A finite subgroup of the exceptional Lie group G_2

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Abstract. With a view to further refining the use of the exceptional group G_2 in atomic and nuclear spectroscopy, it is confirmed that a simple finite subgroup $L_{168} \sim PSL_2(7)$ of order 168 of the symmetric group S_8 is also a subgroup of G_2. It is established by character theoretic and other methods that there are two distinct embeddings of L_{168} in G_2, analogous to the two distinct embeddings of $SO(3)$ in G_2. Relevant branching rules, tensor products and symmetrised tensor products are tabulated. As a stimulus to further applications the branching rules are given for the restriction from L_{168} to the octahedral crystallographic point group O.

Dedicated to the memory of Giulio Racah 1909 - 1965

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1. Introduction

The exceptional group G_2, while known to mathematicians since Cartan's thesis[1] of 1894 was not introduced to physicists until some fifty years ago in Racah's[2] 1949 paper "The Theory of Complex Spectra IV" in which he used the group G_2 in his classification of the states of the f^n electron configurations and in the simplification of the calculation of the matrix elements of the Coulomb and spin-orbit interactions *. Racah was able to exploit the fact that G_2 occurs as a subgroup of the rotation group in seven dimensions, $SO(7)$, and furthermore, that the physical rotation group $SO(3)$ associated with the angular momentum of electrons in $f-$orbitals occurs as a natural subgroup of G_2. Racah's work has been well described by Judd[3]. In nuclear physics[4] it was also possible to exploit the group G_2 in the classification of the states of the $f-$shell. The group $SU(3)$ also occurs as a subgroup of G_2, a fact that has been exploited in the interacting boson model of nuclei[5].

Judd[6] has suggested that it might be useful to find and exploit some non-trivial finite subgroups of G_2 noting in particular the subgroups of the symmetric group S_8 listed by Littlewood[7]. This possibility is encouraged by the occurrence of irreducible representations of dimension 7 in Littlewood's groups of order 1344 and 168. The group of order 168 is of particular interest herein, it contains not only a dimension 7 irreducible representation but also a complex pair of irreducible representationsof dimension 3 which is suggestive of an embedding in $SU(3)$. We shall designate Littlewood's order 168 group as L_{168} noting that it appears in the Atlas of Finite Groups[8] as $L_3(2) \sim L_2(7)$, meaning that it is isomorphic to $GL_3(2)$, the general linear group in 3-dimensions over a field of order 2, and to $PSL_2(7)$, the projective special linear group in 2-dimensions over a field of order 7. In addition the Atlas indicates that it is isomorphic to $PSU_2(7)$ and $O_3(7)$. The Atlas indicates that L_{168} may be specified in terms of generators and relations by $\langle R, S, T, \mid R^2 = S^3 = T^7 = RST = (TSR)^4 = 1\rangle$. It can then be realised as a subgroup of the symmetric group S_8 through the identification $R = (18)(27)(34)(56)$, $S = (128)(375)$ and $T = (1234567)$. Since these permutations are all of even parity, L_{168} is also a subgroup of the alternating group A_8.

In the Atlas account of L_{168} nothing is said about G_2, nor indeed $O(7)$, the orthogonal group in 7 dimensions. However, a search of the literature reveals[9−11] that L_{168} is indeed a subgroup of G_2. Moreover, it has been established[10] that any two embeddings of a finite group in G_2 are conjugate if and only if they afford the same character on the natural 7-dimensional representation of $SO(7)$. In this paper it is confirmed by character theoretic and other methods that there exist two non-conjugate embeddings of L_{168} in G_2. These embeddings are explored in some detail

* The introduction of G_2 in physics is said to have arisen in one of Racah's lectures in Jerusalem where he spoke about the problem of labelling for f and d electrons. He discussed the power of group theory in physics in general and spectroscopy in particular. One thing led to another and then he suddenly left the class. Later he told students that he was so consumed by the idea and that he did not notice leaving.....

along with the corresponding branching rules.

We first establish in §2 some simple properties of the group-subgroup pair $S_8 \supset L_{168}$ that follow directly from their respective character tables and then explicitly identify L_{168} as a finite subgroup of both G_2 and $SU(3)$ by means of techniques based on the consideration of various plethysms, or symmetrised tensor and indeed spinor products. This is augmented in §4 by an alternative derivation of the explicit form of the embeddings that is based on the use of a rather simple necessary and sufficient condition for any finite group H to be a subgroup of G_2. This derivation involves a consideration not just of the characters of representations, but of the eigenvalues of group elements in those representations.

