# Growing binary trees

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LIP6

(joint with Olivier Bodini and Antoine Genitrini)

ALEA 2026

March 12, 2026

# Settings

Vertex types

- ● internal node
- ○ anchor (active leaf)
- □ leaf (dead leaf)

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- At the beginning ($t = 0$), our tree is an anchor ○
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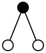
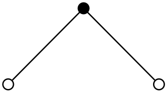
  - or a subtree 

$t = 1$



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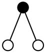
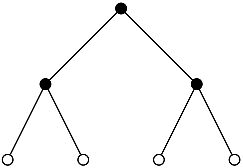
  - or a subtree 

$t = 2$



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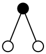
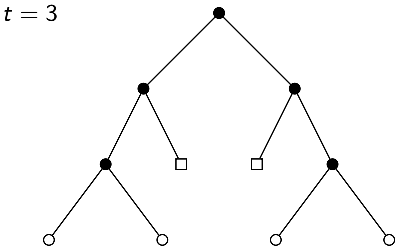
  - or a subtree 



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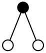
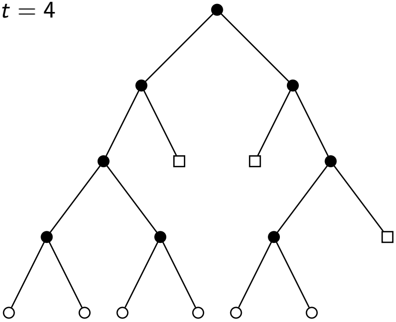
  - or a subtree 

$t = 4$



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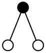
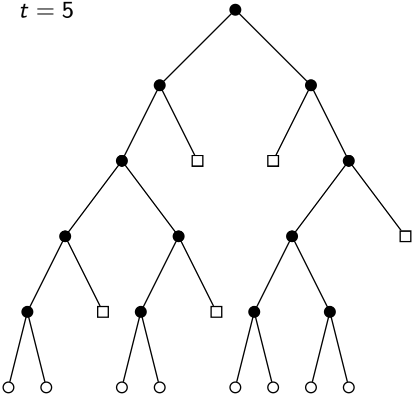
  - or a subtree 

$t = 5$



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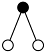
  - or a subtree 

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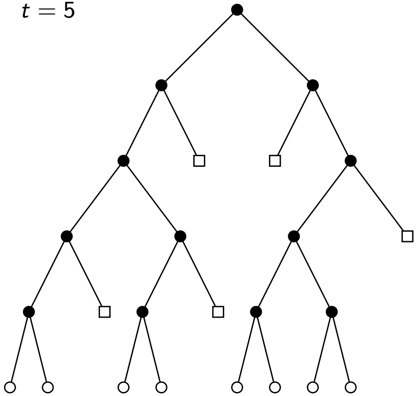
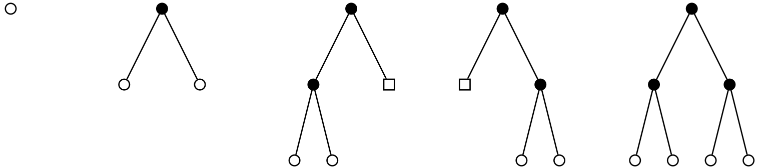


Studied objects: active trees (i.e. trees that have anchors)

# Counting

$$t_{n,m} = \{\text{active trees with } n \text{ internal nodes } m \text{ anchors}\}$$



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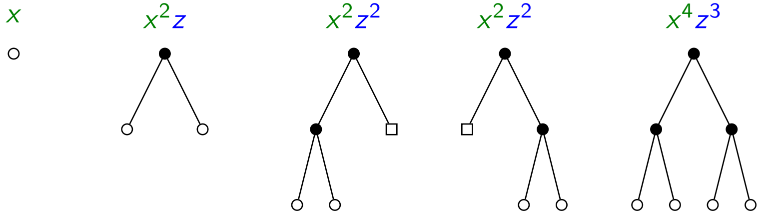


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Marking variables:

