Group of substitution with prefunction and boson normal ordering problem

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Outline of the talk

1. Motivations: Boson normal ordering problem
   - $1D$ harmonic oscillator
   - Boson operators

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   - $1D$ harmonic oscillator
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2. Infinite matrices
3. Group of substitution with prefunction

One-dimensional harmonic oscillator: quantum mechanics

In quantum mechanics, the Hamiltonian operator of a particle of mass m subject to a certain potential is given by the following operators equation

$$H = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 X^2$$

where X is the **position** operator and P the **momentum** operator both of them acting on a given Hilbert space.

One-dimensional harmonic oscillator: quantum mechanics

Solving the equation for the Hamiltonian operator is in fact equivalent to the spectral analysis of H that is the determination of the eigenvalues of the Hamiltonian

$$Hv = \lambda v$$

where the eigenvalue λ is the energy associated to the eigenvector v.

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Therefore the eigenvectors of H and N are the same.

One-dimensional harmonic oscillator: quantum mechanics

Some properties :

1. $a^\dagger$ is the adjoint of a;
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Conclusion: In the quantum case, there is a discrete range of possible values for the energy of the harmonic oscillator.

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Fock space

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Interpretation: they are exactly n bosons in the state v_n. The one-particle Hilbert space of (quantum) states, called **Fock space**, is spanned by the **number states** v_n for $n \in \mathbb{N}$.

Occupation number representation

In this situation the operators a and $a^\dagger$ act on the number states as follows

$$\begin{array}{cccrl} a & v_n & = & \sqrt{n} & v_{n-1} \; ; \\ a^\dagger & v_n & = & \sqrt{n+1} & v_{n+1} \; . \end{array}$$

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To solve these problems one has to fix the order of the operators involved in a sequence of Boson operators.
This leads to the notion of **normal ordered form** of the boson operators in which all $a^{\dagger}$ stand to the left of all the factors a.

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Note that in general, as operators $:P(a,a^\dagger): \neq P(a,a^\dagger)$. The equality holds only for operators which are already in normal form.

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The application of the normal ordering operation on $P(a, a^\dagger)$ leads to the polynomial $\mathcal{N}(P(a, a^\dagger))$ which is obtained by moving all the annihilation operators a to the right using the commutation relation $[a, a^\dagger] = 1$.

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Note that as operators, $P(a, a^\dagger) = \mathcal{N}(P(a, a^\dagger))$.

Boson normal ordering: Normal ordering problem

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Boson normal ordering: powers of a word

Let $\omega \in \{a, a^{\dagger}\}^*$. Let e be the difference between the number of creation operators and annihilation operators in ω.

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$$\mathcal{N}(\omega^n) = \begin{cases} (a^\dagger)^{ne} \left(\sum_{k \geq 0} S_\omega(n,k)(a^\dagger)^k a^k \right) & \text{if } e \geq 0 \,, \\ \left(\sum_{k \geq 0} S_\omega(n,k)(a^\dagger)^k a^k \right) (a)^{n|e|} & \text{if } e < 0 \,. \end{cases}$$

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This is a generalization of the following (Katriel, 1974): if $\omega = a^\dagger a$, then $S_\omega(n,k)$ are the usual Stirling numbers of the second kind.

Boson normal ordering: powers of a word

The normal form of nth powers of a word a very important because they are used to find the normal form of evolution operators $e^{\lambda\omega}$.

Boson normal ordering: powers of a word with only one annihilation operator

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Let us consider a string of boson operators
with only one annihilation operator of the following form:

$$\omega = (a^\dagger)^{r-p} a (a^\dagger)^p \; .$$

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Then S_ω is a unipotent matrix *i.e.* a lower triangular matrix with diagonal elements equal to 1.

Conclusion: We want to study topological, algebraic or combinatorial properties of these doubly-infinite matrices in order to understand the meaning of the coefficients $S_{\omega}(n, k)$ in the particular case of a word with only one annihilation operator.

Topological properties of:

$$\mathbb{C}^{\mathbb{N}\times\mathbb{N}} \supset \mathbb{C}^{\mathbb{N}\times(\mathbb{N})} \simeq \mathcal{L}(\mathbb{C}^{\mathbb{N}}) \supset \mathtt{LT}(\mathbb{N},\mathbb{C}) \supset \mathtt{LT}^{\times}(\mathbb{N},\mathbb{C}) \supset \mathtt{UT}(\mathbb{N},\mathbb{C}) \; .$$

Fréchet space

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$$\begin{array}{rccc} \mathtt{pr}_{n,k} : & \mathbb{C}^{\mathbb{N}\times\mathbb{N}} & \longrightarrow & \mathbb{C} \\ & M & \longmapsto & M_{n,k} \end{array}$$

are continuous (the topology of simple convergence), $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ is a Fréchet space.

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are continuous (the topology of simple convergence), $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ is a Fréchet space.
$\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ is not a Banach space (because every neighborhood of zero contains a non trivial subvector space).

Row-finite matrices

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$\mathbb{C}^{\mathbb{N}\times(\mathbb{N})}$ is an associative unital noncommutative algebra.

Row-finite matrices

The vector space $\mathbb{C}^{\mathbb{N}}$ of complex sequences is a Fréchet (but not a Banach) space when equiped with the weakest topology for which every natural projection $\mathrm{pr}_k((u_n)_{n\in\mathbb{N}}) = u_k$ is continuous.

