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THE HIVE MODEL AND THE FACTORISATION OF KOSTKA COEFFICIENTS

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Abstract. The hive model is used to explore the properties of both Kostka coefficients and stretched Kostka coefficient polynomials. It is shown that both of these may factorise, and that they can then be expressed as products of certain primitive coefficients and polynomials, respectively. It is further shown how to determine a sequence of linear factors $(t+m)$ of the primitive polynomials, where t is the stretching parameter, as well as a bound on their degree in the form of a simple formula which is conjectured to be exact.

Résumé. Nous utilisons le modèle des ruches pour étudier les propriétés des cœfficients de Kostka et des polynômes associés aux cœfficients de Kostka dilatés. Nous montrons que les uns et les autres peuvent se factoriser : ils s'écrivent comme des produits de cœfficients (respectivement polynômes) primitifs. En outre, nous montrons comment établir une suite de facteurs linéaires $(t+m)$ des polynômes primitifs (t est le paramètre de dilatation), et proposons une formule simple donnant une borne supérieure de leur degré.

1. Introduction

There is no doubt that Kostka coefficients, $K_{\lambda\mu}$ are interesting combinatorial objects. They are indexed by pairs of partitions λ and μ, and are non-zero if and only if these partitions have the same weight and λ precedes μ with respect to the dominance partial order on partitions [JK]. They count the number of semistandard Young tableaux of shape determined by λ and of weight determined by μ, see for example [L, M, S2]. They also count Gelfand-Tsetlin patterns, as described for example in [GT, S2]. These patterns are in bijective correspondence with semistandard Young tableaux and also with certain K-hives, introduced comparatively recently [KTT1] as a variation on the hives used to calculate Littlewood-Richardson coefficients [KT, KTW, B]. These K-hives are triangular arrays of non-negative integers with borders of length n labelled by the parts of the partitions 0, λ and μ, where $0 = (0, 0, \ldots, 0)$. Counting such K-hives gives $K_{\lambda\mu}$.

Multiplying all the parts of the partitions λ and μ by a stretching parameter t, with t a positive integer, gives new partitions $t\lambda$ and $t\mu$. Since the weights of $t\lambda$ and $t\mu$ are simply those of λ and μ multiplied by t, and this scaling preserves dominance partial ordering, it follows that $K_{t\lambda,t\mu}$ is non-zero if and only if $K_{\lambda\mu}$ is non-zero.

By way of a non-trivial example, for $n = 9$, $\lambda = (9, 5, 2, 2, 2, 2, 1)$ and $\mu = (5, 5, 5, 3, 1, 1, 1, 1, 1)$ it is found quite typically that

$$K_{t\lambda,t\mu} = \frac{1}{24}(t+1)^2(t+2)(5t+2)(t^2+3t+6) \tag{1.1}$$

These functions $K_{t\lambda,t\mu}$ are necessarily quasi-polynomials in t since they enumerate integer points of rational polytopes subject to a scaling by t [E, S1]. However, contrary to initial expectations [BK], these rational polytopes may, and indeed sometimes do, possess non-integral vertices [KTT1, DeLM]. Despite this, it has been proved [KR, K1, BGR] that $P_{\lambda\mu}(t) = K_{t\lambda,t\mu}$ is always a polynomial in t.

A study of such K-polynomials has revealed a number of interesting features that are illustrated in the above example. In particular, it appears that the coefficients in the expansion of $P_{\lambda\mu}(t)$ are all positive rational numbers. This remains a conjecture [KTT1].

Secondly, $P_{\lambda\mu}(t)$ often contains a sequence of factors $(t+m)$ for $m = 1, 2, \ldots, M$, with M some non-negative integer. In the above example $M = 2$. More generally, both the value of M and the degree, d of the K-polynomial appear to be difficult to predict from a knowledge of n, λ and μ. Although the hive model does give an immediate bound, $d \leq (n-1)(n-2)/2$, on the degree, it can be seen in the above $n = 9$ example that we have $d = 6$, so that the bound is far from being saturated.

It is also found that under certain circumstances there exists a factorisation of the form

$$P_{\lambda\mu}(t) = P_{\sigma\zeta}(t)\ P_{\tau\eta}(t) \tag{1.2}$$

for some σ, τ, ζ and η such that $\lambda = (\sigma, \tau)$ and $\mu = (\zeta, \eta)$, where the notation is intended to signify that the list of parts of λ are simply those of σ followed by those of τ, and similarly for μ. In the above example we have $P_{9522221,555311111}(t) = P_{9522,5553}(t)\ P_{221,11111}(t)$ with

$$P_{9522,5553}(t) = \frac{1}{2}(t+1)(5t+2) \quad \text{and} \quad P_{221,11111}(t) = \frac{1}{12}(t+1)(t+2)(t^2+3t+6). \tag{1.3}$$

In this presentation, our intention is to exploit the hive model to study the properties of stretched Kostka coefficient polynomials. We first derive combinatorially the precise conditions under which such K-polynomials factorise as products of certain primitive K-polynomials. In the case of any primitive K-polynomial we then show how to determine the precise range of values, $1 \leq m \leq M$, such that the K-polynomial contains a factor $(t+m)$. This is done by giving an interpretation of $P_{\lambda\mu}(t)$ for negative integer values of t. Finally, we obtain a formula for an explicit bound on the degree d of a primitive K-polynomial which we conjecture is always saturated.

Our analysis covers not only the Kostka coefficients $K_{\lambda\mu}$ and the K-polynomials $P_{\lambda\mu}(t)$ in which μ is a partition, but also $K_{\lambda\beta}$ and $P_{\lambda\beta}(t)$ in which β is, more generally, a weight. The combinatorial proof of the factorisation theorem in this case is presented in an Appendix.

2. Kostka coefficients

Let n be a fixed positive integer. For any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ let $|\alpha| = \alpha_1 + \cdots + \alpha_n$ be the weight of α, let $\#\alpha = n$ be the number of components of α and let the symmetric group S_n act naturally on the components of α. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition of length $\ell(\lambda) \leq n$ and weight $|\lambda|$ with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. Then $|\lambda| = \lambda_1 + \cdots + \lambda_n$ with $\lambda_i > 0$ for $i \leq \ell(\lambda)$ and $\lambda_i = 0$ for $i > \ell(\lambda)$. It is sometimes convenient to write λ in terms of its distinct parts, that is to set $\lambda = (\kappa_1^{v_1}, \ldots, \kappa_m^{v_m})$ with $\kappa_1 > \cdots > \kappa_m \geq 0$ and $v_j > 0$

for $j = 1, \ldots, m$. Then $|\lambda| = v_1\kappa_1 + \cdots + v_m\kappa_m$ with $v_1 + \cdots + v_m = n$, and $\ell(\lambda) = n$ if $\kappa_m > 0$ and $\ell(\lambda) = n - v_m$ if $\kappa_m = 0$.

Definition 2.1. *Let* $\mathbf{x} = (x_1, \ldots, x_n)$. *Then to each partition* λ *with* $\ell(\lambda) \leq n$ *there corresponds a Schur function* $s_\lambda(\mathbf{x})$ *defined by*

$$s_\lambda(\mathbf{x}) = \frac{\left|x_i^{n+\lambda_j-j}\right|_{1\leq i,j\leq n}}{\left|x_i^{n-j}\right|_{1\leq i,j\leq n}}. \tag{2.1}$$

Since $s_\lambda(\mathbf{x})$ is a ratio of two alternants, it is a symmetric polynomial in the components $x_1, \ldots, x_n$ of $\mathbf{x}$. Indeed, the Schur functions $\{s_\lambda(\mathbf{x}) \,|\, \ell(\lambda) \leq n\}$ constitute a linear basis of the algebra of symmetric polynomials in the set of variables $\{x_1, \ldots, x_n\}$.

Definition 2.2. *The expansion of each Schur function in terms of monomials takes the form*

$$s_\lambda(\mathbf{x}) = \sum_\beta K_{\lambda\beta}\mathbf{x}^\beta, \tag{2.2}$$

where the summation is over all $\beta = (\beta_1, \ldots, \beta_n)$ *and* $\mathbf{x}^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$. *The coefficients* $K_{\lambda\beta}$ *are known as Kostka coefficients.*

The Schur functions defined by (2.1) also have a combinatorial interpretation from which it is possible to evaluate the Kostka coefficients.

Each partition λ with $\ell(\lambda) \leq n$ specifies a corresponding Young diagram F^λ of shape λ. This consists of $|\lambda|$ boxes arranged in $\ell(\lambda)$ left-adjusted rows of lengths $\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}$. Furthermore, there exists a set, $\mathcal{T}^\lambda(n)$, of semistandard tableaux T. Each such T is a numbering of the boxes of F^λ with entries from $\{1, 2, \ldots, n\}$ such that they are weakly increasing across rows and strictly increasing down columns. Any such $T \in \mathcal{T}^\lambda(n)$ is said to have weight $\mathrm{wgt}\,(T) = (\#1'\text{s} \in T, \#2'\text{s} \in T, \ldots, \#n'\text{s} \in T)$ where $\#k'\text{s} \in T$ is the number of entries k in T for $k = 1, 2, \ldots, n$.