- $x$ marks anchors
- $z$ marks internal nodes

$$T(x, z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} t_{n,m} x^m z^n$$

$$T(x, z) = x + x^2 z + 2x^2 z^2 + \ldots$$

# Equation

- Growing process replacement:

$$x \mapsto 1 \qquad\qquad x \mapsto zx^2$$

- Equation:

$$T(x,z) = x + T(1 + zx^2, z) - T(1, z)$$

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  - $C(z) = 1 + zC^2(z)$

$$\sum_{m=1}^{\infty} t_{n,m} = C_n$$

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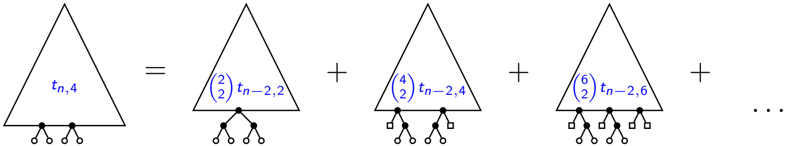
- Recurrent relation $(n, k > 0)$:

$$t_{n,2k-1} = 0 \qquad \text{and} \qquad t_{n,2k} = \sum_{m=k}^{\infty} \binom{m}{k} t_{n-k,m}$$

# Recurrent relations

- Exact: $$t_{n,2k} = \sum_{m=\lceil k/2 \rceil}^{\infty} \binom{2m}{k} t_{n-k,2m} \qquad (n > 0)$$



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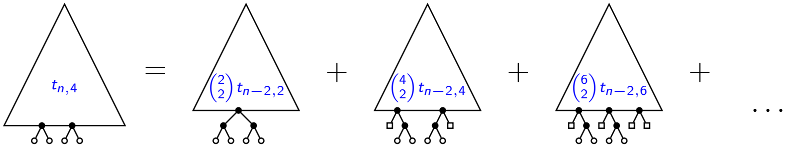
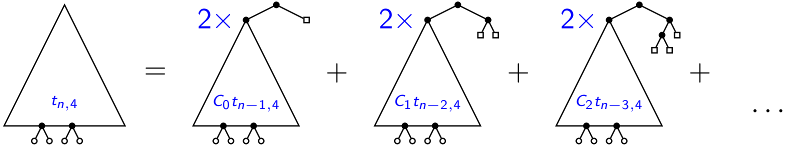



- Asymptotic: $$t_{n,2k} > 2 \sum_{m=0}^{r} C_m t_{n-m-1,2k} \qquad (n \gg r)$$



# Mandelbrot Polynomials

- Define

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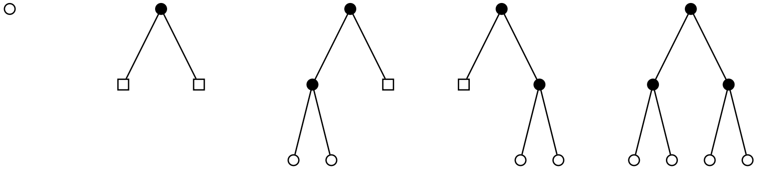
- $p_n(x, z)$ counts binary trees:
  - of height at most $n$,
  - anchors are at level $n$,
  - leaves are at levels $k < n$



$$p_2(x, z) = 1 + z + 2x^2z^2 + x^4z^3$$

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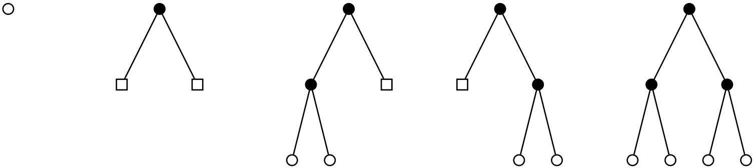

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$$q_n(x,z) = zp_n(x,z)$$

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$$q_n(x,z) = zp_n(x,z)$$

- $q_n(1,z)$ are known as Mandelbrot Polynomials

- Corollary: for $k < n$, $[z^{k+1}]q_n(1,z) = C_k$

# Cumulative value of anchors

- Denote

$$\widetilde{T}(x,z) := \frac{\partial T}{\partial x}(x,z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} m t_{n,m} x^m z^n$$

- Equation:

$$\widetilde{T}(x,z) = 1 + 2xz\,\widetilde{T}(1+zx^2, z)$$

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- In terms of Mandelbrot polynomials:

$$\widetilde{T}(1, z) = 1 + \sum_{k=1}^{\infty} 2^k \prod_{\ell=0}^{k-1} q_\ell(1, z)$$

