STRETCHED LITTLEWOOD-RICHARDSON AND KOSTKA COEFFICIENTS

R C KING, C TOLLU, AND F TOUMAZET

This paper is dedicated to the memory of Professor R. T. Sharp, a great colleague and friend.

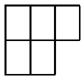
ABSTRACT. Littlewood-Richardson and Kostka coefficients are specified by means of partitions. If the parts of the partitions are all multiplied by some fixed positive integer N, then the values of the coefficients will change as a function of the stretching parameter N. It is conjectured that this behaviour is always polynomial in N. The study makes use of a hive model for evaluating the coefficients, thereby making a connection with integer points of certain rational convex polytopes. Many examples of the relevant polynomials are given, along with further conjectures regarding their general form and that of their generating functions.

1. INTRODUCTION

The objective here is to describe some speculative work, still in progress, on the behaviour of Littlewood-Richardson coefficients, $c^{\lambda}_{\mu\nu}$, and Kostka coefficients, $K_{\lambda\mu}$, under the simultaneous stretching of the partitions λ, μ and ν by some scale factor N, where N is a positive integer. The work was motivated by Fulton's Conjecture that $c^{N\lambda}_{N\mu,N\nu} = 1$ for all positive integers N if and only if $c^{\lambda}_{\mu\nu} = 1$, and by the observation that $c^{N\lambda}_{N\mu,N\nu}$ appears always to be $N+1$ if $c^{\lambda}_{\mu\nu} = 2$.

We offer here, along with a considerable amount of supporting evidence, a series of stronger conjectures. The most important of these is that the stretched Littlewood-Richardson coefficients are polynomial in the stretching parameter. A similar conjecture applies to stretched Kostka coefficients.

In order to define both Littlewood-Richardson coefficients and Kostka coefficients it is necessary to introduce some standard notation and terminology [7, 8]. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0$ be a partition of weight $|\lambda| = \lambda_1 + \cdots + \lambda_p$ and length $\ell(\lambda) = p$. The corresponding Young diagram F^{λ} consists of $|\lambda|$ boxes arranged in $\ell(\lambda) = p$ left-adjusted rows of lengths $\lambda_1, \lambda_2, \ldots, \lambda_p$. For example, for $\lambda = (3, 2)$ we have

(1.1) $F^{3,2} =$ 

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A standard Young tableau, T^λ, is a numbering of the boxes of F^λ with entries from $\{1,2,\ldots,n\}$, weakly increasing across rows and strictly increasing down columns. Typically, for $\lambda=(3,2)$ we have amongst others the following standard Young tableaux, T^{32} of shape F^{32} provided that $n\geq 3$:

$$\begin{array}{ccc}1&1&1\\2&2&\end{array}\qquad\begin{array}{ccc}1&1&1\\2&3&\end{array}\qquad\begin{array}{ccc}1&1&2\\2&3&\end{array}\qquad\begin{array}{ccc}1&1&3\\2&2&\end{array}\qquad\cdots\quad . \tag{1.2}$$

The Schur function $s_\lambda(\mathbf{x})$ is the symmetric function of the components of $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ that may be defined by

$$s_\lambda(\mathbf{x})=\sum_{T^\lambda} x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}, \tag{1.3}$$

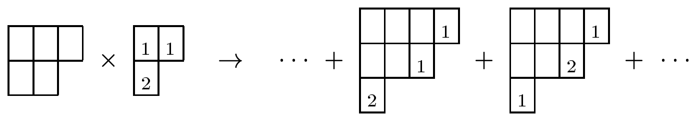
where m_i is the number of entries i in T^λ for $i=1,2,\ldots,n$.

The Littlewood-Richardson coefficients, $c^\lambda_{\mu\nu}$, are then defined by

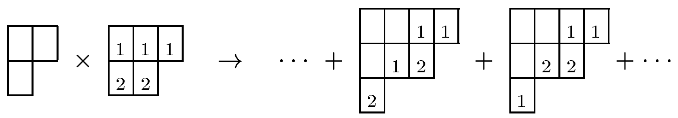
$$s_\lambda(\mathbf{x})s_\mu(\mathbf{x})=\sum_\nu c^\nu_{\lambda\mu}\ s_\nu(\mathbf{x}). \tag{1.4}$$

They may be evaluated by means of the famous Littlewood-Richardson rule [7]p94, [8]p142. This states that $c^\nu_{\lambda\mu}$ is the number of distinctly labelled diagrams of shape F^ν obtained by adding to F^λ the boxes of F^μ labelled by their row number in such a way that at every stage: (i) the entries increase weakly across rows; (ii) the entries increase strictly down columns; (iii) the sequence of entries read from right to left across rows taken in turn from top to bottom is such that $\#1's\geq\#2's\geq\cdots\geq\#n's$ (the lattice permutation rule).

For example, if $\lambda=(3,2)$, $\mu=(2,1)$ and $\nu=(4,3,1)$, then $c^{431}_{32,21}=2$ since the Littlewood-Richardson rule gives rise to just to just two appropriately labelled diagrams:



Similarly, if $\lambda=(2,1)$, $\mu=(3,2)$ and $\nu=(4,3,1)$, then $c^{431}_{21,32}=2$, where the only relevant diagrams are:



Kostka coefficients may also be defined combinatorially [7]p191, [8]p101 : $K_{\lambda\mu}$ is the number of distinctly labelled standard Young tableaux T^λ of shape F^λ and weight μ, that is to say with μ_i entries i for $i=1,2,\ldots,\ell(\mu)$.

For example, if $\lambda=(3,2)$ and $\mu=(2,2,1)$ then there are just two standard Young tableaux T^{32} of weight $(2,2,1)$:

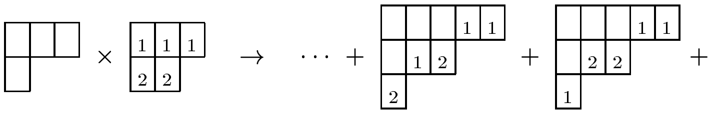
$$\begin{array}{ccc}1&1&2\\2&3&\end{array}\qquad\begin{array}{ccc}1&1&3\\2&2&\end{array}$$

This implies that $K_{32,221}=2$.

