ irreducible representations corresponding to physical Spin-Isospin states.

Here we consider the case of $N = 16$. We need the decomposition of the fundamental irreducible representations of $Sp(96, R) \to Sp(6, R) \times O(16)$. Only irreducible representations of $Sp(6, R)$ involving three non-zero parts can yield physical states. For the even parity states the partitions will need to be at least of weight 12 and for the odd parity states at least of weight 13. Using the program SCHUR we find for the even parity states of weight 12 using the branching rule $Sp(96, R) \to Sp(6, R) \times O(16)$ for $\langle \frac{1}{2}(0) \rangle$

$$\langle \frac{1}{2}(0) \rangle \to$$

$$\begin{array}{llll} \langle 8; (4^3) \rangle [4^3] & + \langle 8; (543) \rangle [543] & + \langle 8; (5^2 2) \rangle [5^2 2] & + \langle 8; (63^2) \rangle [63^2] \\ + \langle 8; (642) \rangle [642] & + \langle 8; (651) \rangle [651] & + \langle 8; (732) \rangle [732] & + \langle 8; (741) \rangle [741] \\ + \langle 8; (82^2) \rangle [82^2] & + \langle 8; (831) \rangle [831] & + \langle 8; (921) \rangle [921] & + \langle 8; (10\ 1^2) \rangle [10\ 1^2] \end{array} \tag{B.1}$$

and for the odd parity states applying the same rule to $\langle \frac{1}{2}(1) \rangle$

$$\langle \frac{1}{2}(1) \rangle \to$$

$$\begin{array}{llll} \langle 8; (54^2) \rangle [54^2] & + \langle 8; (5^2 3) \rangle [5^2 3] & + \langle 8; (643) \rangle [643] & + \langle 8; (652) \rangle [652] \\ + \langle 8; (6^2 1) \rangle [6^2 1] & + \langle 8; (73^2) \rangle [73^2] & + \langle 8; (742) \rangle [742] & + \langle 8; (751) \rangle [751] \\ + \langle 8; (832) \rangle [832] & + \langle 8; (841) \rangle [841] & + \langle 8; (92^2) \rangle [92^2] & + \langle 8; (931) \rangle [931] \\ + \langle 8; (10\ 21) \rangle [10\ 21] & + \langle 8; (11\ 1^2) \rangle [11\ 1^2] & & \end{array} \tag{B.2}$$

Since in the groundstate of ${}^{16}_{8}O$ one must have just one physical state with $I = 0$ and $S = 0$, only one of the irreducible representations in (B.1) can yield such a state. This is the $O(16)$ irreducible representation $[4^3]$. All the other irreducible representations in (B.1) cannot yield a $S(16)$ irreducible representation containing all parts lower or equal to 4 and may be discarded.

The odd parity states of weight 13 must come from the nuclear configuration $s^4p^{11}(s+d)$. The p^{11} configuration with respect to $U(3)$, belongs to $\{4^23\}$ and the $(s+d)$ to $\{2\}$ and hence the only relevant $U(3)$ irreducible representations of weight 13 must be $\{643\}$ et $\{54^2\}$.

Under the restriction $O(16) \to S(16) \sim U(4)$ we find

$$O(16) \to S(16) \sim U(4)$$

$O(16)$	$S(16)$	$U(4)$
$[4^3]$	$\{4^4\}$	$\{4^4\}$
$[54^2]$	$\{4^331\} + \{4^4\}$	$\{54^23\} + \{4^4\}$
$[643]$	$\{4^331\} + \{4^4\}$	$\{54^23\} + \{4^4\}$

and under $U(4) \to SU(4) \to SU(2)^I \times SU(2)^S$ we obtain

$$U(4) \to SU(4) \to SU(2)^I \times SU(2)^S$$

$U(4)$	$SU(4)$	$(2I+1, 2S+1)$
$\{4^4\}$	$\{0\}$	$(1,1)$
$\{54^23\}$	$\{21^2\}$	$(1,3)+(3,1)+(3,3)$

All other irreducible representations in equations (B.1) and et (B.2) can be regarded as non-physical states and can be said to be associated with "*dead partitions*").

The above exercise shows that one needs a critereon that would allow one to immediately identify the dead partitions (or alternatively the live partitions). The next step in this case is to parametrize this partitions and give the algorithm to sort these partitions.

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