# Small values of $t_{n,2k}$

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $t_{n,2}$ | 1 | 2 | 4 | 12 | 32 | 104 | 328 | 1 080 | 3 648 | 12 544 |
| $t_{n,4}$ | 0 | 0 | 1 | 2 | 10 | 24 | 92 | 308 | 1 028 | 3 584 |
| $t_{n,6}$ | 0 | 0 | 0 | 0 | 0 | 4 | 8 | 40 | 176 | 584 |
| $t_{n,8}$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 10 | 84 |
| $t_{n,10}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| $n$ | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|
| $t_{n,2}$ | 43 600 | 153 504 | 546 272 | 1 960 368 | 7 085 456 | 25 773 296 |
| $t_{n,4}$ | 12 736 | 45 160 | 161 152 | 581 632 | 2 114 504 | 7 727 656 |
| $t_{n,6}$ | 2 144 | 8 192 | 30 720 | 112 496 | 416 528 | 1 553 776 |
| $t_{n,8}$ | 282 | 1 048 | 4 368 | 18 224 | 69 676 | 265 220 |
| $t_{n,10}$ | 24 | 104 | 352 | 1 616 | 8 208 | 34 704 |
| $t_{n,12}$ | 0 | 4 | 36 | 96 | 456 | 2 936 |
| $t_{n,14}$ | 0 | 0 | 0 | 8 | 16 | 80 |
| $t_{n,16}$ | 0 | 0 | 0 | 0 | 1 | 2 |

# Anchor distributions

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| $t_{n,12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $C_n$ | 1 | 2 | 5 | 14 | 42 | 132 | 429 | 1 430 | 4 862 | 16 796 | 58 786 |

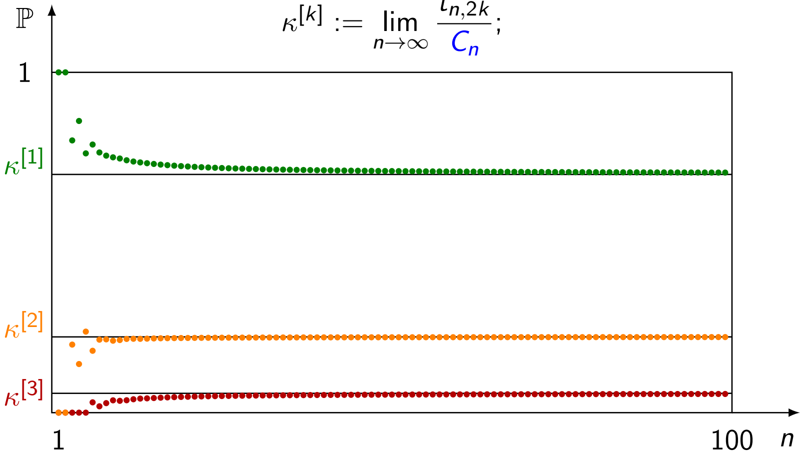
It looks like eventually

- $\frac{t_{n,2}}{C_n}$ is decreasing,
- $\frac{t_{n,2k}}{C_n}$ is increasing for $k > 1$,

and there are some limits.

# Proportions of the first three lines

$$\kappa^{[k]} := \lim_{n\to\infty} \frac{t_{n,2k}}{C_n};$$




$\kappa^{[1]} = 0.70092\ldots;$ $\kappa^{[2]} = 0.22256\ldots;$ $\kappa^{[3]} = 0.05684\ldots$

# Fixed number of anchors

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$$T(x,z) = x + \sum_{k=1}^{\infty} Q^{[k]}(z) x^{2k}$$

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- $p_h(1,z)$ GF of trees of height at most $h$,

$$e_h(z) := C(z) - p_h(1,z)$$

$$\boxed{Q_{h+1}^{[1]}(z) = 2zC(z)\left(1 - \frac{e_h(z)}{C(z)}\right) Q_h^{[1]}(z)}$$

# Scaling limits

Let

- $0 < U_1 < U_2$,
- $\tau = \sqrt{1 - 4z}$,
- $\tau \to 0^+ \quad (z \to 1/4), \quad h\tau \in [U_1, U_2]$.