Kostka coefficients can be viewed as special types of Littlewood-Richardson coefficients. To be precise $K_{\lambda\mu} = c^{\tau}_{\sigma\lambda}$ where

$$\begin{cases} \tau_i = \mu_i + \mu_{i+1} + \cdots \\ \sigma_i = \mu_{i+1} + \mu_{i+2} + \cdots \end{cases} \qquad \text{for} \quad i = 1, 2, \ldots \tag{1.5}$$

For example, if $\lambda = (3,2)$ and $\mu = (2,2,1,)$, then $\tau = (5,3,1)$ and $\sigma = (3,1)$. The calculation of $c^{\tau}_{\sigma\lambda}$ by means of the Littlewood-Richardson rule then leads to the diagrams:

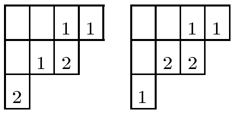


It follows that $K_{32,221} = c^{531}_{31,32} = 2$, as obtained before.

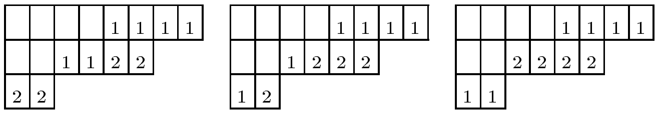
Now we are in a position to define and evaluate stretched Littlewood-Richardson and Kostka coefficients. The partition obtained from $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ by multiplying all of its parts by the same positive integer, N, is denoted by $N\lambda = (N\lambda_1, N\lambda_2, \ldots, N\lambda_p)$. With this notation, we refer to $c^{N\lambda}_{N\lambda,N\mu}$ and $K_{N\lambda,N\mu}$ as stretched Littlewood-Richardson and Kostka coefficients, respectively, where N is said to be the stretching parameter. It is not difficult to evaluate these stretched coefficients.

For example, if $\lambda = (2,1)$, $\mu = (3,2)$ and $\nu = (4,3,1)$ then the stretched Littlwood-Richardson coefficient $c^{N\nu}_{N\lambda,N\mu}$ may be evaluated for various N as follows:

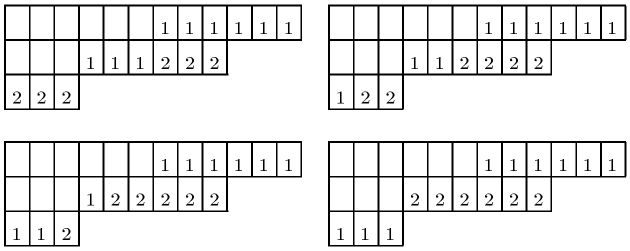
$N = 1:$



$N = 2:$



$N = 3:$



It follows that $c^{\nu}_{\lambda\mu} = 2$, $c^{2\nu}_{2\lambda,2\mu} = 3$, and $c^{3\nu}_{3\lambda,3\mu} = 4$. Moreover, by noting that the only freedom in these diagrams lies in the labelling of the boxes in the third row, it is easy to see that in this case $c^{N\nu}_{N\lambda,N\mu} = N + 1$.

Similarly, if $\lambda = (3,2)$ and $\mu = (2,1,1,1)$ then the stretched Kostka coefficients $K_{N\lambda,N\mu}$ are determined for various N by the enumeration of the following standard tableaux.

$$N=1: \quad \begin{ytableau} 1 & 1 & 2 \\ 3 & 4 \end{ytableau} \quad \begin{ytableau} 1 & 1 & 3 \\ 2 & 4 \end{ytableau} \quad \begin{ytableau} 1 & 1 & 4 \\ 2 & 3 \end{ytableau}$$

$$N=2: \quad \begin{ytableau} 1 & 1 & 1 & 1 & 2 & 2 \\ 3 & 3 & 4 & 4 \end{ytableau} \quad \begin{ytableau} 1 & 1 & 1 & 1 & 2 & 3 \\ 2 & 3 & 4 & 4 \end{ytableau} \quad \begin{ytableau} 1 & 1 & 1 & 1 & 3 & 3 \\ 2 & 2 & 4 & 4 \end{ytableau}$$

$$\begin{ytableau} 1 & 1 & 1 & 1 & 2 & 4 \\ 2 & 3 & 3 & 4 \end{ytableau} \quad \begin{ytableau} 1 & 1 & 1 & 1 & 3 & 4 \\ 2 & 2 & 3 & 4 \end{ytableau} \quad \begin{ytableau} 1 & 1 & 1 & 1 & 4 & 4 \\ 2 & 2 & 3 & 3 \end{ytableau}$$

$$N=3: \quad \begin{ytableau} 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ 3 & 3 & 3 & 4 & 4 & 4 \end{ytableau} \quad \begin{ytableau} 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 3 & 4 & 4 & 4 \end{ytableau}$$

$$\begin{ytableau} 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 3 & 4 & 4 & 4 \end{ytableau} \quad \begin{ytableau} 1 & 1 & 1 & 1 & 1 & 1 & 3 & 3 & 3 \\ 2 & 2 & 2 & 4 & 4 & 4 \end{ytableau}$$

$$\begin{ytableau} 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 4 \\ 2 & 3 & 3 & 3 & 4 & 4 \end{ytableau} \quad \begin{ytableau} 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 4 & 4 & 4 \end{ytableau}$$

$$\begin{ytableau} 1 & 1 & 1 & 1 & 1 & 1 & 3 & 3 & 4 \\ 2 & 2 & 2 & 3 & 4 & 4 \end{ytableau} \quad \begin{ytableau} 1 & 1 & 1 & 1 & 1 & 1 & 2 & 4 & 4 \\ 2 & 2 & 3 & 3 & 4 & 4 \end{ytableau}$$

$$\begin{ytableau} 1 & 1 & 1 & 1 & 1 & 1 & 3 & 4 & 4 \\ 2 & 2 & 2 & 3 & 3 & 4 \end{ytableau} \quad \begin{ytableau} 1 & 1 & 1 & 1 & 1 & 1 & 4 & 4 & 4 \\ 2 & 2 & 2 & 3 & 3 & 3 \end{ytableau}$$

It follows that in this case $K_{\lambda\mu}=3$, $K_{2\lambda,2\mu}=6$ and $K_{3\lambda,3\mu}=10$. Quite generally, the right-most N boxes in the first row can only be filled with entries 2, 3 or 4, in standard order. Once these are chosen the remaining entries are all fixed. It follows that

$$K_{N\lambda,N\mu} = \frac{1}{2}(N+1)(N+2). \tag{1.6}$$

2. Known results

Having exposed the nature of the problem, it is worth listing some known results.