We then compute in §5 some explicit branching rules for $G_2 \to L_{168}$ and for $SU(3) \to L_{168}$ giving sufficient data for possible future applications to such things as a study of the properties of f-electrons in an octahedral environment. To this end the branching rules for L_{168} to the octahedral point group O are tabulated in §6.

2. L_{168} as a subgroup of S_8

The characters of the group L_{168} as given by Littlewood[7] are reproduced in Table 1. We label the 6 irreducible representations of L_{168} by the letter sequence a, b, h, j, k, l. In the notation of the Atlas[8] the corresponding characters are denoted by $\chi_1, \chi_4, \chi_5, \chi_6, \chi_2, \chi_3$, respectively. In particular the irreducible representation a with character χ_1 is the trivial, identity representation of L_{168}.

Table 1. Character table for L_{168}.

cycles	(1^8)	(2^4)	$(1^2 3^2)$	(4^2)	(17)	(17)
order	1	21	56	42	24	24
a	1	1	1	1	1	1
b	6	2	0	0	-1	-1
h	7	-1	1	-1	0	0
j	8	0	-1	0	1	1
k	3	-1	0	1	$\frac{1}{2}(-1+i\sqrt{7})$	$\frac{1}{2}(-1-i\sqrt{7})$
l	3	-1	0	1	$\frac{1}{2}(-1-i\sqrt{7})$	$\frac{1}{2}(-1+i\sqrt{7})$

Making use of the character table[7] for S_8 together with Table 1 readily gives the $S_8 \to L_{168}$ branching rules given in Table 2. Since under $S_8 \to L_{168}$ the decompositions of irreducible representations labelled by conjugate partitions are the same we give only the decomposition in the case of one partition of each conjugate pair.

Table 2. Branching rules for $S_8 \to L_{168}$

Dim	$S_8 \quad \to$	L_{168}
1	$\{8\}$	a
7	$\{71\}$	h
20	$\{62\}$	$2b + j$
21	$\{61^2\}$	$h + j + k + l$
28	$\{53\}$	$2h + j + k + l$
64	$\{521\}$	$2b + 2h + 4j + k + l$
35	$\{51^3\}$	$a + 2b + 2h + j$
14	$\{4^2\}$	$2a + 2b$
70	$\{431\}$	$2b + 4h + 3j + k + l$
56	$\{42^2\}$	$a + 4b + h + 3j$
90	$\{421^2\}$	$2b + 4h + 4j + 3k + 3l$
42	$\{3^22\}$	$2h + 2j + 2k + 2l$

Next we need to know the decomposition of Kronecker products of all the irreducible representations of L_{168}, and in the case of Kronecker squares their resolution into their symmetric and antisymmetric parts. Again these decompositions may be readily calculated from the character tables. The non-trivial products and their resolution are given in Table 3.

Table 3. The resolution of Kronecker products and plethysms for the non-trivial irreducible representations of L_{168}.

Product	*Resolution*
$b \times h$	$b + 2h + 2j + k + l$
$b \times j$	$2b + 2h + 2j + k + l$
$b \times k$	$h + j + l$
$b \times l$	$h + j + k$
$h \times j$	$2b + 2h + 3j + k + l$
$h \times k$	$b + h + j$
$h \times l$	$b + h + j$
$j \times k$	$b + h + j + k$
$j \times l$	$b + h + j + l$
$k \times l$	$a + j$

Plethysm	*Resolution*
$b \otimes \{2\}$	$a + 2b + j$
$b \otimes \{1^2\}$	$h + j$
$h \otimes \{2\}$	$a + 2b + h + j$
$h \otimes \{1^2\}$	$h + j + k + l$
$j \otimes \{2\}$	$a + 2b + h + 2j$
$j \otimes \{1^2\}$	$2h + j + k + l$
$k \otimes \{2\}$	b
$k \otimes \{1^2\}$	l
$l \otimes \{2\}$	b
$l \otimes \{1^2\}$	k

It can be deduced from Table 3 that all irreducible representations other than the trivial representation a are faithful representations of L_{168}. This is a consequence of the fact that a representation λ_H of a finite group H is faithful if and only if all the irreducible representations of H appear in the decomposition of some Kronecker power of λ_H. The same conclusion is reached by noting that L_{168} is simple.

Furthermore, an irreducible representation λ_H of a group H is orthogonal if and only if the symmetrised square $\lambda_G \otimes \{2\}$ contains the trivial, identity representation of H. ¿From Table 3 it can be seen, therefore, that b, h, j are all orthogonal, while k and l are not orthogonal. This can also be deduced of course from the fact that their characters are complex. Indeed k and l constitute a mutually complex conjugate pair of unitary irreducible representations of L_{168}. This is sufficient to conclude that their direct sum $k + l$ is orthogonal.