In this case:

- Error:

$$\frac{e_h(z)}{\tau} = \frac{4}{e^{h\tau} - 1} + O\left(\tau \log \frac{1}{\tau}\right)$$

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- Trees with two anchors:
$$\frac{Q_h^{[1]}(z)}{\tau^2} = \frac{\kappa^{[1]}}{\sinh^2(h\tau/2)} + O\left(\tau \log \frac{1}{\tau}\right)$$

with $\kappa^{[1]} = 4 \lim_{h\to\infty} h^2 Q_h^{[1]}(1/4)$

# Asymptotics for trees with two anchors

- Singular expansion:

$$Q^{[1]}(z) = Q^{[1]}(1/4) + 2\kappa^{[1]}\sqrt{1-4z} + O\left(\sqrt{1-4z}\right),$$

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- Asymptotics:

$$t_{n,2} = [z^n]Q^{[1]}(z) \sim \frac{\kappa^{[1]}}{\sqrt{\pi}} 4^n n^{-3/2}$$

- Limit proportions:

$$\frac{t_{n,2}}{C_n} \sim \kappa^{[1]}$$

  (since $C_n \sim 4^n n^{-3/2}/\sqrt{\pi}$)

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$$\frac{t_{n,2}}{C_n} \sim \kappa^{[1]} \qquad \frac{t_{n,2k}}{C_n} \sim \kappa^{[k]} \sim 4^{1-k}$$

(since $C_n \sim 4^n n^{-3/2}/\sqrt{\pi}$)

# Column nonzero values

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
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| $t_{n,2}$ | 1 | 2 | 4 | 12 | 32 | 104 | 328 | 1 080 | 3 648 | 12 544 | 43 600 |
| $t_{n,4}$ | 0 | 0 | 1 | 2 | 10 | 24 | 92 | 308 | 1 028 | 3 584 | 12 736 |
| $t_{n,6}$ | 0 | 0 | 0 | 0 | 0 | 4 | 8 | 40 | 176 | 584 | 2 144 |
| $t_{n,8}$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 10 | 84 | 282 |
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- Define $$a_n = \max\{k \colon t_{n,2k} > 0\}$$

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|---|---|---|---|---|---|---|---|---|---|---|
| $a_n$ | 1 | 1 | 2 | 2 | 2 | 3 | 4 | 4 | 4 | 4 |
| $a_{n+10}$ | 5 | 6 | 6 | 7 | 8 | 8 | 8 | 8 | 8 | 9 |
| $a_{n+20}$ | 10 | 10 | 11 | 12 | 12 | 12 | 13 | 14 | 14 | 15 |
| $a_{n+30}$ | 16 | 16 | 16 | 16 | 16 | 16 | 17 | 18 | 18 | 19 |

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| $a_{n+10}$ | 5 | 6 | 6 | 7 | 8 | 8 | 8 | 8 | 8 | 9 |
| $a_{n+20}$ | 10 | 10 | 11 | 12 | 12 | 12 | 13 | 14 | 14 | 15 |
| $a_{n+30}$ | 16 | 16 | 16 | 16 | 16 | 16 | 17 | 18 | 18 | 19 |

- Lemma: The sequence $(a_n)$ satisfies

$$a_1 = 1, \qquad a_n = \max\{k\colon\, k \leqslant 2a_{n-k}\}$$

# One property of $(a_n)$

**Lemma**

*The sequence $(a_n)$ satisfies*

$$a_1 = 1, \qquad a_n = \max\{k\colon k \leqslant 2a_{n-k}\}$$

We have

$$t_{n,2k} = \sum_{m=\lceil k/2 \rceil}^{a_{n-k}} \binom{2m}{k} t_{n-k,2m}$$

and

$$t_{n,2k} \neq 0 \qquad \Leftrightarrow \qquad \exists\, m\colon\ k \leqslant 2m \leqslant 2a_{n-k}$$

