Asymptotiques des motifs consécutifs dans les permutations et les couplages parfaits

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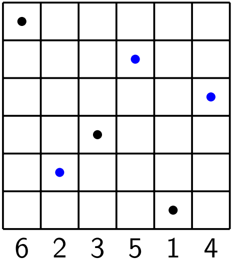
Part I

Patterns in permutations

Patterns in permutations

Let $\tau = \tau_1 \ldots \tau_p$ and $\sigma = \sigma_1 \ldots \sigma_n$ be two permutations, $p < n$.

- τ **occurs in** σ if $\exists\ i_1 < \ldots < i_p : \quad \sigma_{i_j} < \sigma_{i_s} \Leftrightarrow \tau_j < \tau_s$



<u>Example:</u> $p = 3, \quad \tau = 132, \qquad n = 6, \quad \sigma = 623514.$

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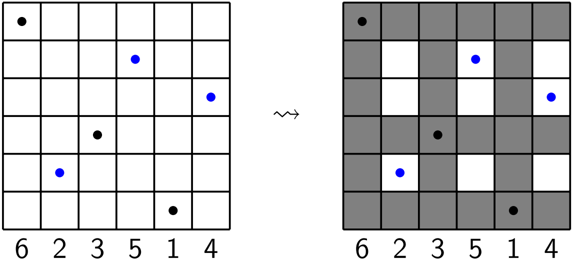

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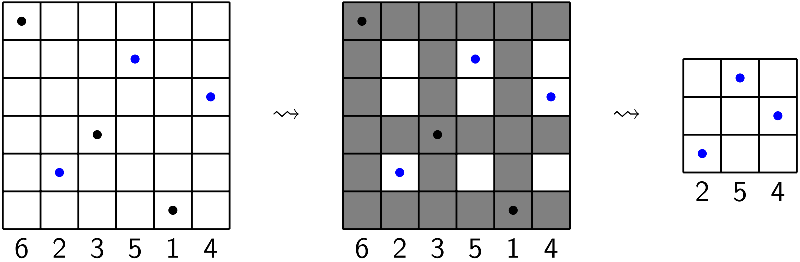

Example: $\quad p = 3, \quad \tau = 132, \qquad n = 6, \quad \sigma = 623514.$

Tight patterns in permutations

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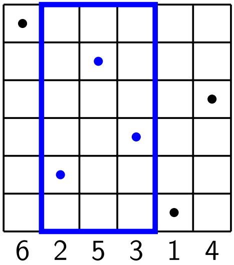

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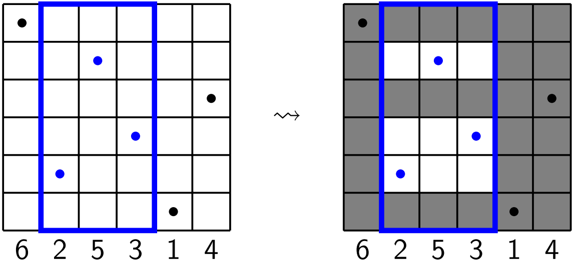

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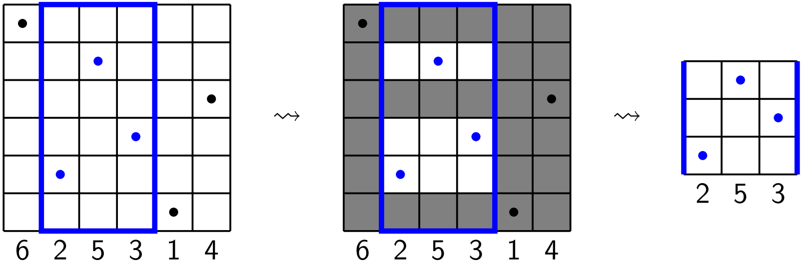

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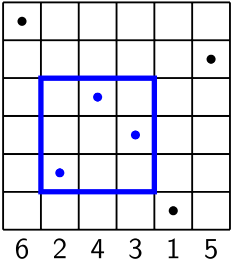
Very tight patterns in permutations

Let $\tau = \tau_1 \dots \tau_p$ and $\sigma = \sigma_1 \dots \sigma_n$ be two permutations, $p < n$.

- τ **very tightly occurs in** σ if $\quad \exists\, i, h \; \forall\, j : \quad \sigma_{i+j} = \tau_j + h$



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In this talk: very tight patterns only.