3. L_{168} as a subgroup of G_2

It is well-known that the symmetric group S_{n+1} occurs as a subgroup of the full orthogonal group $O(n)$ and hence $O(7) \supset S_8$. In such an embedding the vector irreducible representation [1] of $O(7)$ decomposes irreducibly into the $\{71\}$ irreducible representation of S_8. The irreducible representation $\{71\}$ is orthogonal but not unimodular[12] and hence while S_8 may be embedded in $O(7)$ it cannot be embedded in $SO(7)$. The irreducible representations of G_2 are all orthogonal and unimodular, including the defining 7-dimensional irreducible representation. It follows that G_2 is subgroup of $SO(7)$. This is sufficient to show that S_8 is certainly not a subgroup of G_2.

In order to show that L_{168} is a subgroup of G_2 the key observation is to note the characterisation[13] of G_2 as the subgroup of $SO(7)$ which leaves a spinor invariant and to determine whether any unimodular, orthogonal, faithful 7-dimensional representation of L_{168} shares this characteristic. In what follows two such representations are found, corresponding to two distinct embeddings of L_{168} in G_2. Since $SU(3)$ is a subgroup of G_2 the question arises as to whether or not an embedding of L_{168} in G_2 corresponds to an embedding of L_{168} in $SU(3)$. This can be established through the identification, if possible, of some unimodular, unitary, faithful 3-dimensional representation of L_{168}. One of the two embeddings identified below is of this type.

It is straightforward to dispose of the question of unimodularity. Quite generally any irreducible representation λ_H of a group H having dimension N is unimodular if and only if the Nth-fold antisymmetrised Kronecker power, $\lambda_H \otimes \{1^N\}$ is just the trivial, identity representation of H. Since $\lambda_H \otimes \{1^N\}$ necessarily has dimension 1, it follows that every irreducible representation λ_H of H is unimodular if H possesses no 1-dimensional irreducible representation other than the identity representation. This is true, for example, of the alternating group A_n, and as can be seen from Table 1 it is also true of L_{168}. It follows that every irreducible representationof L_{168} is unimodular. Moreover, since every representation is a direct sum of irreducible representations, all representations of L_{168} are unimodular.

Recalling the conclusions regarding the orthogonality of representations of L_{168} which we drew from Table 3, we can conclude that the only 7-dimensional unimodular orthogonal representations of L_{168} are h, $a+b$, $a+k+l$ and $7a$. Of these all are faithful except, of course, $7a$. Each of the others defines an embedding of L_{168} in $SO(7)$.

Before proceeding further we must address the labelling of the irreducible representations of G_2. Two schemes exist in the literature, the traditional scheme of Racah[2] and that based upon[14] the $SU(3)$ subgroup of G_2. In the latter scheme the irreducible representations of G_2 are labelled by (λ_1, λ_2) where λ_1 and λ_2 are non-negative integers such that $\lambda_1 \geq 2\lambda_2$. Racah's labels (u_1, u_2) are such that $u_1 = \lambda_1 - \lambda_2$ and $u_2 = \lambda_2$. Throughout this paper we shall use the $SU(3)$ labelling scheme with separational commas and trailing zeros normally omitted.

The defining representation of G_2 is the fundamental 7-dimensional irreducible representation (1). Separating the Kronecker square of this irreducible representation of G_2 into its symmetric and antisymmetric parts gives

$$(1) \otimes \{2\} = (2) + (0) \qquad \text{and} \qquad (1) \otimes \{1^2\} = (21) + (1).$$

It follows that if G_2 contains H as a subgroup with $(1) \rightarrow \lambda_H$, where λ_H is a representation, not necessarily irreducible, of H, then it follows that the identity representation, 1_H, of H must occur in the decomposition of the symmetric part $\lambda_H \otimes \{2\}$ of the Kronecker square of λ_H, and that λ_H itself must occur in the decomposition of the antisymmetric part $\lambda_H \otimes \{1^2\}$ of its Kronecker square, ie.

$$\langle \lambda_H \otimes \{2\}, 1_H \rangle \geq 1 \qquad \text{and} \qquad \langle \lambda_H \otimes \{1^2\}, \lambda_H \rangle \geq 1. \tag{1}$$

These are necessary, though not sufficient, conditions for H to be a subgroup of G_2.