Theorem 2.1. *(Saturation Conjecture)* [5]

$$c^{N\nu}_{N\lambda,N\mu} > 0 \iff c^{\nu}_{\lambda\mu} > 0. \tag{2.1}$$

Corollary 2.2. *(K-Saturation Conjecture)*

$$K_{N\lambda,N\mu} > 0 \iff K_{\lambda\mu} > 0. \tag{2.2}$$

Theorem 2.3. *(Fulton's Conjecture)* [4, 6]

$$c^{N\nu}_{N\lambda,N\mu} = 1 \iff c^{\nu}_{\lambda\mu} = 1. \tag{2.3}$$

Corollary 2.4. *(K-Fulton's Conjecture)*

$$K_{N\lambda,N\mu} = 1 \iff K_{\lambda\mu} = 1. \tag{2.4}$$

The dominance ordering of partitions is such that λ is said to dominate μ, and we write $\lambda \geq \mu$ if and only if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \tag{2.5}$$

for all $i = 1, 2, \ldots$. With this notation it is possible to establish the following:

Lemma 2.5.

$$K_{\lambda\mu} > 0 \iff |\lambda| = |\mu| \textit{ and } \lambda \geq \mu. \tag{2.6}$$

Thanks to this Lemma one can write down a rather simple direct proof of Corollary 2.2, namely the argument

$$\begin{aligned} K_{\lambda\mu} > 0 \iff |\lambda| = |\mu| \text{ and } \lambda \geq \mu \\ \iff |N\lambda| = |N\mu| \text{ and } N\lambda \geq N\mu \iff K_{N\lambda,N\mu} > 0. \end{aligned} \tag{2.7}$$

where use has been made of the rather obvious fact that the simulataneous scaling of all the parts of the partitions λ and μ by N preserves both the equality of weights of these partitions and their dominance order.

Results of a quite different nature may be found by noting the connection between Littlewood-Richardson coefficients and tensor products, and between Kostka coefficients and weight multiplicities, all within the context of the representation theory of the general linear group, $GL(n)$.

To be precise, for each partition λ with $\ell(\lambda) \leq n$ there exists an irreducible representation V^λ of $GL(n)$ of highest weight λ and character $\mathrm{ch}\, V^\lambda$. With a suitable identification of the indeterminates $x_1, x_2, \ldots, x_n$, the character $\mathrm{ch}\, V^\lambda$ is nothing other than the Schur-function $s_\lambda(\mathbf{x})$. It follows that the Littlewood-Richardson coefficients govern the decomposition of tensor products of irreducible representations of $GL(n)$ in accordance with the formula

$$V^\lambda \otimes V^\mu = \sum_\nu c_{\lambda\mu}^\nu \; V^\nu. \tag{2.8}$$

Moreover, the weight space decomposition of the irreducible representation V^λ takes the form $V^\lambda = \oplus_{\mathbf{m}} V_{\mathbf{m}}^\lambda$, where the sum is taken over all weights $\mathbf{m} = (m_1, m_2, \ldots, m_n)$. This decomposition is such that the multiplicity of the weight specified by $\mathbf{m} = \mu$ in the irreducible representation V^λ is given by $\dim V_\mu^\lambda = K_{\lambda\mu}$.

This allows us to lean on some well known results in the physics literature. First, the quantum theory of angular momentum is such that the coupling of two angular momenta is multiplicity free. This implies that all tensor product and weight decompositions are multiplicity free for not only the rotation group $SO(3)$, but also for $SU(2)$ and even $GL(2)$. This leads to the following:

Observation 2.6. *For all λ, μ, ν with $\ell(\lambda), \ell(\mu), \ell(\nu) \leq 2$*

$$c_{N\lambda,N\mu}^{N\nu} = c_{\lambda\mu}^\nu = 1 \textit{ and } K_{N\lambda,N\mu} = K_{\lambda\mu} = 1. \tag{2.9}$$

In connection with the special unitary group $SU(3)$ Wigner [10] established a formula for weight multiplicities which applies equally well to $GL(3)$, namely

$$K_{\lambda\mu} = 1 + \min\{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \lambda_1 - \mu_1, \mu_3 - \lambda_3\}. \tag{2.10}$$

This implies that for any positive integer N

$$K_{N\lambda,N\mu}-1=\min\{N\lambda_1-N\lambda_2,N\lambda_2-N\lambda_3,N\lambda_1-N\mu_1,N\mu_3-N\lambda_3\} \\ = N\,\min\{\lambda_1-\lambda_2,\lambda_2-\lambda_3,\lambda_1-\mu_1,\mu_3-\lambda_3\}=N\,(K_{\lambda\mu}-1)\,. \tag{2.11}$$

Hence we have

Observation 2.7. *If $K_{\lambda\mu}=k>0$ with $\ell(\lambda),\ell(\mu)\leq 3$ then*

$$K_{N\lambda,N\mu}=1+N(k-1). \tag{2.12}$$

3. Polynomial conjectures

We are now in a position to state our main conjecture, namely

Conjecture 3.1. *For all partitions λ,μ,ν such that $c^{\nu}_{\lambda\mu}>0$ there exists a polynomial $P^{\nu}_{\lambda\mu}(N)$ in N with non-negative rational coefficients such that $P^{\nu}_{\lambda\mu}(0)=1$ and $P^{\nu}_{\lambda\mu}(N)=c^{N\nu}_{N\lambda,N\mu}$ for all positive integers N.*

An immediate corollary of this conjecture, if proved true, but one that also stands in its own right as a new conjecture, is the following:

Conjecture 3.2. *For all partitions λ,μ such that $K_{\lambda\mu}>0$ there exists a polynomial $P_{\lambda\mu}(N)$ in N with non-negative rational coefficients such that $P_{\lambda\mu}(0)=1$ and $P_{\lambda\mu}(N)=K_{N\lambda,N\mu}$ for all positive integers N.*

The nature of the polynomials encountered in these two conjectures is certainly constrained in some way, and as preliminary comment we offer the following:

Conjecture 3.3. *Given k such that $c^{\nu}_{\lambda\mu}=k$ or $K_{\lambda\mu}=k$ there exists only a restricted number of distinct polynomials $P(N)$ such that $P(N)=P^{\nu}_{\lambda\mu}(N)$ or $P(N)=P_{\lambda\mu}(N)$. In particular:*

$$\begin{array}{lll} k=0 & \Longrightarrow P(N)=0; & \textit{(Saturation Conjecture)} \\ k=1 & \Longrightarrow P(N)=1; & \textit{(Fulton's Conjecture)} \\ k=2 & \Longrightarrow P(N)=N+1; & \\ k=3 & \Longrightarrow P(N)=2N+1 & \textit{or}\quad P(N)=\tfrac{1}{2}(N+1)(N+2). \end{array} \tag{3.1}$$

Evidence in support of our polynomial conjectures comes to a large extent from fitting data on Littlewood-Richardson and Kostka coefficients calculated for many partitions λ, μ and ν for a wide range of values of N. For example, for $\lambda=(4,2,1)$ and $\mu=(5,3,2)$ the unstretched Littlewood-Richardson coefficient $c^{\nu}_{421,532}=k$ takes values ranging from $k=1$ to $k=7$ for various ν. The data on the corresponding stretched Littlewood-Richardson coefficients for representative partitions ν can all be fitted by the polynomials $P(N)=c^{N\nu}_{N(421),N(532)}$ given below.

$$\begin{array}{lll} k=1 & \nu=(953) & P(N)=1 \\ k=2 & \nu=(9431) & P(N)=(N+1) \\ k=3 & \nu=(8441) & P(N)=(N+1)(N+2)/2 \\ k=4 & \nu=(8531) & P(N)=(N+1)(N+2)(N+3)/6 \\ k=4 & \nu=(7442) & P(N)=(N+1)^2 \\ k=5 & \nu=(7541) & P(N)=(N+1)(N+2)(2N+3)/6 \\ k=6 & \nu=(7532) & P(N)=(N+1)^2(N+2)/2 \\ k=7 & \nu=(74321) & P(N)=(N+1)(N+2)(N^2+3N+6)/6 \end{array} \tag{3.2}$$

It is to be noted that in the case $k = 4$, not only do two different polynomials arise, but they are of different degrees.

Similarly, for various values of the unstretched Kostka coefficients $K_{\lambda\mu} = k$ specified for some quite representative partitions λ and μ, we find that the data on the corresponding stretched Kostka coefficients can all be fitted by polynomials $P(N) = K_{N\lambda,N\mu}$ as illustrated below.

(3.3)

$$\begin{array}{llll}
k=1 & \lambda=(32) & \mu=(31^2) & P(N)=1 \\
k=2 & \lambda=(32) & \mu=(2^21) & P(N)=(N+1) \\
k=3 & \lambda=(32) & \mu=(21^3) & P(N)=(N+1)(N+2)/2 \\
k=4 & \lambda=(41) & \mu=(1^5) & P(N)=(N+1)(N+2)(N+3)/6 \\
k=5 & \lambda=(32) & \mu=(1^5) & P(N)=(N+1)(N^2+2N+2))/2 \\
k=5 & \lambda=(221) & \mu=(1^5) & P(N)=(N+1)(N+2)(N^2+3N+6)/12 \\
k=6 & \lambda=(321) & \mu=(1^6) & P(N)=(N+1)(N+2)(N+3)\cdot \\
 & & & \quad \cdot(17N^2+80N^3+187N^2+220N+168)/1008
\end{array}$$

In general, even for the stretched Kostka coefficients, let alone the more general Littlewood-Richardson coefficients, it appears to be rather difficult to predict the nature of the corresponding polynomial $P(N)$, or even its degree.

However, in the case of a hook shaped partition $\lambda = (a+1, 1^b)$ we do have the following:

Proposition 3.4. *Let $\lambda = (a+1, 1^b)$ and μ be partitions of m, so that $\ell(\lambda) = b+1$, and let $\ell(\mu) = c+1$. Then*

$$K_{\lambda\mu} = \binom{c}{b} \quad \text{and} \quad K_{N\lambda,N\mu} = \prod_{i=1}^{b}\prod_{j=1}^{c-b} \frac{N+i+j-1}{i+j-1}. \tag{3.4}$$

Proof Each hook-shaped standard tableau T^λ contains μ_i i's for $i = 1, 2, \ldots, \ell(\mu)$. All the 1's must go in the top row, so that in particular entry at the top row of the first column must be 1. The b entries β_j for $j = 1, \ldots, b$ in the leg are all distinct and must be chosen from $\{2, 3, \ldots, c+1\}$ without restriction since $\mu_i \geq 1$ for $i = 2, \ldots, c+1$. The remaining a entries α_i for $i = 1, \ldots, a$ are then all placed in the arm in weakly increasing order. This is illustrated below.

$$T^\lambda = \begin{array}{|c|c|c|c|c|}
\hline
1 & \alpha_1 & \alpha_2 & \cdots & \alpha_a \\
\hline
\beta_1 \\
\cline{1-1}
\beta_2 \\
\cline{1-1}
\vdots \\
\cline{1-1}
\beta_b \\
\cline{1-1}
\end{array} \tag{3.5}$$

It follows by a simple counting argument that $K_{\lambda\mu} = \binom{c}{b}$.