Typically, for $n = 3$ and $\lambda = (3, 2, 0)$ we have

$$F^{32} \;=\; \begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2}\end{array} \qquad \mathcal{T}^{32}(3) \ni T \;=\; \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$$

with $\mathrm{wgt}\,(T) = (2, 1, 2)$.

It can be shown from the definition (2.1) that

$$s_\lambda(\mathbf{x}) = \sum_{T\in\mathcal{T}^\lambda(n)} x^{\mathrm{wgt}\,(T)}. \tag{2.3}$$

It then follows that we have:

Property 2.3. *(see for example* [L, p. 191], [M, p. 101].*)* $K_{\lambda\beta}$ *is the number of distinctly labelled semistandard tableaux* $T \in \mathcal{T}^\lambda(n)$ *of shape* F^λ *and weight* $\mathrm{wgt}\,(T) = \beta$, *that is to say with* β_k *entries* k *for* $k = 1, 2, \ldots, n$.

The fact that $s_\lambda(\mathbf{x})$ is symmetric implies that the coefficients $K_{\lambda\beta}$ are insensitive to the permutations of the components of β, that is they possess:

Property 2.4. *For all $w \in S_n$ we have $K_{\lambda,w(\beta)} = K_{\lambda\beta}$, and there must exist a partition μ such that $w(\beta) = \mu$ for some $w \in S_n$, in which case $K_{\lambda\beta} = K_{\lambda\mu}$.*

To specify those partitions λ and weights β for which $K_{\lambda\beta}$ is non-zero it is convenient to introduce the notion of partial sums of the parts of partitions and weights, and the dominance partial order.

Definition 2.5. *For any partition $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ and any weight $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, their partial sums are defined by*

$$ps(\nu)_i = \nu_1 + \nu_2 + \cdots + \nu_i \quad \textit{and} \quad ps(\alpha)_i = \alpha_1 + \alpha_2 + \cdots + \alpha_i \quad \textit{for all } i = 1, 2, \ldots, n. \tag{2.4}$$

More generally, for any subset $I = \{i_1, i_2, \ldots, i_r\}$ of $N = \{1, 2, \ldots, n\}$ of cardinality $\#I = r$ with $1 \le r \le n$ let

$$ps(\nu)_I = \nu_{i_1} + \nu_{i_2} + \cdots + \nu_{i_r} \quad \textit{and} \quad ps(\alpha)_I = \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_r}. \tag{2.5}$$

Definition 2.6. *Given partitions λ and μ of lengths $\ell(\lambda), \ell(\mu) \le n$, then λ is said to dominate μ, and we write $\lambda \succeq \mu$, or more precisely $\lambda \succeq_n \mu$, if*

$$ps(\lambda)_i \ge ps(\mu)_i \quad \textit{for all} \ \ i = 1, 2, \ldots, n \ \textit{and} \ |\lambda| = |\mu|. \tag{2.6}$$

Moreover, λ is said to strongly dominate μ, and we write $\lambda \succ \mu$, or more precisely $\lambda \succ_n \mu$, if

$$ps(\lambda)_i > ps(\mu)_i \quad \textit{for all} \ \ i = 1, 2, \ldots, n-1 \quad \textit{and} \quad |\lambda| = |\mu|. \tag{2.7}$$

We now have the following important condition for the non-vanishing of Kostka coefficients:

Theorem 2.7. *Let λ and μ be partitions of lengths $\ell(\lambda), \ell(\mu) \le n$, and let $N = \{1, 2, \ldots, n\}$. Then*

$$K_{\lambda\mu} > 0 \iff \lambda \succeq_n \mu \tag{2.8}$$

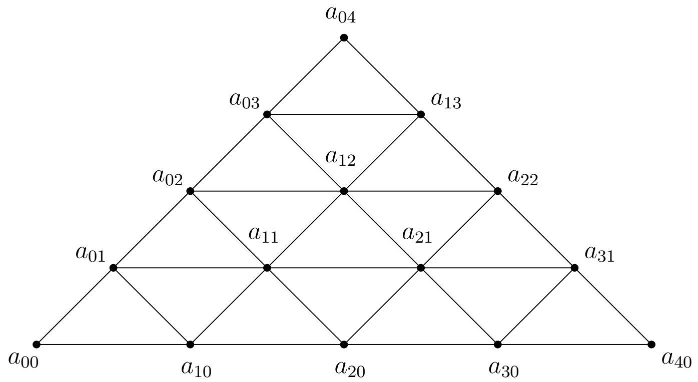
More generally, let λ be a partition of length $\ell(\lambda) \le n$ and let β be a weight with $\#\beta = n$. Then

$$K_{\lambda\beta} > 0 \iff |\lambda| = |\beta| \quad \textit{and} \quad ps(\lambda)_i \ge ps(\beta)_I \quad \textit{for all} \ \ I \subseteq N \ \ \textit{with} \ \ i = \#I > 0. \tag{2.9}$$

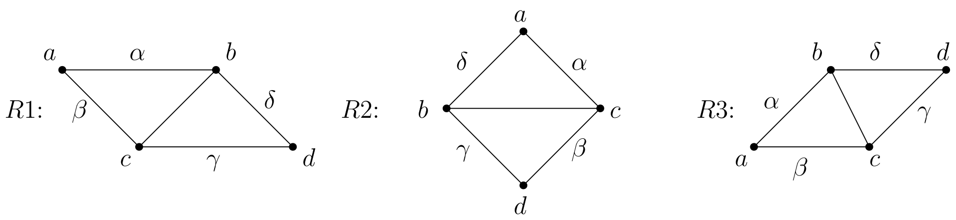
The first set of conditions (2.8) is well-known (see for example [JK, p. 44]), and the second set of conditions (2.9) is a simple corollary following from the fact that if $\mu = w(\beta)$ for some $w \in S_n$ then $ps(\mu)_i = ps(\beta)_I$ for some $I \subseteq N$ with $i = \#I$, and $ps(\mu)_i \ge ps(\beta)_J$ for all $J \subseteq N$ with $i = \#J$.

3. The hive model

An n-hive is an array of numbers a_{ij} with $0 \leq i, j, i+j \leq n$ placed at the vertices of an equilateral triangular graph. Typically, for $n = 4$ their arrangement is as shown below:



Such an n-hive is said to be an integer hive if all of its entries are non-negative integers. Neighbouring entries define three distinct types of rhombus, each with its own constraint condition.



In each case, with the labelling as shown, the hive condition takes the form:

$$b + c \geq a + d \tag{3.1}$$

In what follows we make use of edge labels more often than vertex labels. Each edge in the hive is labelled by means of the difference, $\epsilon = q - p$, between the labels, p and q, on the two vertices connected by this edge, with q always to the right of p. In all the above cases, with this convention, we have $\alpha + \delta = \beta + \gamma$, and the hive conditions take the form:

$$\alpha \geq \gamma \quad \text{and} \quad \beta \geq \delta, \tag{3.2}$$

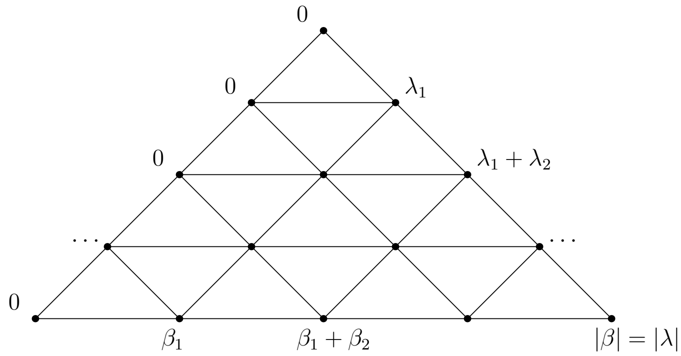
where, of course, either one of the conditions $\alpha \geq \gamma$ or $\beta \geq \delta$ is sufficient to imply the other.

Although, for completeness, we have included hive conditions for all three types of rhombus that appear in a general hive, it is only the type R1 and R2 hive conditions that apply to what we call K-hives. For such K-hives the type R3 hive conditions will, in general, be violated.

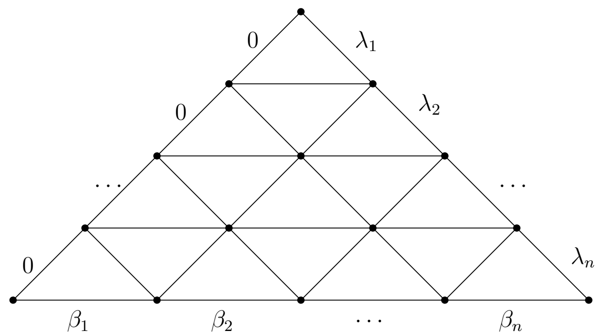
Definition 3.1. *A K-hive is an integer hive satisfying the hive conditions (3.1), or equivalently (3.2) for all its constituent rhombi of type R1 and R2 (but not R3), with border labels determined by the zero partition or weight* $0 = (0, 0, \ldots, 0)$ *with* $\#0 = n$*, a partition*

λ with $\ell(\lambda) \le n$ and a weight β with $\#\beta = n$, satisfying the constraint $|\lambda| = |\beta|$, in such a way that $a_{0i} = 0$ for $i = 0, 1, \ldots, n$, $a_{j,n-j} = ps(\lambda)_j = \lambda_1 + \lambda_2 + \cdots + \lambda_j$ for $j = 1, 2, \ldots, n$, $a_{k,0} = ps(\beta)_k = \beta_1 + \beta_2 + \cdots + \beta_k$ for $k = 1, 2, \ldots, n$.