The first of these conditions is guaranteed if λ_H is orthogonal, but this is not enough to guarantee the satisfaction of the second condition. For example, the fact that for S_8 we have

$$\{71\} \otimes \{2\} = \{8\} + \{71\} + \{62\} \qquad \text{and} \qquad \{71\} \otimes \{1^2\} = \{61^2\}$$

immediately precludes the possibility of S_8 being a subgroup of G_2 with an embedding such that $(1) \rightarrow \{71\}$. In the case of L_{168} and our three candidate 7-dimensional representations h, $a+b$ and $a+k+l$, it can be deduced from Table 3 that:

$$\begin{aligned}
h \otimes \{2\} &= a + 2b + h + j; && \text{(2a)}\\
h \otimes \{1^2\} &= h + j + k + l; && \text{(2b)}\\
(a+b) \otimes \{2\} &= 2a + 3b + h + j; && \text{(2c)}\\
(a+b) \otimes \{1^2\} &= b + h + j; && \text{(2d)}\\
(a+k+l) \otimes \{2\} &= 2a + 2b + j + k + l; && \text{(2e)}\\
(a+k+l) \otimes \{1^2\} &= a + j + 2k + 2l. && \text{(2f)}
\end{aligned}$$

It follows that consistency with (1) is violated in the case of the representation $a+b$, but that both h and $a+k+l$ satisfy the requisite consistency conditions.

It remains now to determine whether or not in the reduction of the 8-dimensional spin irreducible representation Δ of $SO(7)$ to L_{168} a spinor is left invariant, just as it is in the case of the reduction from $SO(7)$ to G_2. This defining property[13] of G_2 is such that $\Delta \to (0) + (1)$, or equivalently

$$(1) \otimes \Delta = (0) + (1). \tag{3}$$

It follows that in the case of any faithful, unimodular, orthogonal 7-dimensional representation λ_H of H, a necessary and sufficient condition for H to be a subgroup of G_2 is that

$$\lambda_H \otimes \Delta = 1_H + \lambda_H. \tag{4}$$

Using the algebra of plethysms, the known plethysm of the spin representation Δ of $SO(7)$[7] and (4) we have:

$$\begin{aligned}(\lambda_H \otimes \Delta) \otimes \{2\} &= \lambda_H \otimes (\Delta \otimes \{2\}) \\ &= \lambda_H \otimes \{1^3\} + 1_H;\end{aligned} \tag{5a}$$

and

$$\begin{aligned}(\lambda_H \otimes \Delta) \otimes \{2\} &= (1_H + \lambda_H) \otimes \{2\} \\ &= 1_H + \lambda_H + \lambda_H \otimes \{2\}.\end{aligned} \tag{5b}$$

Comparison of (5a) and (5b) then implies that for H to be a subgroup of G_2 it is necessary that

$$\lambda_H \otimes \{1^3\} = \lambda_H + \lambda_H \otimes \{2\} \tag{5c}$$

The fact that, conversely, (5c) implies (4) is dealt with in §5. By virtue of (1) we have

$$\langle \lambda_H \otimes \{1^3\}, 1_H \rangle \geq 1. \tag{6}$$

This coincides with the condition (ii) of Lemma 2 listed by Cohen and Wales[9] as necessary for H to be a subgroup of G_2. Condition (5c) is stronger, and when coupled with the orthogonality and unimodularity conditions on λ_H, is sufficient to ensure that H is a subgroup of G_2.

Remarkably, the plethysm identity (5c) can only be true if

$$Dim(\lambda_H) = 7, \tag{7}$$

which is the case here.

Explicit evaluation of plethysms then yields the results:

$$\begin{aligned}
&h \otimes \{1^3\} &&= a + 2b + 2h + j, && (8a)\\
&h + (h \otimes \{2\}) &&= a + 2b + 2h + j; && (8b)\\
&(a+b) \otimes \{1^3\} &&= 3h + j + k + l, && (8c)\\
&(a+b) + ((a+b) \otimes \{2\}) &&= 3a + 4b + j; && (8d)\\
&(a+k+l) \otimes \{1^3\} &&= 3a + 2b + j + 2k + 2l, && (8e)\\
&(a+k+l) + ((a+k+l) \otimes \{2\}) &&= 3a + 2b + j + 2k + 2l. && (8f)
\end{aligned}$$

It follows that (5c) is satisfied in the case of both h and $a+k+l$, but not in the case of $a+b$. This is entirely in line with the conclusion following (2).