Hence,

$$k \leqslant 2a_{n-k}$$

# Sequence of repeating elements of $(a_n)$

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $a_n$ | 1 | 1 | 2 | 2 | 2 | 3 | 4 | 4 | 4 | 4 |
| $a_{n+10}$ | 5 | 6 | 6 | 7 | 8 | 8 | 8 | 8 | 8 | 9 |
| $a_{n+20}$ | 10 | 10 | 11 | 12 | 12 | 12 | 13 | 14 | 14 | 15 |
| $a_{n+30}$ | 16 | 16 | 16 | 16 | 16 | 16 | 17 | 18 | 18 | 19 |

Define

$$\ell_n = \#\{k \colon a_k = n\}$$

# Sequence of repeating elements of $(a_n)$

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $a_n$ | 1 | 1 | 2 | 2 | 2 | 3 | 4 | 4 | 4 | 4 |
| $a_{n+10}$ | 5 | 6 | 6 | 7 | 8 | 8 | 8 | 8 | 8 | 9 |
| $a_{n+20}$ | 10 | 10 | 11 | 12 | 12 | 12 | 13 | 14 | 14 | 15 |
| $a_{n+30}$ | 16 | 16 | 16 | 16 | 16 | 16 | 17 | 18 | 18 | 19 |

Define

$$\ell_n = \#\{k \colon a_k = n\}$$

We have

$$(\ell_n) = 2,\ 3,\ 1,\ 4,\ 1,\ 2,\ 1,\ 5,\ 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1,\ 6,\ 1,\ 2,\ 1,\ \ldots$$

# Description of $(\ell_n)$

## Proposition

The sequence $(\ell_n)$ satisfies

$$\ell_n = \begin{cases} p+2 & \text{if } n = 2^p, \\ p+1 & \text{if } n = 2^p a, \ a \text{ is odd}, \ a > 1\,. \end{cases}$$

In particular,

- $\ell_{2n} = \ell_n + 1$ for even indices,
- $\ell_{2n+1} = 1$ for odd indices greater than 1,
- $\ell_1 = 2$.

Induction based on $a_n = \max\{k \colon k \leqslant 2a_{n-k}\}$

# Description of $(a_n)$

**Proposition**

The sequence $(a_n)$ satisfies

$$a_n = a_{n-1-a_{n-1}} + a_{n-2-a_{n-2}}, \qquad a_0 = a_1 = a_2 = 1$$

- Question. How to explain this recurrence combinatorially?

# Description of $(a_n)$

## Proposition

The sequence $(a_n)$ satisfies

$$a_n = a_{n-1-a_{n-1}} + a_{n-2-a_{n-2}}, \qquad a_0 = a_1 = a_2 = 1$$

- Question. How to explain this recurrence combinatorially?
- $(a_n)$ is known as a meta-Fibonacci sequence

## Corollary (Tanny, 1992)

- $\lim_{n\to\infty} \frac{a_n}{n} = \frac{1}{2}$
- $\sum_{n=0}^{\infty} a_n z^n = z \sum_{n=0}^{\infty} \prod_{i=1}^{n} (z + z^{2^i})$

# Trees of fixed height

$$t_{n,2k,h} = \#\{\text{active trees with } n \text{ internal nodes } 2k \text{ anchors of height } h\}$$

| $k$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 3 | 0 | 0 | 0 | 0 | 0 | 4 | 0 | 0 |
| 2 | 0 | 0 | 0 | 2 | 6 | 0 | 0 | 0 |
| 1 | 0 | 0 | 4 | 4 | 0 | 0 | 0 | 0 |
| $h = 3$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

$n$

# Trees of fixed height

$t_{n,2k,h} = \#\{\text{active trees with } n \text{ internal nodes } 2k \text{ anchors of height } h\}$

| $k$ | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 28 | 0 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 24 | 56 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 6 | 60 | 70 | 0 | 0 | 0 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 24 | 80 | 56 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 | 4 | 16 | 36 | 60 | 28 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0 | 0 | 0 | 8 | 16 | 24 | 24 | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $h = 4$ | | 2 | | 4 | | 6 | | 8 | | 10 | | 12 | | 14 | | 16 |