Schematically, we have



Alternatively, in terms of edge labels we have:



With this definition, we then have:

Proposition 3.2. [KTT1] *The Kostka coefficient $K_{\lambda\beta}$ is the number of K-hives with border labels determined as above by λ and β.*

Proof. Thanks to Property 2.3, it is only necessary to establish a bijection between the set of all semistandard tableaux $T \in \mathcal{T}^\lambda(n)$ of weight $\mathrm{wgt}\,(T) = \beta$ and the set of all K-hives with boundary specified as in Definition 3.1. This bijection comes about through the existence of a sequence of maps exemplified as shown below.

$$T = \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2}\end{array} \quad \Longleftrightarrow \quad G = \begin{array}{ccccc} 3 & & 2 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{array}$$

$$\Longleftrightarrow \quad Z = \begin{array}{ccccccc} 0 & & 3 & & 5 & & 5 \\ & 0 & & 2 & & 3 & \\ & & 0 & & 2 & & \\ & & & 0 & & & \end{array} \qquad \Longleftrightarrow \quad H = \begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 3 & & \\ & 0 & & 2 & & 5 & \\ 0 & & 2 & & 3 & & 5 \end{array}$$

Each step from the semistandard tableau T to the K-hive H is a bijection. G is just the Gelfand-Tsetlin pattern corresponding to T whose top row is the partition specifying the shape of T, and whose successive rows thereafter are the partitions specifying the shape of the sub-diagrams obtained from T by deleting all boxes containing the entries n, then $n-1$, and so on. The array Z is then formed by adding a diagonal of 0's and forming the cumulative rows sums of the entries in G. Finally, H is just a re-orientation of Z, and corresponds to a K-hive, with all edges removed for the sake of clarity.

The upshot, quite generally, is that for all (i,j) with $0 \le i,j,i+j \le n$ the entries of the K-hive H are given by

$$a_{ij} = \#\text{ of entries} \le (i+j) \text{ in first } i \text{ rows of } T\,. \tag{3.3}$$

The fact that $a_{0j}=0$ follows immediately. Since the entries of T are strictly increasing down each column, it follows that no entry i may lie below the ith row, and thus (3.3) implies that $a_{i0} = ps\,(\beta)_i$ with $\beta_i = \text{wgt}\,(T)_i = \#i'\text{s} \in T$, as required. Similarly, since all entries of T are no larger than n, it follows that (3.3) implies that $a_{i,n-i} = ps\,(\lambda)_i$ where λ is the shape of T.

Again using (3.3), the hive decrements $\Delta = b+c-a-d = \alpha-\gamma = \beta-\delta$ arising from rhombi of type R1 and R2 with a in the position (i,j) are given by:

$$\begin{aligned} &R1: \quad \Delta = \#\text{ of entries} \le (i+j) \text{ in row } i+1 \;-\; \#\text{ of entries} \le (i+j+1) \text{ in row } i+2\,;\\ &R2: \quad \Delta = \#\text{ of entries equal to } (i+j) \text{ in row } i+1\,. \end{aligned}$$

The fact that numbers of entries k in each row r of T are non-negative, ensures that $\Delta \ge 0$ in the R2 case, while the fact that they are weakly increasing across rows and strictly increasing down columns ensures that $\Delta \ge 0$ in the R1 case. Thus if T is semistandard, then the map to H yields a K-hive.

Conversely, the map from any K-hive H to a tableau T always yields a semistandard tableau, as can be seen by reversing the above arguments. The fact that $\Delta \ge 0$ in the R2 case ensures that each row of T contains non-negative numbers of each distinct entry. If these are arranged in weakly decreasing order across each row, then the fact that $\Delta \ge 0$ in the R1 case implies that all the entries in row $i+1$ immediately above the k's in row $i+2$ are less than k, that is the entries are strictly increasing down columns.

Thus the maps described above are bijective, and the Proposition 3.2 follows. $\square$

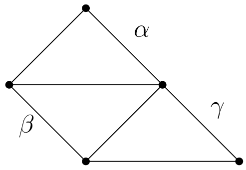
As an example of the application of this Proposition, if $n=3$, $\lambda=(3,2,0)$ and $\beta=(2,1,2)$ then the corresponding K-hives take the form:

$$\begin{array}{ccccccc} &&&0&&& \\ &&0&&3&& \\ &0&&a&&5& \\ 0&&2&&3&&5 \end{array} \tag{3.4}$$

where once again for the sake of clarity all the hive edges have been omitted. Since the only integer values of a satisfying the hive conditions (3.1) for all the constituent rhombi of type R1 and R2 are $a = 3$ and $a = 2$, it follows that $K_{32,212} = 2$. The corresponding semistandard tableaux are, of course, given by:

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 3 & 3 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2}\end{array} \tag{3.5}$$

It might be pointed out that when expressed in terms of edge labels, the hive conditions (3.2) for all constituent rhombi of types R1 and R2 imply that in every K-hive the edge labels along any line parallel to the right-hand edge of the hive are weakly decreasing from top left to bottom right. This can be seen from the following 5-vertex sub-diagram.



The edge conditions on the rhombi of type R1 and R2 in the above diagram give $\beta \geq \gamma$ and $\alpha \geq \beta$, respectively, so that $\alpha \geq \gamma$, as claimed. This is of course consistent with the edges of the right-hand boundary of each K-hive being specified by a partition λ.

4. Stretched coefficients

4.1. **Polynomial conjectures.** Now we are in a position to define and evaluate stretched Kostka coefficients. The partition obtained from $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ by multiplying all of its parts by the same positive integer, t, is denoted by $t\lambda = (t\lambda_1, t\lambda_2, \ldots, t\lambda_p)$. With this notation, we refer to $K_{t\lambda,t\beta}$ as stretched Kostka coefficients, where t is said to be the stretching parameter. It is not difficult to evaluate these stretched coefficients, particularly through the use of the hive model, for a range of positive integer values of t.

For example, for $\lambda = (3, 2)$ and $\mu = (1, 1, 1, 1, 1)$ the corresponding stretched Kostka coefficients are given by

$$K_{t\lambda,t\mu} = \frac{1}{2}(t + 1)(t^2 + 2t + 2). \tag{4.1}$$

The generating function for these coefficients takes the form:

$$F_{\lambda\mu}(z) = \sum_{t=0}^{\infty} K_{t\lambda,t\mu}\ z^t = \frac{1 + z + z^2}{(1 - z)^4}. \tag{4.2}$$

On the basis of this and many other examples we were led to the following:

Conjecture 4.1. [KTT1] *For all partitions λ and μ such that $K_{\lambda\mu} > 0$ there exists a polynomial $P_{\lambda\mu}(t)$ in t with positive rational coefficients such that $P_{\lambda\mu}(0) = 1$ and $P_{\lambda\mu}(t) = K_{t\lambda,t\mu}$ for all positive integers t.*

Conjecture 4.2. [KTT1] *Given that the degree of the polynomial $K_{t\lambda,t\mu}$ is d, the generating function for $K_{t\lambda,t\mu}$ takes the form $F_{\lambda\mu}(z) = G_{\lambda\mu}(z)/(1-z)^{d+1}$, with $G_{\lambda\mu}(z)$ a polynomial of degree $\leq d$ having non-negative integer coefficients.*

Much of these conjectures has now been proved. In particular the fact that $P_{\lambda\mu}(t)$ is polynomial appears to have been proved first by Kirillov and Reshetikhin [KR], with a more recent proof provided by Billey, Guillemin and Rassart [BGR]. In fact all that remains to be answered are the questions of the positivity of the coefficients in $P_{\lambda\mu}(t)$ and $G_{\lambda\mu}(z)$.

Before looking in more detail at the nature of the polynomials $P_{\lambda\mu}(t)$ we note:
Saturation Condition 4.3. [KTT1]

$$K_{t\lambda,t\mu} > 0 \iff K_{\lambda\mu} > 0. \tag{4.3}$$

Proof. This can be seen immediately from Theorem 2.7 by noting that $|t\lambda| = t\,|\lambda|$ and $ps(t\lambda)_r = t\,ps(\lambda)_r$ for all partitions λ and all r. It follows that

$$K_{\lambda\mu} > 0 \iff |\lambda| = |\mu| \text{ and } \lambda \succeq \mu \iff |t\lambda| = |t\mu| \text{ and } t\lambda \succeq t\mu \iff K_{t\lambda,t\mu} > 0, \tag{4.4}$$

as required. □

Turning to the Conjecture 4.1 itself, the following key component has been established:
Theorem 4.4. [K1, BGR] *Let λ and μ be partitions of lengths $\ell(\lambda), \ell(\mu) \leq n$ such that $K_{\lambda\mu} > 0$. Then $P_{\lambda\mu}(t) = K_{t\lambda,t\mu}$ is a polynomial of degree at most $(n-1)(n-2)/2$ in t.*

Clearly, Property 2.4 then implies as a corollary of Theorem 4.4 that if $K_{\lambda\beta} > 0$ then $P_{\lambda\beta}(t) = K_{t\lambda,t\beta}$ is also a polynomial in t with the same upper bound on its degree.