If we now consider each fat hook-shaped tableaux $T^{N\lambda}$ it must contains $N\mu_i$ i's for $i = 1, 2, \ldots, \ell(\mu)$. There are at least N 1's, all to go in the top row, filling in particular all the boxes at the top of the fat column. The remaining Nb entries βij with $i = 1, \ldots, b$ and $j = 1, \ldots, N$ in the fat leg are chosen from $\{2, 3, \ldots, c+1\}$ with repetitions allowed across rows but not down columns, but without restriction of choice since $N\mu_i \geq N$ for $i = 2, \ldots, \ell(\mu)$. Finally the remaining Na entries α_{ij}

for $i = 1, \dots, N$ and $j = 1, 2, \dots, a$ are placed in weakly increasing order in the stretched thin arm. Once again this is illustrated by:

$$T^{N\lambda} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & \cdots & 1 & \alpha_{11} & \alpha_{12} & & \cdots & & \alpha_{Na} \\ \hline \beta_{11} & \beta_{12} & \cdots & \beta_{1N} \\ \cline{1-4} \beta_{21} & \beta_{22} & \cdots & \beta_{2N} \\ \cline{1-4} \vdots & \vdots & \cdots & \vdots \\ \cline{1-4} \beta_{b1} & \beta_{b2} & \cdots & \beta_{bN} \\ \cline{1-4} \end{array} \tag{3.6}$$

It follows that $\#\{T^{N\lambda}\} = \#\{T^{N^b}\} = \dim_{GL(c)} V^{N^b}$. The famous hook-length formula [8]p45 then gives the required result, which is a polynomial in N.

4. The hive model and convex polytopes

To try to understand the nature of the various polynomials fits that we have found it is worth considering a model for all the inequalities underlying the calculation of Littlewood-Richardson and Kostka coefficients. This model is known as the hive model [2, 5, 6].

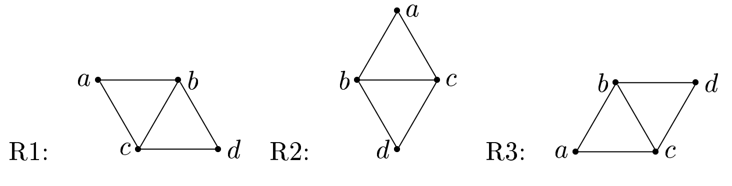
An n-hive is a triangular array of numbers a_{ij} with $0 \le i, j \le n$. Typically, for $n = 4$ their arrangement is as shown below

$$\begin{array}{ccccccccccc} & & & & & a_0^0 & & & & & \\ & & & & / & & \backslash & & & & \\ & & & & a_0^1 & — & a_1^0 & & & & \\ & & & / & \backslash & / & \backslash & & & & \\ & & & a_0^2 & — & a_1^1 & — & a_2^0 & & & \\ & & / & \backslash & / & \backslash & / & \backslash & & & \\ & & a_0^3 & — & a_1^2 & — & a_2^1 & — & a_3^0 & & \\ & / & \backslash & / & \backslash & / & \backslash & / & \backslash & & \\ & a_0^4 & — & a_1^3 & — & a_2^2 & — & a_3^1 & — & a_4^0 & \\ / & \backslash & / & \backslash & / & \backslash & / & \backslash & / & \backslash \\ a_0^5 & — & a_1^4 & — & a_2^3 & — & a_3^2 & — & a_4^1 & — & a_5^0 \end{array} \tag{4.1}$$

Such an n-hive is said to be an integer hive if all of its entries are non-negative integers.

Neighbouring entries define three distinct types of rhombus, each with its own constraint condition.

(4.2)



In each case, with the labelling as shown, the hive condition takes the form:

$$\text{HC:} \qquad b + c \ge a + d \tag{4.3}$$

Definition 4.1. *An LR-hive is an integer hive satisfying the hive condition HC for all its constituent rhombi of type R1, R2 and R3, with border entries determined by partitions λ, μ, ν in such a way that $a_0^0 = 0$, $a_j^0 = \lambda_1 + \lambda_2 + \cdots + \lambda_j$ for $j = 1, 2, \dots, n$,*

$a_{i0} = \nu_1 + \nu_2 + \cdots + \nu_i$ for $i = 1, 2, \ldots, n$, $a^k_{n-k} = a^0_n + \mu_1 + \mu_2 + \cdots + \mu_k$ for $k = 1, 2, \ldots, n$, with $\ell(\lambda), \ell(\mu), \ell(\nu) \le n$ and $|\lambda| + |\mu| = |\nu|$.

Schematically, we have

$$
\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & / & & \backslash & & & & \\
 & & & \nu_1 & & — & & \lambda_1 & & & \\
 & & / & & \backslash & & / & & \backslash & & \\
 & \nu_1+\nu_2 & & — & & a^1_1 & & — & & \lambda_1+\lambda_2 & \\
\cdots & — & a^2_1 & — & a^1_2 & — & \cdots & & & & \\
\cdots & — & \cdots & — & \cdots & — & \cdots & — & \cdots & & \\
|\nu| & — & |\nu|-\mu_n & — & \cdots & — & \cdots & — & |\lambda|+\mu_1 & — & |\lambda|
\end{array}
\tag{4.4}
$$

With the above definition

Proposition 4.2. *The Littlewood-Richardson coefficient $c^\nu_{\lambda\mu}$ is the number of LR-hives with border labels determined as above by λ, μ, ν.*

By way of example if $\lambda = (3,2)$, $\mu = (2,1)$ and $\nu = (4,3,1)$ the corresponding LR-hives take the form

$$
\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 4 & — & 3 & & \\
 & 7 & — & a & — & 5 & \\
8 & — & 8 & — & 7 & — & 5
\end{array}
\tag{4.5}
$$

It follows that $c^{431}_{32,21} = 2$ since the only integer values of a satisfying the hive condition HC for all constituent rhombi of type R1, R2 and R3 are $a = 6$ and $a = 7$.