We conclude that L_{168} is a subgroup of G_2. Indeed there exist two distinct embeddings of L_{168} in G_2 defined by

$$G_2 \to L_{168}: \quad (1) \to h \quad \text{and} \quad (1) \to a+k+l. \tag{9}$$

These two embeddings are analogous to the two embeddings of $SO(3)$ in G_2 defined by the restrictions $G_2 \to SO(3)$ with $(1) \to [3]$ and $G_2 \to SU(3) \to SO(3)$ with $(1) \to \{1\} + \{1^2\} + \{0\} \to 2[1] + [0]$. In fact the analogy can be extended further by noting that the second of our two embeddings of L_{168} in G_2 defined in (9) is such that $G_2 \supset SU(3) \supset L_{168}$ with

$$(1) \to \{1\} + \{1^2\} + \{0\} \to k+l+a. \tag{10}$$

This follows from the fact that the irreducible representation k of L_{168} is a faithful, unimodular, unitary 3-dimensional representation. It therefore defines an embedding of L_{168} in $SU(3)$ through the restriction

$$SU(3) \to L_{168}: \quad \{1\} \to k. \tag{11}$$

Since $\{1^2\}$ is complex conjugate to $\{1\}$ and l is the complex conjugate of k, we also have $\{1^2\} \to l$, as required to give (10).

4. The use of eigenvalues of representation matrices of L_{168}

As noted earlier, the group denoted here by L_{168} can be defined in terms of generators and relations by $\langle R, S, T, \mid R^2 = S^3 = T^7 = RST = (TSR)^4 = 1 \rangle$. It can be realised as a subgroup of the symmetric group S_8. The cycle structure of elements in its 6 conjugacy classes are given by $(1^8), (2^4), (1^2 3^2), (4^2), (17), (17)$, so that the elements themselves have order $1, 2, 3, 4, 7, 7$, respectively. This implies that the eigenvalues of the matrices representing these elements can only be powers of the corresponding primitive roots of unity. This observation is sufficient in almost all cases to write down the complete set of eigenvalues of each irreducible matrix representation simply from a knowledge of the characters in Table 1, which are of course the sums of the eigenvalues. The only additional observation that is required in order to complete this exercise is that the cycle structure of TSR is (4^2) and that of $(TSR)^2$ is (2^4). The corresponding matrices can be simultaneously diagonalised so that in any given irreducible representation the eigenvalues in the conjugacy class labelled by (2^4) are the squares of those in the conjugacy class labelled by (4^2). The results for each of the irreducible representations of L_{168} are displayed in Tables 4a and 4b, where $\omega = e^{i2\pi/3}$ and $\eta = e^{i2\pi/7}$.

Table 4a. Eigenvalues of group elements in each conjugacy class of L_{168} in the irreducible representations a, b and h.

Irrep	a	b	h
Dim	1	6	7
(1^8)	1	$(1,1,1,1,1,1)$	$(1,1,1,1,1,1,1)$
(2^4)	1	$(1,1,1,1,-1,-1)$	$(1,1,1,-1,-1,-1,-1)$
$(1^2 3^2)$	1	$(1,1,\omega,\omega,\omega^2,\omega^2)$	$(1,1,1,\omega,\omega,\omega^2,\omega^2)$
(4^2)	1	$(1,1,-1,-1,i,-i)$	$(1,-1,-1,i,i,-i,-i)$
(17)	1	$(\eta,\eta^2,\eta^3,\eta^4,\eta^5,\eta^6)$	$(1,\eta,\eta^2,\eta^3,\eta^4,\eta^5,\eta^6)$
(17)	1	$(\eta,\eta^2,\eta^3,\eta^4,\eta^5,\eta^6)$	$(1,\eta,\eta^2,\eta^3,\eta^4,\eta^5,\eta^6)$

Table 4b. Eigenvalues of group elements in each conjugacy class of L_{168} in the irreducible representations j, k and l.

Irrep	j	k	l
Dim	8	3	3
(1^8)	$(1,1,1,1,1,1,1,1)$	$(1,1,1)$	$(1,1,1)$
(2^4)	$(1,1,1,1,-1,-1,-1,-1)$	$(1,-1,-1)$	$(1,-1,-1)$
$(1^2 3^2)$	$(1,1,\omega,\omega,\omega,\omega^2,\omega^2,\omega^2)$	$(1,\omega,\omega^2)$	$(1,\omega,\omega^2)$
(4^2)	$(1,1,-1,-1,i,i,-i,-i)$	$(1,i,-i)$	$(1,i,-i)$
(17)	$(1,1,\eta,\eta^2,\eta^3,\eta^4,\eta^5,\eta^6)$	(η,η^2,η^4)	(η^3,η^5,η^6)
(17)	$(1,1,\eta,\eta^2,\eta^3,\eta^4,\eta^5,\eta^6)$	(η^3,η^5,η^6)	(η,η^2,η^4)

By taking products of the tabulated eigenvalues it is easy to confirm, as stated in §3, that each irreducible representation of L_{168} is unimodular. These tabulations may also be used to confirm the orthogonality of each representation. Here, however, we wish to show that the tabulations offer a very quick way to determine which of the 7-dimensional representations h, $a+b$ and $a+k+l$ define embeddings of L_{168} in G_2.