5. Factorisation

Berenstein and Zelevinsky [BZ] introduced the notion of *primitive* Kostka coefficients and pointed out that every Kostka coefficient may be expressed as a product of primitive Kostka coefficients. First we deal with the case $K_{\lambda\mu}$ where λ and μ are both partitions. Following Berenstein and Zelevinsky, we make the following definition.

Definition 5.1. *Let λ and μ be partitions such that $|\lambda| = |\mu|$, $\ell(\lambda), \ell(\mu) \leq n$ and $\lambda \succeq_n \mu$, so that $K_{\lambda\mu} > 0$. Then $K_{\lambda\mu}$ is said to be primitive if $\lambda \succ_n \mu$, that is $ps\,(\lambda)_r > ps\,(\mu)_r$ for all $r = 1, 2, \ldots, n-1$. Conversely, if $K_{\lambda\mu}$ is not primitive then there exists at least one value of r, with $1 \leq r < n$, such that $ps\,(\lambda)_r = ps(\mu)_r$.*

If $K_{\lambda\mu} > 0$ but $K_{\lambda\mu}$ is not primitive then $K_{\lambda\mu}$ factorises in accordance with the following theorem.

Theorem 5.2. *Let λ, μ be partitions such that $|\lambda| = |\mu|$ and $\ell(\lambda), \ell(\mu) \leq n$, with $\lambda \succeq_n \mu$ and $ps(\lambda)_r = ps(\mu)_r$ for some r such that $1 \leq r < n$. Let $\lambda = (\sigma, \tau)$ and $\mu = (\zeta, \eta)$, with $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, $\tau = (\lambda_{r+1}, \lambda_{r+2}, \ldots, \lambda_n)$, $\zeta = (\mu_1, \mu_2, \ldots, \mu_r)$ and $\eta = (\mu_{r+1}, \mu_{r+2}, \ldots, \mu_n)$, then*

$$K_{\lambda\mu} = K_{\sigma\zeta}\ K_{\tau\eta}. \tag{5.1}$$

We first give a combinatorial proof based on the use of semistandard tableaux, and then a second combinatorial proof based on the use of K-hives.

Proof 1. It follows from the definition of semistandard tableaux that no entry k of $T \in \mathcal{T}^\lambda(n)$ may appear lower than the kth row of T. If $ps\,(\lambda)_r = ps\,(\mu)_r$ then all entries $1, 2, \ldots, r$ completely fill the first r rows of T. This means that all the remaining entries $r{+}1, r{+}2, \ldots, n$ must fill the rows of T below the rth. Thus the top r rows of T constitute a semistandard tableau $T_r \in \mathcal{T}^\sigma(r)$ of shape F^σ with entries taken from the set $\{1, 2, \ldots, r\}$, while the bottom $(n-r)$ rows of T constitute a second semistandard tableau $T \in \mathcal{T}^\tau(n-r)$ of shape F^τ with entries taken from the set $\{r+1, r+2, \ldots, n\}$. Moreover, if any such semistandard tableaux T_r of shape F^σ and T_{n-r} of shape F^τ are joined together to create a tableau T of shape F^λ with $\lambda = (\sigma, \tau)$, then T is itself semistandard since all the entries in the bottom row of T_r, which are necessarily $\leq r$, are strictly less than all the entries in the top row of T_{n-r}, which are necessarily $\geq (r+1)$.

Setting, $\mathbf{x} = \mathbf{y}\,\mathbf{z}$ with $\mathbf{y} = (x_1, x_2, \ldots, x_r)$ and $\mathbf{z} = (x_{r+1}, x_{r+2}, \ldots, x_n)$ it follows from (2.3) that

$$s_\lambda(\mathbf{x}) = \sum_{T \in \mathcal{T}^\lambda(n)} \mathbf{x}^{\mathrm{wgt}\,(T)} = \sum_{T_r \in \mathcal{T}^\sigma(r)} \; \sum_{T_{n-r} \in \mathcal{T}^\tau(n-r)} \mathbf{y}^{\mathrm{wgt}\,(T_r)}\; \mathbf{z}^{\mathrm{wgt}\,(T_{n-r})}.$$

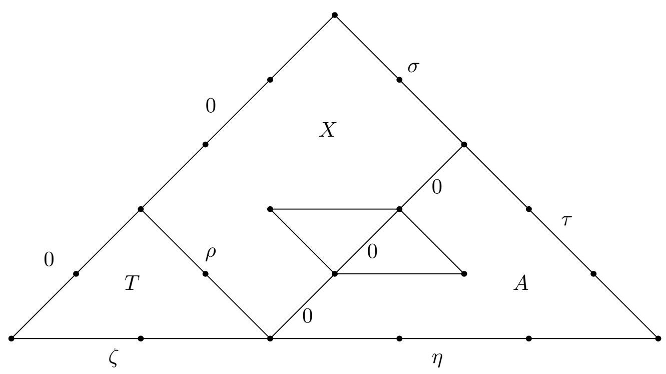
It then follows from Property 2.3 that

$$\begin{aligned} K_{\lambda\mu} &= \#T \in \mathcal{T}^\lambda(n) \text{ with } \mathrm{wgt}\,(T) = \mu \\ &= (\#T_r \in \mathcal{T}^\sigma(r) \text{ with } \mathrm{wgt}\,(T_r) = \zeta) \cdot (\#T_{n-r} \in \mathcal{T}^\tau(n-r) \text{ with } \mathrm{wgt}\,(T_{n-r}) = \eta) \\ &= K_{\sigma\zeta}\; K_{\tau\eta}\,, \end{aligned}$$

as required. □

Proof 2. First it should be noted that the hypothesis $\lambda \succeq_n \mu$ implies that $K_{\lambda\mu} > 0$. Moreover, we have $ps(\sigma)_i = ps(\lambda)_i \geq ps(\mu)_i = ps(\zeta)_i$ for $i = 1, 2, \ldots, r-1$ and $|\sigma| = ps(\sigma)_r = ps(\lambda)_r = ps(\mu)_r = ps(\zeta)_r = |\zeta|$. Thus $\sigma \succeq_r \zeta$ and hence $K_{\sigma\zeta} > 0$. In addition we have $ps(\tau)_i = ps(\lambda)_{r+i} - ps(\lambda)_r = ps(\lambda)_{r+i} - |\sigma| \geq ps(\mu)_{r+i} - |\zeta| = ps(\mu)_{r+i} - ps(\mu)_r = ps(\eta)_i$ for $i = 1, 2, \ldots, n-r$, and $|\tau| = |\lambda| - |\sigma| = |\mu| - |\zeta| = |\eta|$. Thus $\tau \succeq_{n-r} \eta$ and hence $K_{\tau,\eta} > 0$.

Now consider the K-hive with boundary determined by $\lambda = (\sigma, \tau)$

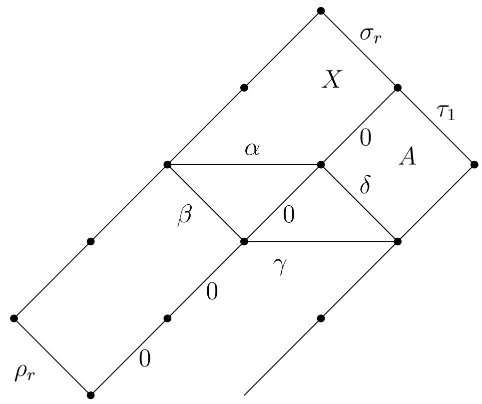


The rules about edge sums are such that on the boundary of the triangular region T we have $|\zeta| = |0| + |\rho| = |\rho|$. Since $|\zeta| = |\sigma|$ we have $|\rho| = |\sigma|$. The same rules applied to the parallelogram X then imply that the sum of the edge lengths on the boundary between X and A must be $|0| + |\sigma| - |\rho| = 0$. Since X can be viewed as a collection of rhombi of type R2, applying the hive condition $\beta \geq \delta$ to each of these rhombi implies that the edge lengths on the lower right boundary of X must be non-negative. Since their sum is zero, they must all be zero, as indicated in the diagram. The $\alpha \geq \gamma$ hive condition for R2 may then be applied to these same rhombi constituting X, yielding the constraints $\rho_i \leq \sigma_i$ for $i = 1, 2, \ldots, r$. Since $|\rho| = |\sigma|$, we must have $\rho_i = \sigma_i$ for $i = 1, 2, \ldots, r$, that is $\rho = \sigma$.

Thus each K-hive H with boundary λ and μ consists of a parallelogram X in which all edge lengths are fixed, together with a K-hive T with boundary σ and ζ, and a second K-hive A with boundary τ and η. The hive conditions for H imply those appropriate to T and A.

In order to prove the required factorisation (5.1) it only remains to show that combining all possible hives T and A with a parallelogram X of the appropriate boundary gives a hive H in which all the hive conditions corresponding to rhombi crossing the boundaries of T with X, and A with X, are automatically satisfied. Since, for K-hives we only use the rhombi of type R1 and R2, it is clear that no such rhombus crosses the boundary between T and X, while only rhombi of type R1 cross the boundary between A and X. One such rhombus has been shown above.