In the same way we introduce

Definition 4.3. *A K-hive is an integer hive satisfying the hive condition HC for all its constituent rhombi of type R1 and R2 (but not R3), with border labels determined by partitions λ, μ in such a way that $a_{i0} = 0$ for $i = 0, 1, \ldots, n$, $a_{0j} = \lambda_1 + \lambda_2 + \cdots + \lambda_j$ for $j = 1, 2, \ldots, n$, $a_{n-k,k} = \mu_1 + \mu_2 + \cdots + \mu_k$ for $k = 1, 2, \ldots, n$, with $\ell(\lambda), \ell(\mu) \le n$ and $|\lambda| = |\mu|$.*

Schematically, we have

$$
\begin{array}{c}
0 \\
/ \quad \backslash \\
0 \,—\, \lambda_1 \\
/ \quad \backslash \quad / \quad \backslash \\
0 \,—\, a_1^1 \,-\, \lambda_1+\lambda_2 \\
/ \quad \backslash \quad / \quad \backslash \quad / \quad \backslash \\
0 \,—\, a_1^2 \,—\, a_2^1 \,—\, \cdots \\
/ \quad \backslash \quad / \quad \backslash \quad / \quad \backslash \quad / \quad \backslash \\
\cdots \,—\, \cdots \,—\, \cdots \,—\, \cdots \,—\, \cdots \\
/ \quad \backslash \quad / \quad \backslash \quad / \quad \backslash \quad / \quad \backslash \quad / \quad \backslash \\
0 \,—\, \mu_1 \,-\, \mu_1+\mu_2 \,-\, \cdots \,—\, \cdots \,—\, |\lambda|
\end{array}
\tag{4.6}
$$

Proposition 4.4. *The Kostka coefficient $K_{\lambda\mu}$ is the number of K-hives with border labels determined as above by λ, μ.*

For example, if $\lambda = (3,2)$ and $\mu = (2,2,1)$ the corresponding K-hives take the form:

$$
\begin{array}{c}
0 \\
/ \quad \backslash \\
0 \,—\, 3 \\
/ \quad \backslash \quad / \quad \backslash \\
0 \,—\, a \,—\, 5 \\
/ \quad \backslash \quad / \quad \backslash \quad / \quad \backslash \\
0 \,—\, 2 \,—\, 4 \,—\, 5
\end{array}
\tag{4.7}
$$

Since the only integer values of a satisfying the hive condition HC for all the constituent rhombi of type R1 and R2 are $a = 2$ and $a = 3$, it follows that $K_{32,221} = 2$

The adoption of the hive model, as a way of dealing with all the inequalities implicit in the definition of Littlewood-Richardson and Kostka coefficients, has the advantage that it is easy to determine individual coefficients rather than full tensor product or weight space decompositions. The problem is reduced to solving the rhombus constraints for $m = (n-1)(n-2)/2$ interior vertex labels a_{ij}. The corresponding set of linear constraints with integer coefficients defines a convex rational polytope, $\mathcal{P}$, in $\mathbb{R}^m$, known as the hive polytope. The unstretched coefficient k is just the number of points in the intersection of the hive polytope, $\mathcal{P}$, with the integer lattice, $\mathbb{Z}^m$, that is $k = \#\{\mathcal{P} \cap \mathbb{Z}^m\}$. Similarly, the stretched coefficient $k(N)$ is the number of points in the intersection of the stretched convex rational polytope $N\mathcal{P} = \{Nv | v \in P\}$ with the integer lattice, that is $k(N) = \#\{N\mathcal{P} \cap \mathbb{Z}^m\}$.

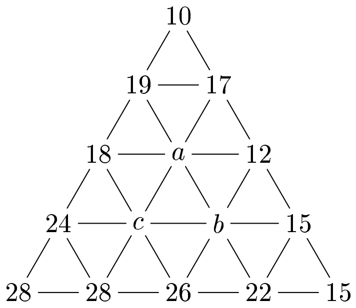
By way of a simple application of this approach we can derive the following:

Proposition 4.5. *For partitions of lengths ≤ 3, if the unstretched Littlewood-Richardson or Kostka coefficient is k then the stretched coefficient is $1 + N(k-1)$.*

Proof For $n = 3$ there is just one interior vertex label $a = a_{11}$. All the unstretched constraints must reduce to a single constraint of the form $a_1 \leq a \leq a_2$ with $a_1, a_2 \in \mathbb{Z}$ both linear in the parts of the relevant partitions. With this notation, $k = 1 + a_2 - a_1$. Correspondingly, the stretched constraints must reduce to the single constraint $Na_1 \leq a \leq Na_2$ so that $k(N) = 1 + Na_2 - Na_1 = 1 + N(k-1)$, as required.

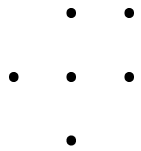
As a more complicated example, the LR-hives showing that $c_{753,742}^{9964} = 6$ take the form:

(4.8)



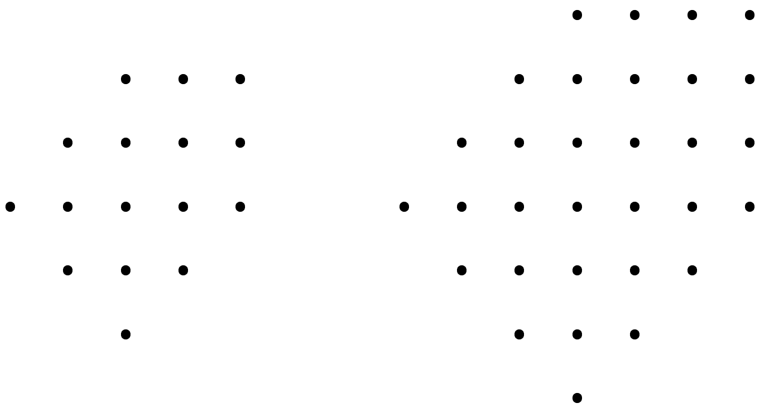
where the constraints are such that $a_{11} = 16$ while $(a_{12}, a_{21}) = (19,23)$, $(20,22)$, $(20,23)$, $(20,24)$, $(21,23)$ or $(21,24)$. These solutions (a_{11}, a_{12}, a_{21}) to the LR-hive constraints define a two-dimensional convex polytope $\mathcal{P}$, whose integer points in $P \cap \mathbb{Z}^3$ are given by the following diagram, which lies in the plane $a_{11} = 16$ with the a_{12} and a_{21} axes horizontal and vertical, respectively.

(4.9)



Expanding $\mathcal{P}$ to give $N\mathcal{P}$ yields the following sets of integer points for $N = 2$ and $N = 3$.