Proposition *Let λ_H be a faithful, orthogonal, unimodular, 7-dimensional represention of a compact or finite group H. If the eigenvalues of every group element in the representation λ_H are of the form $\{1,x,y,z,x^{-1},y^{-1},z^{-1}\}$ for some x, y and z such that $|x|=|y|=|z|=1$ and $xyz=1$, then H is a subgroup of G_2 with an embedding defined by $(1)\to\lambda_H$.*

Proof First it should be noted that since λ_H is orthogonal, unimodular and 7-dimensional the image of each group element in the representation λ_H is an element of $SO(7)$. All such elements necessarily have eigenvalues $\{1,x,y,z,x^{-1},y^{-1},z^{-1}\}$, where x, y and z are each of the form $e^{i\phi}$ for some real ϕ.

With this notation the characters of λ_H and $\lambda_H \otimes \Delta$ are given by

$$\lambda_H \qquad = 1 + x + y + z + x^{-1} + y^{-1} + z^{-1}; \tag{12a}$$

$$\lambda_H \otimes \Delta = \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)\left(y^{\frac{1}{2}} + y^{-\frac{1}{2}}\right)\left(z^{\frac{1}{2}} + z^{-\frac{1}{2}}\right), \tag{12b}$$

Now let

$$f(x,y,z) = \lambda_H \otimes \Delta - 1_H - \lambda_H. \tag{13}$$

Substituting (12a) and (12b) into (13) gives

$$\begin{aligned} f(x,y,z) &= (x^{\frac{1}{2}} + x^{-\frac{1}{2}})(y^{\frac{1}{2}} + y^{-\frac{1}{2}})(z^{\frac{1}{2}} + z^{-\frac{1}{2}}) - (2 + x + y + z + x^{-1} + y^{-1} + z^{-1}) \\ &= x^{-1}(x^{\frac{1}{2}}y^{\frac{1}{2}}z^{\frac{1}{2}} - 1)(x^{\frac{1}{2}}y^{\frac{1}{2}}z^{-\frac{1}{2}} - 1)(x^{\frac{1}{2}}y^{-\frac{1}{2}}z^{\frac{1}{2}} - 1)(x^{\frac{1}{2}}y^{-\frac{1}{2}}z^{-\frac{1}{2}} - 1) \end{aligned} \tag{14}$$

It follows that $f(x,y,z) = 0$ if and only if

$$x^\alpha y^\beta z^\gamma = 1 \quad \text{for some } \alpha, \beta, \gamma \in \{1, -1\}. \tag{15}$$

Since the identification of x, y and z amongst the set $\{x, y, z, x^{-1}, y^{-1}, z^{-1}\}$ is entirely a matter of choice, it follows from (13) that

$$\lambda_H \otimes \Delta = 1_H + \lambda_H \tag{16}$$

if and only if there exists some choice of x, y and z such that $xyz = 1$. To complete the proof of the Proposition it is merely necessary to note that (16) coincides with the necessary and sufficient condition identified in (4). •

To avoid the use of spinor characters it is possible to arrive at the same result through the use of the necessary and sufficient condition (5c). This can be seen by noting that

$$\lambda_H \otimes \{1^3\} - \lambda_H - \lambda_H \otimes \{2\} = f(x^2, y^2, z^2). \tag{17}$$

The condition (5c) then takes the form $f(x^2, y^2, z^2) = 0$. The same factorisation as noted in (14) coupled with an appropriate identification of x, y and z, then leads, as before, to the condition $xyz = 1$.

The condition $xyz = 1$ is nothing other than a re-statement of condition (iv) of Lemma 2 of Cohen and Wales[9], from which can be derived their conditions (ii), (v) and (vi).