We may look in more detail at such a rhombus by means of the following diagram which shows a strip one edge length wide on either side of the $A - X$ boundary.



With the edge labels as shown, the R2 rhombus condition applied in the region A just below the $A - X$ boundary gives $\delta \leq \tau_1$, while the same rhombus condition applied in the region X just above the $A - X$ boundary gives $\rho_r \leq \beta \leq \sigma_r$. Since $\rho_r = \sigma_r$ and $\tau_1 = \lambda_{r+1} \leq \lambda_r = \sigma_r$, we have $\delta \leq \tau_1 \leq \sigma_r = \beta$. However, this is just what is required to satisfy the R1 condition (3.2).

This completes the proof of the K-hive factorisation theorem. □

Repeated use of this theorem allows any non-primitive Kostka coefficient $K_{\lambda\mu}$ to be expressed as a unique product of primitive Kostka coefficients $K_{\sigma\zeta}$. It is only necessary at each stage to factor out the term meeting the hypotheses of the above Theorem 5.2 with the smallest possible value of r.

Applying the above argument to stretched Kostka coefficients immediately gives the following:

Theorem 5.3. *Let λ, μ be partitions such that $|\lambda| = |\mu|$ and $\ell(\lambda), \ell(\mu) \leq n$, with $\lambda \succeq_n \mu$ and $ps(\lambda)_r = ps(\mu)_r$ for some r for which $1 \leq r < n$. Let $\lambda = (\sigma, \tau)$ and $\mu = (\zeta, \eta)$, with $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, $\tau = (\lambda_{r+1}, \lambda_{r+2}, \ldots, \lambda_n)$, $\zeta = (\mu_1, \mu_2, \ldots, \mu_r)$ and $\eta = (\mu_{r+1}, \mu_{r+2}, \ldots, \mu_n)$, then for all positive integers t we have*

$$P_{\lambda\mu}(t) = P_{\sigma\zeta}(t)\ P_{\tau,\eta}(t). \tag{5.2}$$

An example of this type of factorisation has been provided in the introduction in (1.3).

It is quite instructive from a combinatorial point of view, although not strictly necessary by virtue of Property 2.4, to extend the above analysis to the case of Kostka coefficients $K_{\lambda\beta}$ for which the weight β is not necessarily a partition. In this case $K_{\lambda\beta}$ is said to be primitive if $|\lambda| = |\beta|$ and $ps(\lambda)_r \geq ps(\beta)_I$ for any proper subset I of $N = \{1, 2, \ldots, n\}$ with $r = \#I$.

In the non-primitive case, we have:

Theorem 5.4. *Let λ be a partition with $\ell(\lambda) \leq n$, and let $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ be a weight such that $|\lambda| = |\beta|$ and $ps(\lambda)_i \geq ps(\beta)_I$ for any $I \subset N$, with $i = \#I$. Then $K_{\lambda\beta}$ is not primitive if there exists a proper subset $I = \{i_1, i_2, \ldots, i_r\}$ of N such that $ps(\lambda)_r = ps(\beta)_I$ with $r = \#I$ for some r for which $1 \leq r < n$. Let the complement of I in N be denoted by $\overline{I} = \{j_1, j_2, \ldots, j_{n-r}\}$. In such a case let $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, $\tau = (\lambda_{r+1}, \lambda_{r+2}, \ldots, \lambda_n)$, $\zeta = (\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_r})$ and $\eta = (\beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_{n-r}})$, so that $\lambda = (\sigma, \tau)$, with σ and τ partitions, and $\beta = \zeta \cup \eta$, with ζ and η weights, not necessarily partitions, satisfying the partial sum constraints $|\zeta| = |\sigma|$ and $|\eta| = |\tau|$. Then*

$$K_{\lambda\beta} = K_{\sigma\zeta}\ K_{\tau\eta}. \tag{5.3}$$

A complete combinatorial proof of this theorem, based on the use of hives, is presented in an Appendix below.

Once again scaling everything by t is straightforward. If $K_{\lambda\beta}$ is primitive then so is $K_{t\lambda,t\beta}$ since all the partial sum conditions are scaled by the same factor t. By the same token the factorisation occurs in the stretched non-primitive case just as it does in the unstretched non-primitive case – all boundary edges are simply scaled by t, and as we have shown, it is these boundary edges that completely determine the factorisation. Thus under the hypotheses of Theorem 5.4 we have

$$P_{\lambda\beta}(t) = P_{\sigma\zeta}(t)\ P_{\tau\eta}(t). \tag{5.4}$$

This is illustrated for $n = 6$ and $r = 3$ by $\lambda = (9, 6, 4, 4, 2, 0)$ and $\beta = (2, 6, 3, 7, 6, 1)$, for which $ps(\lambda)_3 = 19 = ps(\beta)_I$ with $I = \{2, 4, 5\}$. In this case $\sigma = (9, 6, 4)$, $\tau = (4, 2, 0)$, $\zeta = (6, 7, 6)$ and $\eta = (2, 3, 1)$. Correspondingly we find

$$P_{96442,263761} = (t + 1)(2t + 1),\ \ P_{964,676} = (2t + 1),\quad P_{42,231} = (t + 1), \tag{5.5}$$

thereby exemplifying the factorisation (5.4).

It is interesting to note that the above factorisation of Kostka coefficients can be extended to the case of q-dependent Kostka–Foulkes polynomials, $K_{\lambda\mu}(q)$. These have a combinatorial definition in terms of the charge statistic on semistandard Young tableaux [M, p. 242]. By way of illustration, in the case of the above example we find

$$\begin{aligned} K_{t(96442),t(766321)}(q) &= q^{3t}\,\frac{q^{t+1}-1}{q-1}\,\frac{q^{2t+1}-1}{q-1} \\ K_{t(964),t(766)}(q) &= q^{2t}\,\frac{q^{2t+1}-1}{q-1} \\ K_{t(42),t(321)}(q) &= q^{t}\,\frac{q^{t+1}-1}{q-1}. \end{aligned}$$

6. The zeros of stretched Kostka polynomials

As has been noted the stretched Kostka polynomials $P_{\lambda\mu}(t)$ contain factors $(t+m)$ for some sequence of values $m = 1, 2, \ldots, M$ for some positive integer M. This is no accident. In this section we describe a method of determining M for stretched Kostka polynomials. It will be recalled that the Kostka coefficients are defined in terms of Schur functions by (2.2). It then follows from (2.1) that $P_{\lambda\mu}(t) = K_{t\lambda,t\mu}$ is the coefficient of $x_1^{t\mu_1}x_2^{t\mu_2}\cdots x_n^{t\mu_n}$ in the expansion of

$$s_{t\lambda}(\mathbf{x}) = \frac{\left|x_i^{t\lambda_j+n-j}\right|}{\left|x_i^{n-j}\right|} \tag{6.1}$$

This may be readily extended to the case $t = -m$ with m a positive integer. Then $P_{\lambda\mu}(-m) = K_{-m\lambda,-m\mu}$ is the coefficient of $x_1^{-m\mu_1}x_2^{-m\mu_2}\cdots x_n^{-m\mu_n}$ in the expansion of $s_{-m\lambda}(\mathbf{x})$. However

$$s_{-m\lambda}(\mathbf{x}) = \frac{\left|x_i^{-m\lambda_j+n-j}\right|}{\left|x_i^{n-j}\right|} = \frac{\left|x_i^{-m\lambda_{n-k+1}-n+k}\right|}{\left|x_i^{-n+k}\right|}, \tag{6.2}$$

where first x_i^{n-1} has been extracted as a common factor from the ith row of each determinant for $i = 1, 2, \ldots, n$ and cancelled from numerator and denominator, and then j replaced by $k = n-j+1$ with an appropriate reversal of order of the columns in both determinants. If we now set $\overline{x}_i = x_i^{-1}$ for $i = 1, 2, \ldots, n$ and $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$, this gives

$$s_{-m\lambda}(x_1, x_2, \ldots, x_n) = \frac{\left|\overline{x}_i^{m\lambda_{n-k+1}+n-k}\right|}{\left|\overline{x}_i^{n-k}\right|} = s_{m\lambda_n,\ldots,m\lambda_2,m\lambda_1}(\overline{\mathbf{x}}). \tag{6.3}$$