(4.10)



Extending these diagrams to higher values of N is straightforward and counting points leads to the polynomial formula $k(N) = P(N) = \frac{1}{2}(5N^2 + 5N + 2)$.

More generally, the determination of Littlewood-Richardson and Kostka coefficients by the enumeration of integer hives, and hence by the enumeration of integer points in convex rational polytopes, can be exploited in a manner which at first sight comes close to proving our polynomial conjectures.

Let $\mathcal{P} \in \mathbb{R}^m$ be a convex rational polytope of dimension $d \leq m$, let $N\mathcal{P} = \{Nv \,|\, v \in \mathcal{P}\}$ and let $i(\mathcal{P}, N) = \#\{N\mathcal{P} \cap \mathbb{Z}^m\}$ be the number of integer points in

$N\mathcal{P}$. A convex rational polytope $\mathcal{P} \in \mathbb{R}^m$ is an integer polytope if all of its vertices lie in $\mathbb{Z}^m$.

Then we have the following:

Theorem 4.6. *(Ehrhart)* [3], [9]*p235*

For any convex rational polytope $\mathcal{P} \in \mathbb{R}^m$, $i(\mathcal{P}, N)$ is a quasi-polynomial of degree d in N, that is there exists $r \geq 1$ and polynomials $P_s(N)$ of degree d in N such that $i(P,N) = P_s(N)$ for $N \equiv s \mod r$. If the convex rational polytope $\mathcal{P} \in \mathbb{R}^m$ is an integer polytope, then $i(\mathcal{P}, N)$ is a polynomial in N of degree d.

Corollary 4.7. *If the convex rational polytope $\mathcal{P}$ associated with the enumeration of LR-hives or K-hives of given border is an integer polytope, then the corresponding stretched Littlewood-Richardson or Kostka coefficients are polynomial in the stretching parameter N.*

To exploit this Corollary it is neceesary to determine whether or not the integer hives define convex rational polytopes that are integer. We can prove the following:

Theorem 4.8. *Let λ, μ, ν be partitions of lengths ≤ 4, then the hive polytope $\mathcal{P}$ over the border defined by λ, μ, ν is an integer polytope.*

Method of Proof Since $\ell(\lambda), \ell(\mu), \ell(\nu) \leq 4$ we need only consider hives in which the elements to be enumerated constitute a vector $v = (a_{11}, a_{12}, a_{21})$. To prove the theorem it is shown that for any $v = (x, y, z) \in \mathcal{P}$ such that $(x, y, z) \notin \mathbb{Z}^3$ there exists some non-zero perturbation $(p, q, r) \in \mathbb{R}^3$ such that all $v' = (x', y', z') \in [(x-p, y-q, z-r), (x+p, y+q, z+r)]$ are in $\mathcal{P}$. Hence v is not a vertex of $\mathcal{P}$. Thus the vertices of $\mathcal{P}$ are all integer vertices.

Corollary 4.9. *If $\ell(\lambda), \ell(\mu), \ell(\nu) \leq 4$ then both $c^{N\nu}_{N\lambda, N\mu}$ and $K_{N\lambda, N\mu}$ are polynomial in N.*

However, there do exist non-integer hive polytopes. Moreover, not all convex rational polytopes $\mathcal{P}$ associated with LR-hives or K-hives of given border are integer polytopes. For example, the K-hives used to evaluate $K_{32,1^5}$ take the form

$$
\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 0 & & 3 & & & & \\
 & & & 0 & & a & & 5 & & & \\
 & & 0 & & b & & c & & 5 & & \\
 & 0 & & d & & e & & f & & 5 & \\
0 & & 1 & & 2 & & 3 & & 4 & & 5
\end{array}
\tag{4.11}
$$

There exist precisely 5 distinct K-hives of this type, so that $K_{\lambda\mu} = 5$.

However, it can be shown from the hive inequalities that the vertices $v = (a, b, c, d, e, f)$ of the corresponding convex polytope $\mathcal{P}$ are given by

$$\begin{aligned}
v_1 &= (2,4,4,2,3,1)\\
v_2 &= (2,4,4,2,3,2)\\
v_3 &= (3,4,4,2,3,1)\\
v_4 &= (3,4,4,2,3,2)\\
v_5 &= (3,4,4,3,3,2)\\
v_6 &= (\tfrac{5}{2},4,4,\tfrac{3}{2},3,\tfrac{3}{2})
\end{aligned} \tag{4.12}$$

Of these the first 5 are integral and specify the five K-hives to which we have referred. The sixth vertex v_6 is manifestly not integral. The corresponding hive is not a K-hive, it takes the form

$$\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 0 & & 3 & & & & \\
 & & & 0 & & \frac{5}{2} & & 5 & & & \\
 & & 0 & & \frac{3}{2} & & 4 & & 5 & & \\
 & 0 & & \frac{3}{2} & & 3 & & 4 & & 5 & \\
0 & & 1 & & 2 & & 3 & & 4 & & 5
\end{array} \tag{4.13}$$

The convex polytope $\mathcal{P}$ is not integral, so that Ehrhart's Theorem cannot be used to infer that $k(N) = K_{N\lambda,N\mu}$ is a polynomial, only that it is a quasi-polynomial. However, the direct enumeration of K-hives in this case leads to the conclusion that $k(N)$ is in fact polynomial. We find that the data can be fitted by $k(N) = P(N) = (N+1)(N^2+2N+2)/2$.

5. Generating functions

In the spirit of much of the work of Bob Sharp, we can try to encapsulate our results by means of generating functions. All our data on stretched Littlewood-Richardson and Kostka coefficients, $P(N) = c_{N\lambda,N\mu}^{N\nu}$ or $P(N) = K_{N\lambda,N\mu}$, can be fitted by generating functions of the form

$$F(z) = \frac{G(z)}{(1-z)^{d+1}} = \sum_{N=0}^{\infty} P(N)\; z^N. \tag{5.1}$$

This is of course a consequence of our finding that that in every case $P(N)$ is a polynomial.