$n$

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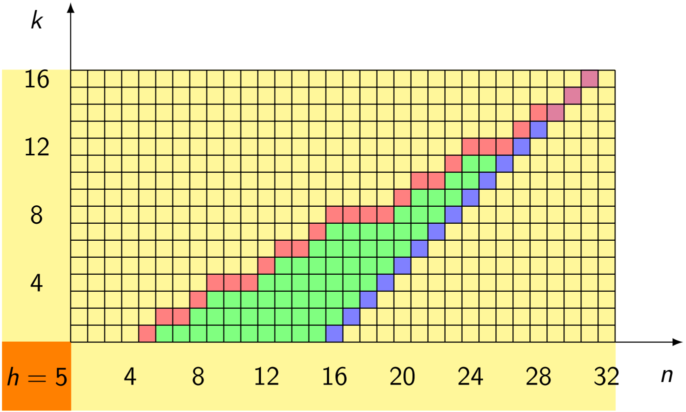


$S_h = \{(n,k) \colon t_{n,2k,h} \neq 0\},$ $|S_h| = 2^{h-2}(2^{h-1} - h + 2)$

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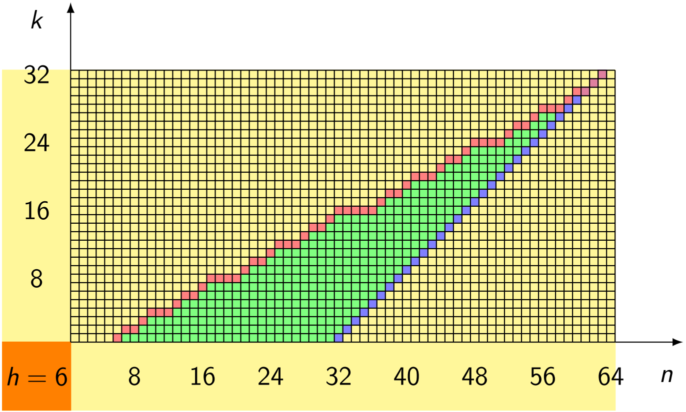


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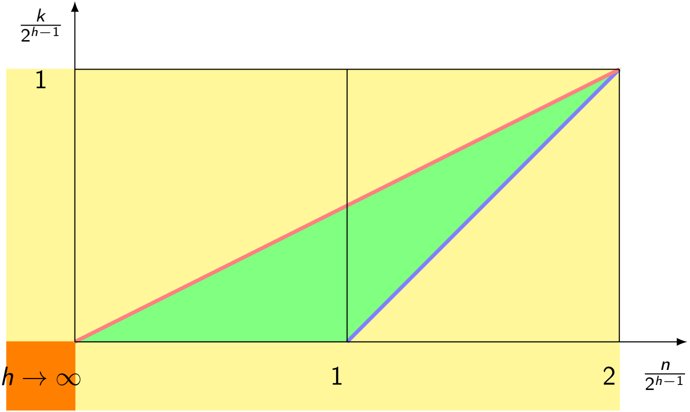


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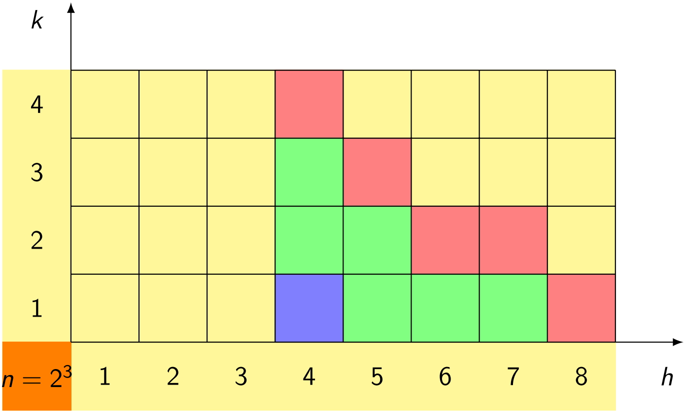
# Trees of fixed size

$t_{n,2k,h} = \#\{$active trees with $n$ internal nodes $2k$ anchors of height $h\}$




$\widehat{S}_n = \{(h,k) \colon t_{n,2k,h} \neq 0\},$ $|\widehat{S}_{2^m}| = 2^{m-2}(2^m - m + 1)$