Since $x_1^{-m\mu_1}x_2^{-m\mu_2}\cdots x_n^{-m\mu_n} = \overline{x}_1^{m\mu_1}\overline{x}_2^{m\mu_2}\cdots\overline{x}_n^{m\mu_n}$ it follows that $P_{\lambda\mu}(-m) = K_{-m\lambda,-m\mu}$ is the coefficient of $\overline{x}_1^{m\mu_1}\overline{x}_2^{m\mu_2}\cdots\overline{x}_n^{m\mu_n}$ in the expansion of the right hand side. It then follows, replacing the dummy variables $\overline{x}_i$ by x_i for $i = 1, 2, \ldots, n$, that $P_{\lambda\mu}(-m) = K_{-m\lambda,-m\mu}$ is the coefficient of $x_1^{m\mu_1}x_2^{m\mu_2}\cdots x_n^{m\mu_n}$ in the expansion of $s_{m\tilde{\lambda}}(\mathbf{x})$, where $\tilde{\lambda} = (\lambda_n, \ldots, \lambda_2, \lambda_1)$

is the weight obtained by reversing the order of the parts of λ, including any trailing zeros. Then to exploit the above it is only necessary to standardise

$$s_{m\tilde{\lambda}}(\mathbf{x}) = s_{m\lambda_n,\ldots,m\lambda_2,m\lambda_1}(x_1,x_2,\ldots,x_n) = \frac{\left|x_i^{m\lambda_{n-k+1}+n-k}\right|}{\left|x_i^{n-k}\right|}. \tag{6.4}$$

This is carried out in the usual way [L] by reordering the columns of the numerator determinant. However there are two quite different possible outcomes:

$$s_{m\tilde{\lambda}}(\mathbf{x}) = \begin{cases} 0 & \text{case (i);} \\ \eta_\rho\, s_\rho(\mathbf{x}) \text{ with } \eta_\rho = \pm 1 & \text{case (ii).} \end{cases} \tag{6.5}$$

For example, in the case $n = 5$ and $\lambda = (4,2,1,0,0)$ we have $\tilde{\lambda} = (0,0,1,2,4)$. For $m = 1$ this gives $s_{m\tilde{\lambda}}(\mathbf{x}) = s_{0,0,1,2,4}(\mathbf{x}) = 0$. Similarly for $m = 2$ we have $s_{m\tilde{\lambda}}(\mathbf{x}) = s_{0,0,2,4,8}(\mathbf{x}) = 0$. However, for $m = 3$ we obtain $s_{m\tilde{\lambda}}(\mathbf{x}) = s_{0,0,3,6,12}(\mathbf{x}) = -s_{8,4,3,3,3}(\mathbf{x})$, while for $m = 5$ we have $s_{m\tilde{\lambda}}(\mathbf{x}) = s_{0,0,4,8,16}(\mathbf{x}) = -s_{12,6,4,3,3}(\mathbf{x})$.

More generally, the case (i) result 0 arises if and only if

$$m\,\lambda_{n-k+1} + n - k = m\,\lambda_{n-l+1} + n - l \tag{6.6}$$

for some k and l such that $1 \le l < k \le n$. In all other cases the formula given in case (ii) applies for some partition ρ of length at most n and weight $m|\lambda|$, and $\eta_\rho = \pm 1$ is a sign factor recording the number of transpositions of columns of the numerator determinant of (6.4) required to standardise the Schur function.

It follows that we can expect there to be a possibility of two types of zero of $P_{\lambda\mu}(t)$ for $t = -m$: type (i) associated with case (i) of (6.5), and type (ii) associated under certain conditions on ρ and $m\,\mu$ with case (ii). Indeed adopting the notation of (6.5) we have the following:

Proposition 6.1. *Let λ and μ be such that $K_{\lambda\mu}$ is primitive. Then $P_{\lambda\mu}(t) = K_{t\lambda,t\mu}$ contains a factor $(t+m)$ if and only if either case (i) applies and*

$$m = (j-i)/(\lambda_i - \lambda_j) \tag{6.7}$$

for some i and j such that $1 \le i < j \le n$, or case (ii) applies and

$$ps(\rho)_k < m\,ps(\mu)_k \tag{6.8}$$

for some k such that $1 \le k < n$.

Proof. We have already noted that $P_{\lambda\mu}(-m) = K_{-m\lambda,-m\mu}$ is the coefficient of $x_1^{m\mu_1}x_2^{m\mu_2}\cdots x_n^{m\mu_n}$ in $s_{m\tilde{\lambda}}(\mathbf{x}) = s_{m\lambda_n,\ldots,m\lambda_2,m\lambda_1}(x_1,x_2,\ldots,x_n)$. Thus if case (i) of (6.5) applies then $P_{\lambda\mu}(-m) = 0$, while if case (ii) applies then $P_{\lambda\mu}(-m) = \eta_\rho K_{\rho,m\mu}$. The conditions for case (i) to apply are given in (6.6). However, setting $i = n-k+1$ and $j = n-l+1$ leads immediately to the required case (i) conditions (6.7) for $P_{\lambda\mu}(-m) = 0$. Turning to case (ii) of (6.5) $P_{\lambda\mu}(-m) = \eta_\rho\, K_{\rho,m\mu}$ will be zero by virtue of Theorem 2.7 if any one of the partial sum conditions $ps(\rho)_k \ge m\,ps(\mu)_k$ for all $k = 1,2,\ldots,n$ is violated. Since $|\rho| = m\,|\mu|$ this leaves precisely the required case (ii) conditions (6.8) for $P_{\lambda\mu}(-m) = 0$. □

We may extend our earlier example with $n = 5$ and $\lambda = (4, 2, 1, 0, 0)$ by taking $\mu = (3, 1, 1, 1, 1)$ for illustrative purposes. We have already seen that there are type (i) zeros, independent of μ that arise for $m = 1$ and $m = 2$. These arise from (6.7) for the pairs $(i, j) = (3, 4)$ and $(3, 5)$, respectively, leading to $(j - i)/(\lambda_i - \lambda_j) = (4 - 3)/(1 - 0) = 1$ and $(5 - 3)/(1 - 0) = 2$. For $m = 3$ we have a type (ii) zero since $P_{\lambda\mu}(-3) = \eta_\rho K_{\rho,m\mu} = -K_{84333,93333} = 0$ by virtue of the fact that $ps(\rho)_1 = 8 < 9 = ps(m\mu)_1$. On the other hand for $m = 4$ we have $P_{\lambda\mu}(-4) = \eta_\rho K_{\rho,m\mu} = -K_{12\,6433,12\,4444} = -3$ which is non-zero, and the same will be true for all $m \geq 4$.

In view of our earlier remarks about the sequence of zeros of $P_{\lambda\mu}(t)$ for $t = -m$ with $m = 1, 2, \ldots, M$, it comes as no surprise that we are able to elaborate on the above and establish that the zeros are indeed consecutive and that for any primitive $K_{\lambda\mu}$ we can always find a finite $M > 0$ such that $P_{\lambda\mu}(-m) = 0$ for all $m = 1, 2, \ldots, M$, but thereafter $P_{\lambda\mu}(-m) \neq 0$ for all $m > M$.

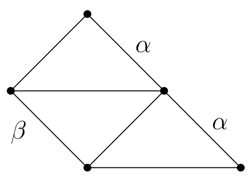
All this is borne out in our example with $n = 5$, $\lambda = (4, 2, 1, 0, 0)$ and $\mu = (3, 1, 1, 1, 1)$ for which

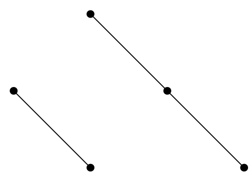
$$P_{\lambda\mu}(t) = K_{t\lambda,t\mu} = \frac{1}{60}(t + 1)(t + 2)(t + 3)(3t^2 + 7t + 10), \tag{6.9}$$

for which $M = 3$ and we have consecutive zeros at $t = -m$ with $m = 1, 2, 3$. Of these, as we have seen, the first two are type (i) and the third is type (ii). The first non-zero case occurs with $t = -4$ and we obtain from the explicit formula (6.9) the result $P_{\lambda\mu}(-4) = -3$ that we had found earlier. In fact for $t = -m$ it is clear that (6.9) gives $P_{\lambda\mu}(-m) < 0$ for all $m > M = 3$.

7. The degrees of stretched Kostka polynomials

As we have seen the calculation of stretched Kostka polynomials may be reduced to that of calculating these polynomials in primitive cases only. Even so the task may be combinatorially formidable. In any given case a knowledge of the degree of the polynomial would be extremely advantageous. Here we establish an upper bound on this degree by means of the following rather innocuous looking result. Let the edge labelling of a particular 5-vertex subset of a K-hive be as shown below in the left hand diagram, with two identical labels α.



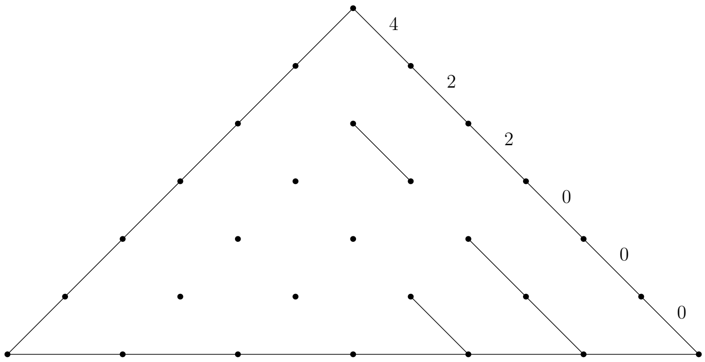


This diagram contains rhombi of type R1 and R2. The hive conditions (3.2) for the former imply $\alpha \leq \beta$, while those for the latter give $\beta \leq \alpha$. It follows that $\beta = \alpha$. This result is displayed more simply by deleting all the edges except those sharing the same label α, and suppressing the label itself. This gives the right hand diagram where the equality of a pair of edge labels in a linear sequence forces an identical edge label in a neighbouring line.