Typically, we find the following generating functions for $K_{N\lambda,N1^6}$ for various λ.

$$
\begin{array}{rl}
(6): & 1/(1-z) \\
(51): & 1/(1-z)^5 \\
(42): & (1+4z+5z^2)/(1-z)^5 \\
(41^2): & (1+3z+z^2)/(1-z)^7 \\
(3^2): & (1+z+z^2)/(1-z)^4 \\
(321): & (1+8z+35z^2+32z^3+9z^4)/(1-z)^8 \\
(31^3): & (1+3z+z^2)/(1-z)^7 \\
(2^3): & (1+z^2)/(1-z)^5 \\
(2^21^2): & (1+z+6z^2+z^3+z^4)/(1-z)^8 \\
(21^4): & 1/(1-z)^5 \\
(1^6): & 1/(1-z)
\end{array}
\tag{5.2}
$$

On the basis of a wealth of similar results we are led to:

Conjecture 5.1. *The generating functions for both $c_{N\lambda,N\mu}^{N\nu}$ and $K_{N\lambda,N\mu}$ take the form $F(z) = G(z)/(1-z)^{d+1}$, with $G(z)$ a polynomial of degree $\deg G \leq d$ having non-negative integral coefficients.*

If true, then it would immediately follow that both $c_{N\lambda,N\mu}^{N\nu}$ and $K_{N\lambda,N\mu}$ are polynomial in N. Of course if the corresponding unstretched K-hives define convex integer polytopes then the conjecture follows from the known properties of Ehrhart polynomials. These properties include the following [9]p208:

Theorem 5.2. *If $\mathcal{P}$ is a convex integer polytope the generating function for the Ehrhart polynomial $i(\mathcal{P}, N)$ takes the form $F(z) = G(z)/(1-z)^{d+1}$, with $G(z)$ a polynomial of degree $\leq d$ having non-negative integral coefficients.*

However, as we have seen not all polytopes $\mathcal{P}$ associated with LR-hives or K-hives of given border are integer. The challenge is therefore, either to prove the validity of our conjectures or provide counter examples.

In the first case the task is to prove that both $P(N) = c_{N\lambda,N\mu}^{N\nu}$ and $P(N) = K_{N\lambda,N\mu}$ are polynomial in N for all partitions λ, μ, ν. If this can be done, it would then be desirable to determine the degree d of $P(N)$ as a function of λ, μ, ν, and to show if possible that all its coefficients are non negative rational numbers. If $F(z) = G(z)/(1-z)^{d+1}$ is the generating function of $P(N)$, it is then necesary to show that $G(z)$ is a polynomial whose coefficients are all non-negative integers, and to determine the degree $\deg G$ of this polynomial.

In the case of counter examples, one must identify partitions λ, μ, ν such that $P(N)$ is quasi-polynomial but not polynomial. In the case of such a quasi-polynomial it would then be desirable to determine the form of the corresponding generating function $F(z)$ and its properties.

We close with an example that might either whet the appetite or dull the senses. In the case of the Kostka coefficients $K_{\lambda\mu}$ with $\lambda = (2^p)$ and $\mu = (1^{2p})$ we believe on the basis of extensive calculations that $P(N) = K_{N\lambda,N\mu}$ is a polynomial of degree $d = (p-1)^2$, and that the numerator of the generating function $G(z)$ is a symmetric polynomial of degree $(p-1)(p-2)$ with non-negative integer coefficients. To exemplify this, in the case $\lambda = (2^4)$ and $\mu = (1^8)$ we can fit the data with the following:

$$
\begin{aligned}
P(N) &= \frac{1}{3240}(N+1)(N+2)(N+3)(N^2+4N+9)\cdot \\
&\qquad \cdot(N^4+8N^3+26N^2+40N+60) \\
F(z) &= (1+4z+31z^2+40z^3+31z^4+4z^5+z^6)/(1-z)^{10}
\end{aligned}
\tag{5.3}
$$

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References

1. A.D. Berenstein and A.V. Zelevinsky, *Triple multiplicities for $sl(r+1)$ and the spectrum of the exterior algebra of the adjoint representation* J. Alg. Comb. **1** (1992), 7–22.
2. A.S. Buch, *The saturation conjecture (after A Knutson and T Tao)* arXive:math.CO/9810180 v1 30 Oct 1998
3. E. Ehrhart, *Polynomes arithmetiques et methode des polyedres en combinatoire* Int. Ser. of Numerical Maths. **35** Birkhauser Verlag, Basel, 1977
4. W. Fulton, *Eigenvalues, invariant factors, highest weight, and Scubert calculus* Bull. Amer. Math. Soc. **37** (2000) 209–249.
5. A. Knutson and T. Tao, *The honeycomb model of $GL_n(\mathbb{C})$ tensor products I: Proof of the saturation conjecture* J. Amer. Math. Soc. **12** (1999) 1055–1090.
6. A. Knutson, T. Tao and C. Woodward, *The honeycomb model of $GL_n(\mathbb{C})$ tensor products II: Puzzles determine facets of the Littlewood-Richardson cone* arXive:math.CO/0107011 2 Jan 2001.
7. D.E. Littlewood, *The Theory of Group Characters* Clarendon Press, Oxford, 1940.
8. I.G. Macdonald, *Symmetric Functions and Hall Polynomials* 2nd Edition, Clarendon Press, Oxford, 1995.
9. R.P. Stanley, *Enumerative Combinatorics* Vol 1. Cambridge University Press, Cambridge, 1997.
10. E. Wigner, *On the consequences of the symmetry of the nuclear Hamiltonian on the spectroscopy of nuclei* Phys. Rev. **51** (1937) 106–119.

R C King, Faculty of Mathematical Studies, University of Southampton, Southampton SO17 1BJ, England

E-mail address: `R.C.King@maths.soton.ac.uk`

C Tollu, Laboratoire d'Informatique de Paris-Nord, Université Paris 13, 93430 Villetaneuse, France

E-mail address: `ct@lipn.univ-paris13.fr`

F Toumazet, Laboratoire d'Informatique de Paris-Nord, Université Paris 13, 93430 Villetaneuse, France

E-mail address: `ft@lipn.univ-paris13.fr`