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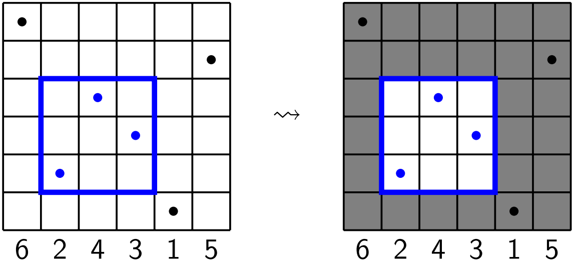

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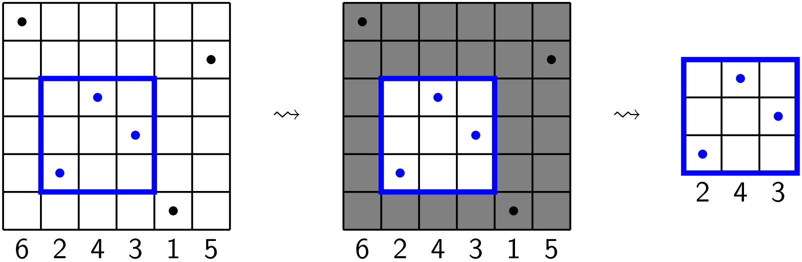

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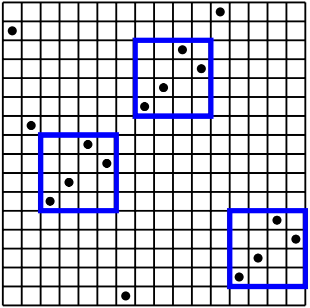
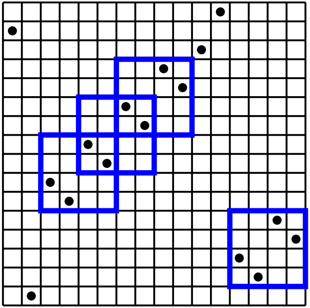
Enumeration methods for very tight patterns

There are two cases.

1. Patterns cannot overlap $\rightsquigarrow$ inclusion-exclusion principle.
2. Patterns can overlap $\rightsquigarrow$ cluster method.



Case 1: $\tau = 1243$

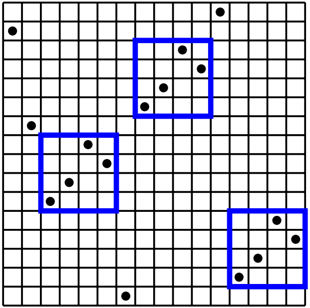


Case 2: $\tau = 2143$

Enumeration of patterns that cannot overlap

- Let $\tau \in S_p$ be a pattern that cannot overlap.

- Let $a_{n,k}(\tau)$ be the number of permutations $\sigma \in S_n$ with k very tight occurrences of τ.



- Theorem (Myers, 2002):

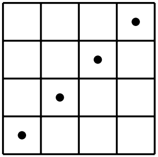
$$a_{n,k}(\tau) = \sum_{i=k}^{\lfloor n/(p-1)) \rfloor} (-1)^{i-k} \binom{i}{k} \binom{n-(p-1)i}{i} (n-(p-1)i)!$$

- Proof: the inclusion-exclusion principle.

Autocorrelation polynomials of patterns in permutations

Autocorrelation polynomial of $\tau \in S_p$ is $A_\tau(z) = \sum_{j=0}^{p-1} c_j z^j$,

$$c_j = \begin{cases} 1 & \text{if } \tau \text{ matchs itself after shifting by } j \text{ positions} \\ 0 & \text{otherwise} \end{cases}$$



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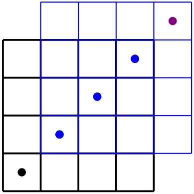

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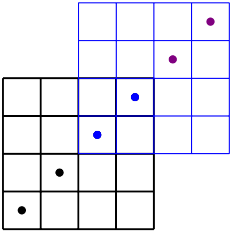

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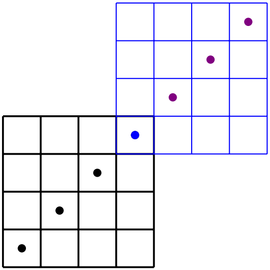

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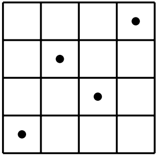