Applying these notions to the case of K-hives with boundaries of length n and with border labels determined by 0, λ and μ, it follows from the above that any equalities of successive parts of λ propagate as equalities of edge labels within each possible K-hive. To be more precise let all the λ-boundary edges be labelled by the parts of λ. If any sequence of parts of λ share the same value, say α, then we can identify an equilateral sub-hive having the sequence of equally labelled edges as one boundary, with its other boundaries parallel to the 0 and μ-boundaries of the original hive. Within this sub-hive all the vertices along lines parallel to the λ-boundary are to be connected by edges indicating that in any K-hive the differences in values between neighbouring entries along these lines are all α.

This process is to be repeated for all sequences of equal edge labels along the λ-boundary. Finally, all neighbouring vertices on all three boundaries are to be connected by edges. In this way we arrive at a graph $G_{n;\lambda}$ that depends only upon n and λ.

For example, for $n = 6$ and $\lambda = (4, 2, 2, 0, 0, 0)$ the graph $G_{n;\lambda}$ takes the form:



With this notation we have:

Proposition 7.1. *Let λ and μ be partitions of lengths $\ell(\lambda), \ell(\mu) \leq n$ such that $K_{\lambda\mu} > 0$. Let $\deg(P_{\lambda\mu}(t))$ be the degree of the corresponding stretched K-polynomial $P_{\lambda\mu}(t) = K_{t\lambda,t\mu}$. Let $d(G_{n;\lambda})$ be the number of connected components of the graph $G_{n;\lambda}$ that are not connected to the boundary. Then*

$$\deg(P_{\lambda\mu}(t)) \leq d(G_{n;\lambda}). \tag{7.1}$$

Moreover, if $\lambda = (w_1^{v_1}, w_2^{v_2}, \ldots, w_r^{v_r})$, with $w_1 > w_2 > \cdots > w_r \geq 0$, $v_s \geq 1$ for all $s = 1, 2 \ldots, r$, with $r \geq 2$, and $v_1 + v_2 + \cdots + v_r = n$, then

$$d(G_{n;\lambda}) = \frac{1}{2}(n-1)(n-2) - \sum_{s=1}^{r} \frac{1}{2} v_s(v_s - 1). \tag{7.2}$$

Proof. Any labelling of the vertices of the hive must be such that the difference α in values along each interior edge of the graph $G_{n;\lambda}$ is, by construction of those edges, equal to a corresponding edge value on the parallel λ-boundary, that is $\alpha = \lambda_i = w_s$ for some i and a corresponding s. For each connected component this fixes the values of all vertex labels in terms of that of any one vertex of that component. In the case of a component connected to the boundary, the value at the boundary vertex fixes all the others in that component.

The maximum number of degrees of freedom in assigning entries to the corresponding set of K-hives with given boundary is then $d(G_{n;\lambda})$.

The application of the stretching parameter t leaves $G_{n;\lambda}$ unaltered, that is $G_{n;t\lambda} = G_{n;\lambda}$, so that the number of degrees of freedom in assigning entries to the stretched K-hives is still $d(G_{n;\lambda})$. For each connected component of $G_{n;\lambda}$ that is not connected to the boundary we may select any one convenient vertex. The values of its label are not fixed by the hive constraints. They must satisfy a set of linear inequalities that are all scaled by t as both the boundary vertex and edge labels are scaled by the stretching parameter t. It follows that the range of allowed values of the label must be either independent of t or linear in t, thereby giving rise to a corresponding contribution to $P_{\lambda\mu}(t)$ that is at most linear in t. The number of degrees of freedom, $d(G_{n;\lambda})$, that we have identified therefore gives an upper bound on the degree of the corresponding stretched K-polynomial in t.

The number of internal vertices of $G_{n;\lambda}$ is $(n-1)(n-2)/2$. For given n and arbitrary λ this provides a preliminary upper bound on the degree of $P_{\lambda\mu}(t)$. However, for $\lambda = (w_1^{v_1}, w_2^{v_2}, \ldots, w_r^{v_r})$, each s such that $v_s > 1$ specifies a sequence of components of λ having the same value, namely w_s. There is a corresponding set of $v_s(v_s-1)/2$ interior edges within $G_{n;\lambda}$. Whether or not these interior edges reach the boundary, as they do in the two cases $s=1$ and $s=r$, their introduction reduces the number of connected components not linked to the boundary by $v_s(v_s-1)/2$. The result (7.2) then follows by noting that for different s the sets of vertices linked by the interior edges are disjoint, so that their contributions to the reduction of $d(G_{n;\lambda})$ are independent. □

In the case of our example with $n=6$ and $\lambda = (4,2,2,0,0,0) = (4,2^2,0^3)$ we can see from the graph of $G_{n;\lambda}$ that $d(G_{n;\lambda}) = 6$, in agreement with the formula (7.2) that gives $d(G_{n;\lambda}) = 5\cdot 4/2 - 2\cdot 1/2 - 3\cdot 2/2 = 6$. Hence for all μ the degree of the corresponding stretched Kostka coefficient polynomial $P_{\lambda\mu}(t)$ must satisfy

$$\deg(P_{\lambda\mu}(t)) \leq 6. \tag{7.3}$$

By way of example, for $\mu = (3,1,1,1,1,1)$ we find

$$P_{\lambda\mu}(t) = \frac{1}{72}(t+1)(t+2)(t+3)(t+4)(t^2+2t+3) \quad \text{so that} \quad F_{\lambda\mu}(z) = \frac{1+3z+6z^2}{(1-z)^7}, \tag{7.4}$$

while for $\mu = (2,2,1,1,1,1)$ we have

$$P_{\lambda\mu}(t) = \frac{1}{60}(t+1)(t+2)(t+3)(3t+5)(t^2+2t+2) \quad \text{so that} \quad F_{\lambda\mu}(z) = \frac{1+9z+19z^2+7z^3}{(1-z)^7}. \tag{7.5}$$

On the other hand if $\mu = (3,2,1,1,1,0)$ we find

$$P_{\lambda\mu}(t) = \frac{1}{24}(t+1)(t+2)(t+3)(t+4) \quad \text{so that} \quad F_{\lambda\mu}(z) = \frac{1}{(1-z)^5}. \tag{7.6}$$

The third case in which $\deg(P_{\lambda\mu}(t)) = 4 < 6$ is one for which $K_{\lambda\mu}$ is not primitive. In both the other cases $K_{\lambda\mu}$ is primitive, and we have $\deg(P_{\lambda\mu}(t)) = 6$ so that the bound (7.1) is saturated.

On the basis of very many calculations of this type we are led to the following:

Conjecture 7.2. *Let λ and μ be partitions of lengths $\ell(\lambda), \ell(\mu) \leq n$ such that $K_{\lambda\mu} > 0$. Let $P_{\lambda\mu}(t) = K_{t\lambda,t\mu}$, then $\deg(P_{\lambda\mu}(t)) = d(G_{n;\lambda})$ for all μ such that $K_{\lambda\mu}$ is primitive.*

We might point out that a formula for the dimension $d = \dim G(\lambda, \mu)$ of the convex Gelfand-Tsetlin polytope $G(\lambda, \mu)$, which is nothing other than our $\deg(P_{\lambda\mu}(t))$, has been given in Section 7 Exercise 3.a. of [K1], and also quoted in Section 5.4 of [K2]. With the summation upper limit extended appropriately, this formula coincides with our $d(G_{n;\lambda})$ if we choose $n = s = \ell(\mu)$. Thus if $K_{\lambda\mu}$ is primitive, in which case it is necessarily true that $n = \ell(\mu)$, the above conjecture is equivalent to the claim made regarding the formula given in [K1]. However, in non-primitive cases such as that exemplified in (1.1), the cited formula in [K1] does not yield the correct value, 6, of $\deg(P_{\lambda\mu}(t))$. Instead, the formula yields the value $d(G_{9;\lambda}) = 21$. To obtain the correct result one must first factorise the non-primitive K-polynomial and then apply the above conjecture to each of the primitive factors to give $\deg(P_{\lambda\mu}(t)) = 2 + 4 = 6$, in accordance with (1.3).

Finally, we should note that the work of Kirillov [K2] includes a wealth of conjectures concerning not just Kostka coefficients and polynomials, but also what he calls parabolic Kostka polynomials and Littlewood-Richardson polynomials. The latter are also discussed in [KTT2].

Note added in proof. The validity of the above Conjecture 7.2, including its restriction to the primitive case, has recently been proved by McAllister [McA].