In the case of $H = L_{168}$ this can be illustrated for each of the faithful, orthogonal, unimodular 7-dimensional representations h, $a + b$ and $a + k + l$ by trying to identify from the Tables 4a and 4b a suitable choice of x y and z for each conjugacy class. They are not necessarily unique since any given eigenvalue may be associated with either one of a pair such as x and x^{-1}. However, the corresponding values of the character Δ are unique and these are displayed in Table 5 for λ_H given by h, $a + b$ and $a + k + l$ alongside one particular choice of (x, y, z) in each conjugacy class.

Table 5. Eigenvalues (x, y, z) for 7-dimensional orthogonal representations λ_H of L_{168} and the corresponding characters of $\lambda_H \otimes \Delta$.

Irrep	$\lambda_H = h$		$\lambda_H = a+b$		$\lambda_H = a+k+l$	
Class	(x,y,z)	Δ	(x,y,z)	Δ	(x,y,z)	Δ
(1^8)	$(1,1,1)$	8	$(1,1,1)$	8	$(1,1,1)$	8
(2^4)	$(1,-1,-1)$	0	$(1,1,-1)$	0	$(1,-1,-1)$	0
$(1^2 3^2)$	$(1,\omega,\omega^2)$	2	$(1,\omega,\omega^2)$	2	$(1,\omega,\omega^2)$	2
(4^2)	$(-1,i,i)$	0	$(1,-1,i)$	0	$(1,i,-i)$	4
(17)	(η,η^2,η^4)	1	(η,η^2,η^4)	1	(η,η^2,η^4)	1
(17)	(η^3,η^5,η^6)	1	(η^3,η^5,η^6)	1	(η^3,η^5,η^6)	1

Examination of this table indicates that $xyz = 1$ for all cases except those of the conjugacy classes (2^4) and (4^2) of the representation $a+b$. In these two cases there is no choice of x, y and z such that $xyz = 1$. Thus only h and $a+k+l$, but not $a+b$, define embeddings of L_{168} in G_2.

Comparison of the results of Table 5 with the character table of L_{168} given in Table 1 also shows that:

$$h \otimes \Delta = a + h; \tag{13a}$$

$$(a+b) \otimes \Delta = a + h; \tag{13b}$$

$$(a+k+l) \otimes \Delta = a + (a+k+l). \tag{13c}$$

Since $a = 1_H$ for $H = L_{168}$ this confirms, yet again, that the necessary and sufficient condition (4) is satisfied in the cases $\lambda_H = h$ and $a+k+l$, but not $a+b$.

5. Branching rules for $G_2 \to L_{168}$ and $SU(3) \to L_{168}$

Branching rules for the decomposition of irreducible representations under the restrictions $G_2 \to L_{168}$ with $(1) \to h$ and $SU(3) \to L_{168}$ with $\{1\} \to k$ may be calculated quite readily. In the former case we may evaluate the decomposition for an arbitrary irreducible representation of G_2 by simply working along the group chain $G_2 \uparrow O(7) \downarrow S_8 \downarrow L_{168}$ to produce the results given in Table 6.

Table 6. Branching rules for the restriction $G_2 \to L_{168}$ defined by $(1) \to h$.

Dim	$G_2 \to$	L_{168}
1	(00)	a
7	(10)	h
27	(20)	$2b + h + j$
14	(21)	$j + k + l$
77	(30)	$a + 2b + 4h + 3j + 2k + 2l$
64	(31)	$2b + 2h + 4j + k + l$
182	(40)	$3a + 8b + 7h + 8j + 3k + 3l$
189	(41)	$6b + 9h + 9j + 3k + 3l$
77	(42)	$a + 4b + 2h + 4j + k + l$
378	(50)	$a + 12b + 17h + 18j + 7k + 7l$
448	(51)	$2a + 16b + 18h + 22j + 8k + 8l$
286	(52)	$2a + 10b + 12h + 123j + 6k + 6l$
714	(60)	$6a + 28b + 30h + 33j + 11k + 11l$
924	(61)	$5a + 32b + 37h + 45j + 18k + 18l$
729	(62)	$5a + 28b + 30h + 35j + 11k + 11l$
273	(63)	$2a + 8b + 13h + 12j + 6k + 6l$
1254	(70)	$8a + 42b + 54h + 59j + 24k + 24l$
1728	(71)	$10a + 62b + 72h + 82j + 31k + 31l$
1547	(72)	$8a + 54b + 65h + 74j + 28k + 28l$
896	(73)	$6a + 32b + 38h + 42j + 16k + 16l$
2079	(80)	$15a + 78b + 84h + 99j + 36k + 36l$
3003	(81)	$15a + 106b + 126h + 144j + 53k + 53l$
2926	(82)	$20a + 108b + 120h + 139j + 51k + 51l$
2079	(83)	$11a + 72b + 88b + 99j + 38k + 38l$
748	(84)	$6a + 30b + 28h + 36j + 13k + 13l$
3289	(90)	$18a + 114b + 141h + 155j + 60k + 60l$
4928	(91)	$28a + 176b + 204h + 236j + 88k + 88l$
5103	(92)	$30a + 180b + 213h + 243j + 93k + 93l$
4096	(93)	$26a + 146b + 172h + 194j + 73k + 73l$
2261	(94)	$11a + 80b + 94h + 109j + 40k + 40l$
5005	(10 0)	$33a + 184b + 208h + 237j + 86k + 86l$