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Proposition

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In particular S_ω defines a continuous operator of $\mathbb{C}^{\mathbb{N}}$.

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3. $\mathtt{LT}(\mathbb{N},\mathbb{C})$ is a topological algebra (*i.e.* its multiplication is continuous);
4. The multiplicative group $\mathtt{LT}(\mathbb{N},\mathbb{C})^\times$ of invertible elements of $\mathtt{LT}(\mathbb{N},\mathbb{C})$, that is lower triangular matrices with nonzero elements on the diagonal, is a Hausdorff topological group.

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We can easily check that $\mathtt{UT}(\mathbb{N},\mathbb{C})$ is a subgroup of $\mathtt{LT}(\mathbb{N},\mathbb{C})^{\times}$.
S_ω is an invertible matrix and therefore it induces a continuous isomorphism of $\mathbb{C}^{\mathbb{N}}$.

It has been proved (G. Duchamp *et al.*, 2003) that for every word $\omega \in \{a, a^\dagger\}^\star$ with only one annihilation operator, there are two formal power series: $P(x)$ an invertible series with $[1]P(x) = 1$ and $S(x)$ without constant term and such that $[x]S(x) = 1$ so that

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$$\sum_{n \geq 0} S_\omega(n, k) \frac{x^n}{n!} = P(x) \frac{S(x)^k}{k!} \ .$$

Sheffer group: prefunctions

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It is well-known that $\mathcal{P}$ is a group for the multiplication of formal power series (sometimes called *Appell group*). Its elements are called *prefunctions*.

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This set is a group for the substitution of formal power series: $(S_1 \circ S_2)(x) = S_1(S_2(x))$ (sometimes called the *associated group*). Its elements are called *substitutions*.
We denote by $S^{[-1]}(x)$ the compositional inverse of $S(x)$: $S(S^{[-1]}(x)) = S^{[-1]}(S(x)) = x$.

Sheffer group: substitutions with prefunctions

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For this law, one has

$$(P(x), S(x))^{-1} = (\frac{1}{P(S^{[-1]}(x))}, S^{[-1]}(x)) \ .$$

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There is a *natural* right linear representation of the semi-direct product $\mathcal{P} \rtimes \mathcal{S}$ on the complex vector space $\mathbb{C}[[x]]$:

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$$f(x).(P(x), S(x)) = P(x)f(S(x)) \ ,$$

$\forall f(x) \in \mathbb{C}[[x]]$ and $\forall (P(x), S(x)) \in \mathcal{P} \rtimes \mathcal{S}$.

Sheffer group: matrix representation

For every $(P(x), S(x)) \in \mathcal{P} \rtimes \mathcal{S}$ there is one and only one $M_{P,S} \in \mathtt{UT}(\mathbb{N}, \mathbb{C})$ defined by

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We call $M_{P,S}$ the *(exponential) Riordan matrix* of $(P(x), S(x))$.

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Obviously it can also be seen as a continuous invertible linear operator on $\mathbb{C}[[x]]$:

$$f(x) = \sum_{n\geq 0} a_n x^n \mapsto \sum_{n\geq 0}(\sum_{k\geq 0} M_{P,S}(n,k)a_k)x^k \ .$$

(The topology on $\mathbb{C}[[x]]$ is not the usual topology induced by the discrete valuation but rather the weakest topology for which the projections $[x^n] : f(x) \mapsto [x^n]f(x)$ are continuous.)

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proposition

Let M be a Riordan matrix. For every $z \in \mathbb{C}$, M^z is also a Riordan matrix.

Unipotent matrices which are Riordan matrices

Let M be a unipotent matrix. For every $k \in \mathbb{N}$, let define the exponential generating function $c_k(x)$ of the kth column of M:

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Then M is a Riordan matrix associated to some element $(P(x), S(x)) \in \mathcal{P} \rtimes \mathcal{S}$ if, and only if, for each $k \in \mathbb{N}$, one has

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(Therefore $c_0(x) = P(x)$ and $\frac{c_1(x)}{c_0(x)} = S(x)$.)

Back to the normal form of ω^n

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In particular for every word $\omega \in \{a, a^\dagger\}$ with only one annihilation operator is associated one and only one $(P(x), S(x)) \in \mathcal{P} \rtimes \mathcal{S}$ such that for every $n \in \mathbb{N}$, $n > 0$,

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$$\mathcal{N}(\omega^n) = (a^\dagger)^{n(|\omega|_{a^\dagger}-1)} \left(\sum_{k \geq 0} [\frac{x^n}{n!}](P(x)\frac{S(x)^k}{k!})(a^\dagger)^k a^k \right)$$

where $|\omega|_{a^\dagger}$ is the number of $a^\dagger$ in ω.

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2. Given a Riordan matrix M and $z \in \mathbb{C}$, which function in a and $a^\dagger$ is associated to M^z (and the converse) ?
3. The "monic" Sheffer group is the "heart" of the machinery of general substitution with prefunction because if $P(x) = \lambda_0 + \lambda_1 x + \ldots$ with $\lambda_0 \neq 0$ and $S(x) = \mu_1 x + \mu_2 x^2 + \ldots$ with $\mu_1 \neq 0$, then $M_{P,S} = \mathtt{diag}(\lambda_0, \lambda_0, \ldots) M_{\frac{P}{\lambda_0}, \frac{S}{\mu_1}} \mathtt{diag}(1, \mu_1, \mu_1^2, \mu_1^3, \ldots)$.