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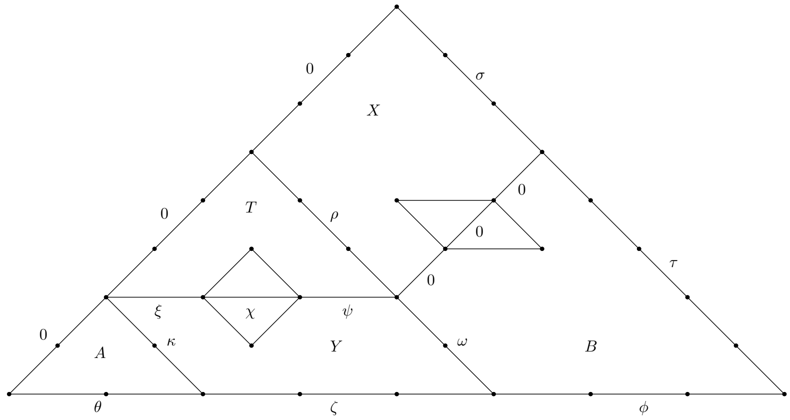
Appendix

Here we present a proof of the following factorisation theorem for $K_{\lambda\beta}$ in the case where β is any weight, not necessarily a partition, such that $K_{\lambda\beta}$ is not primitive. The approach is instructive since it is indicative of the way in which our factorisation result may be extended to the case of Littlewood-Richardson coefficients. An account of this is presented elsewhere [KTT2].

Theorem 5.4. *Let λ be a partition with $\ell(\lambda) \leq n$, and let $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ be a weight such that $|\lambda| = |\beta|$ and $ps(\lambda)_i \geq ps(\beta)_I$ for any $I \subset N = \{1, 2, \ldots, n\}$, with $i = \#I$. Then $K_{\lambda\beta}$ is not primitive if there exists a proper subset $I = \{i_1, i_2, \ldots, i_r\}$ of N such that $ps(\lambda)_r = ps(\beta)_I$ with $r = \#I$ for some r for which $1 \leq r < n$. Let the complement of I in N be denoted by $\overline{I} = \{j_1, j_2, \ldots, j_{n-r}\}$. In such a case let $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, $\tau = (\lambda_{r+1}, \lambda_{r+2}, \ldots, \lambda_n)$, $\zeta = (\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_r})$ and $\eta = (\beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_{n-r}})$, so that $\lambda = (\sigma, \tau)$, with σ and τ partitions, and $\beta = \zeta \cup \eta$, with ζ and η weights, not necessarily partitions, satisfying the partial sum constraints $|\zeta| = |\sigma|$ and $|\eta| = |\tau|$. Then*

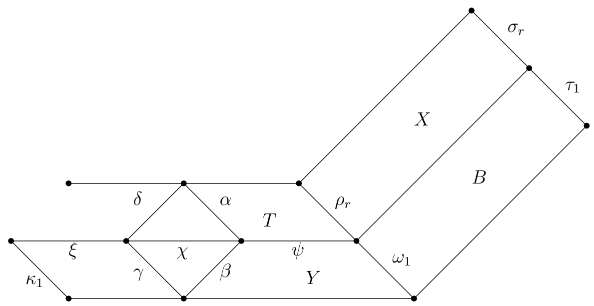
$$K_{\lambda\beta} = K_{\sigma\zeta}\ K_{\tau\eta}. \tag{5.3}$$

Proof Consider first the special case for which $\beta = (\theta, \zeta, \phi)$ and $\eta = (\theta, \phi)$ with $\#\zeta = r$ and $\#\theta + \#\phi = n - r$. The hives H whose enumeration gives $K_{\lambda\beta}$ are illustrated schematically in the following diagram.



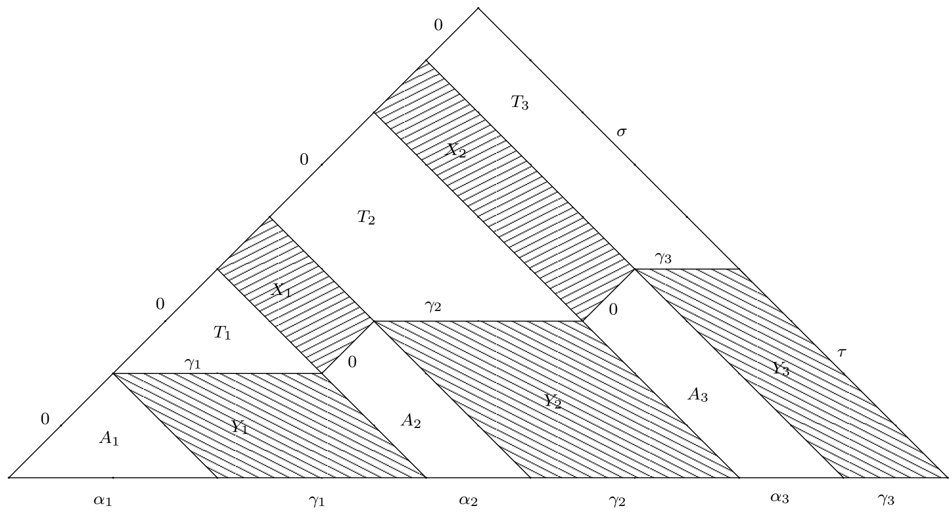
As in our proof of (5.1), covering the parallelograms Y and X with rhombi of type R1 and R2, respectively, and applying the $\gamma \leq \alpha$ hive conditions gives $|\zeta| \leq |\xi| + |\chi| + |\psi| \leq |0| + |\rho| \leq |\sigma|$. Since $|\zeta| = |\sigma|$ the intervening inequalities must be equalities. Then as before the constraint $|\rho| = |\sigma|$ coupled with the R2 hive conditions $\rho_i \leq \sigma_i$ for $i = 1, 2, \ldots, r$ implies that $\rho = \sigma$. In exactly the same way the constraint $|\zeta| = |\xi| + |\chi| + |\psi|$ coupled to the R1 hive conditions of the form $\zeta_j \leq \xi_j, \chi_k, \psi_l$, as appropriate, gives $\zeta = (\xi, \chi, \psi)$. Since $|\rho| = |\sigma|$ and the edge labels along the upper left boundary of X are all 0, it follows, as indicated, that the same is true of the lower right boundary edges of X. We note also that $|\kappa| + |\zeta| = |\xi| + |\chi| + |\psi| + |\omega|$, so that $|\kappa| = |\omega|$. Finally, it follows by using the $\beta \geq \delta$ hive condition for R1 that $\kappa_j \geq \omega_j$ for all j. Hence $\kappa = \omega$ so that the right-hand boundary of A matches up exactly with the left-hand boundary of B. All this shows that each hive H contributing to $K_{\lambda\beta}$ consists of a subhive T contributing to $K_{\sigma\zeta}$ and a subhive formed from the union of A and B contributing to $K_{\tau\eta}$ with $\eta = (\theta, \phi)$. The remaining portions X and Y are completely constrained by the equality of edge labels across each of these parallelograms in each direction.

In order to prove the validity of (5.3) it only remains to show that all cross boundary hive conditions are automatically satisfied once the hives T and $A \cup B$ are specified. The only cases of concern are R1 hives crossing the $X - B$ boundary and the R2 hives crossing the $T - Y$ boundary. The former are covered by the argument used in the proof of Theorem 5.2. The latter are dealt with in a precisely analogous way, through the consideration of a subdiagram following the $T - Y$ and $X - B$ boundary of width equal to two edge lengths, as shown below.



First of all in the portions of the strips overlapping the regions X and Y, we have $\rho_r = \sigma_r$ and $\kappa_1 \geq \gamma \geq \omega_1$. However $\kappa_1 = \omega_1$. In B and T the R2 and R1 hive conditions imply $\omega_1 \leq \tau_1$ and $\rho_r \leq \alpha$. Combining these we have $\gamma = \omega_1 \leq \tau_1 = \lambda_{r+1} \leq \lambda_r = \sigma_r = \rho_r \leq \alpha$. Hence the R2 hive condition $\gamma \leq \alpha$ is automatically satisfied everywhere on the $T - Y$ boundary.

This completes the proof of (5.3) for one class of non-primitive cases. Generalising to the case of any weight β such that $K_{\lambda\beta}$ is not primitive, is now rather straightforward. The most general type of case is illustrated below:



Here, the pattern is clear. We have $\lambda = (\sigma, \tau)$ and $\beta = \zeta \cup \eta$ with $\zeta = \cup_i \gamma_i$ and $\eta = \cup_j \alpha_j$, where in general α_i and γ_j are themselves weights with many parts, as are ζ and η. The regions X_i and Y_i are all parallelograms composed of rhombi if type R2 and R1, respectively. The total number of edges specified by γ_i for all i is r, and the total number of edges specified by α_i for all i is $n - r$, thereby matching the number of parts of the partitions σ and τ, including any trailing zeros. All edges along the $X_i - A_{i+1}$ boundaries are 0, and all along the $Y_i - T_i$ boundaries are specified by the parts of γ_i. The right and left-hand edges of A_i and A_{i+1} match. The same is true of T_i and T_{i+1}. Juxtaposing these edges creates subhives $T = \cup_i T_i$ and $A = \cup_i A_i$ appropriate to $K_{\sigma\zeta}$ and $K_{\tau\eta}$. All the hive conditions are satisfied.

Conversely, reconstituting H by inserting fixed edge parallelograms X_i and Y_i between the various T_i and A_i corresponding to K-hives T and A, always gives rise to a K-hive, as required. This is because the necessary hive conditions for all hives of type R1 and R2 crossing $X_i - A_{i+1}$ and $T_i - Y_i$ boundaries, respectively, are once again automatically satisfied. The argument is precisely the same as that given previously.

This completes the proof of Theorem 5.4. □

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