# Trees of fixed size

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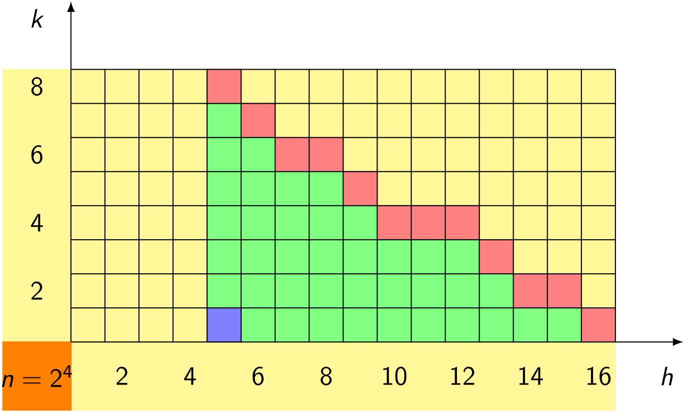


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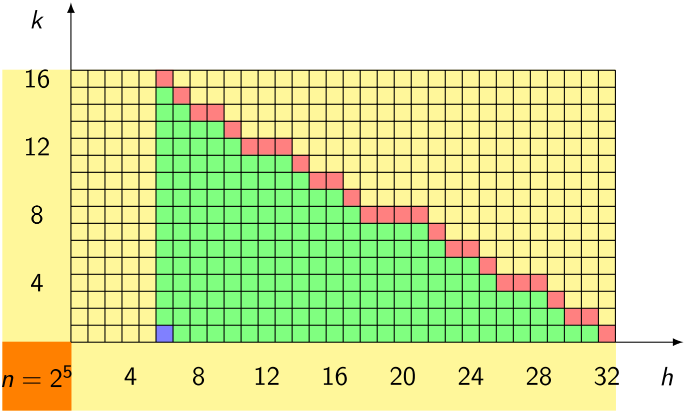


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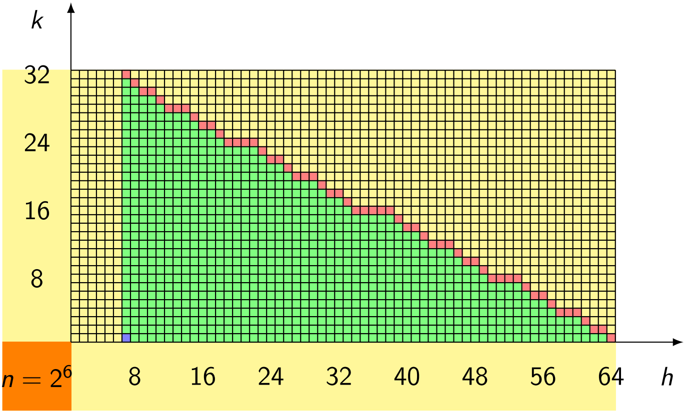


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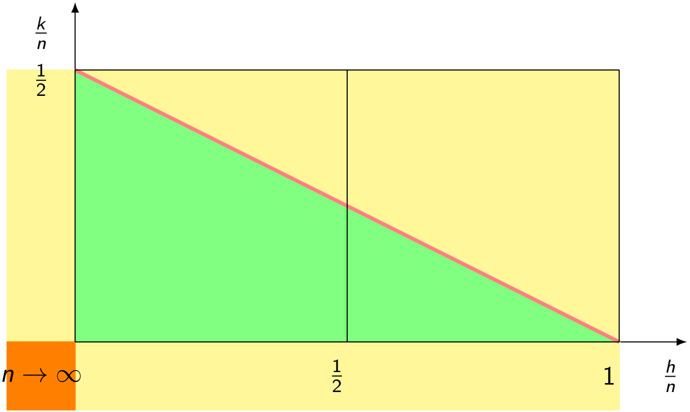


$\widehat{S}_n = \{(h,k)\colon\, t_{n,2k,h} \neq 0\}, \qquad |\widehat{S}_{2^m}| = 2^{m-2}(2^m - m + 1)$

# Conclusion

1. Studied objects:
   - growing binary trees.

2. Related objects:
   - Mandelbrot polynomials,
   - meta-Fibonacci sequences.

3. Results:
   - relations for generating functions,
   - limits for anchor distributions,
   - behavior of the maximal number of anchors,
   - limit forms of nonzero domains.

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   - relations for generating functions,
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Thank you for your attention!

# Literature I


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