In the latter case one may rapidly build up a table by resolving products in $SU(3)$ and L_{168}. This yields the results listed in Table 7. The results for the contragredient partners to those listed in Table 7 may be found by simply making the interchange $k \leftrightarrow l$. Branching rules for the restriction $G_2 \to L_{168}$ using the alternative embedding such that $(1) \to a + k + l$ can be obtained by exploiting the group chain $G_2 \downarrow SU(3) \downarrow L_{168}$, the first stage of which is well documented[15].

Table 5. Branching rules for the restriction $SU(3) \to L_{168}$ defined by $\{1\} \to k$.

Dim	$SU(3) \to$	L_{168}
1	$\{0\}$	a
3	$\{1\}$	k
3	$\{1^2\} = \{\bar{1}\}$	l
6	$\{2\}$	b
8	$\{21\}$	j
10	$\{3\}$	$h + l$
15	$\{31\}$	$h + j$
15	$\{4\}$	$a + b + j$
24	$\{41\}$	$b + h + j + k$
27	$\{42\}$	$2b + h + j$
21	$\{5\}$	$h + j + k + l$
35	$\{51\}$	$b + h + 2j + k + l$
42	$\{52\}$	$b + 2h + 2j + k + l$
28	$\{6\}$	$a + 2b + h + j$
48	$\{61\}$	$2b + 2h + 2j + k + l$
60	$\{62\}$	$a + 3b + 2h + 3j + l$
64	$\{63\}$	$a + 2b + 3h + 3j + k + l$

6. The octahedral subgroup of L_{168}

In terms of potential applications it is worth noting that L_{168} contains the symmetric group S_4 as a subgroup[8] which itself is isomorphic to the crystallographic point group O, commonly known as the octahedral group[3].

For f-electrons in a weak crystal field of octahedral symmetry it is common to consider the group-subgroup chain segment $G_2 \supset SO(3) \supset O$. An alternative group-subgroup chain segment would be $G_2 \supset L_{168} \supset O$, with $SO(3)$ being lost as an approximate speherical symmetry group.

The branching rules for $L_{168} \to S_4 = O$ may be readily found from a knowledge of the characters of L_{168} over the classes of its subgroup S_4 as shown in Table 8. It should be noted that on restriction from L_{168} to S_4 the surviving sets of elements in the classes (1^8), $(1^2 3^2)$ and (4^2) constitute the classes (1^4), (13) and (4), respectivley, of S_4, while the set of those in the class (2^4) splits so as to constitute the two classes $(1^2 2)$ and (2^2) of S_4.

Table 8. Characters of the irreducible representations of L_{168} evaluated on the conjugacy classes of the subgroup S_4.

	(1^4)	$(1^2 2)$	(2^2)	(13)	(4)
a	1	1	1	1	1
b	6	2	2	0	0
h	7	−1	−1	1	−1
j	8	0	0	−1	0
k	3	−1	−1	0	1
l	3	−1	−1	0	1

Comparison of this with the character table[7] of S_4 or, equivalently, that[3] of O, immediately leads to the $L_{168} \to S_4 \sim O$ branching rules displayed in Table 9.

Table 9. Branching rules for $L_{168} \to S_4 \sim O$.

Dim	L_{168}	S_4	O
1	a	$\{4\}$	Γ_1
6	b	$\{4\} + \{31\} + \{2^2\}$	$\Gamma_1 + \Gamma_4 + \Gamma_3$
7	h	$\{31\} + \{21^2\} + \{1^4\}$	$\Gamma_4 + \Gamma_5 + \Gamma_2$
8	j	$\{31\} + \{2^2\} + \{21^2\}$	$\Gamma_4 + \Gamma_3 + \Gamma_5$
3	k	$\{21^2\}$	Γ_5
3	l	$\{21^2\}$	Γ_5

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