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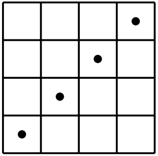

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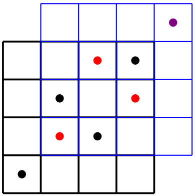

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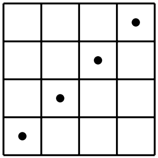

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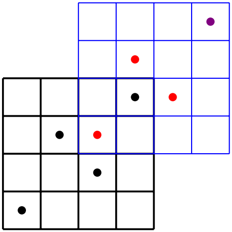

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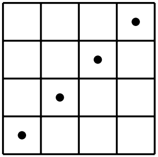

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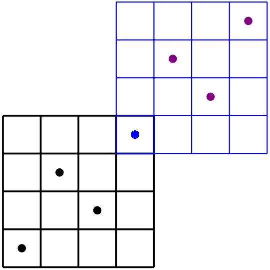

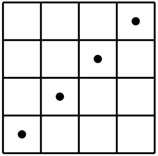

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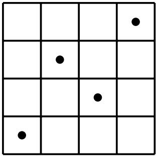

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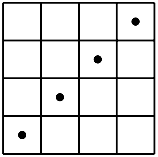

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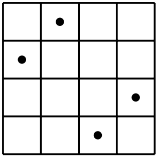

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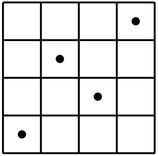

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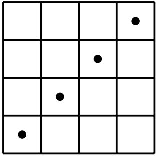

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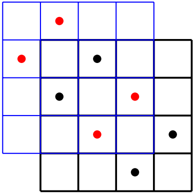

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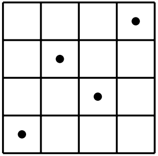

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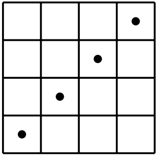

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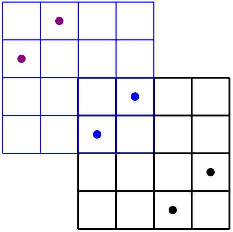

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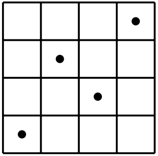

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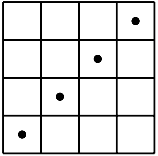

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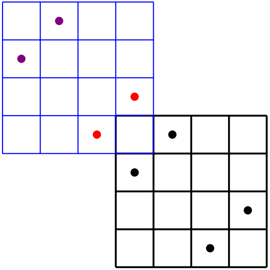

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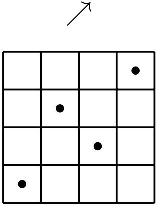
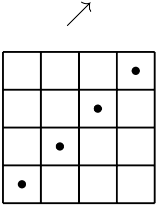
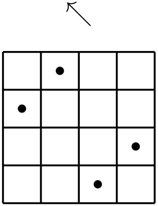
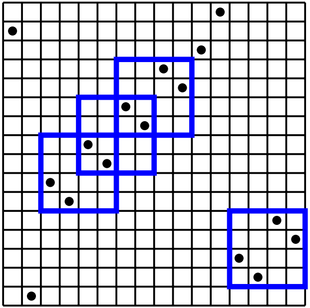
$\tau_1 < \tau_p$	$\tau_1 < \tau_p$	$\tau_1 > \tau_p$
		
$A_{1324}(z) = 1 + z^3$	$A_{1234}(z) = 1 + z + z^2 + z^3$	$A_{3412}(z) = 1 + z^2$

Enumeration of patterns that can overlap

- Let $\tau \in S_p$ be a pattern that can overlap.

- Let $a_{n,k}(\tau)$ be the number of permutations $\sigma \in S_n$ with k very tight occurrences of τ.



- Theorem (Claesson, 2022):

$$\sum_{n,k\geqslant 0} a_{n,k}(\tau) z^n u^k = \sum_{n=0}^{\infty} n! \left(z + \frac{(u-1)z^p}{1-(u-1)\big(A_\tau(z)-1\big)} \right)^n$$

- Proof: the cluster method of Goulden and Jackson.

Asymptotics for $a_{n,k}(12)$

- Asymptotics (Bóna, 2007):

$$\frac{a_{n,k}(12)}{n!} \sim \frac{e^{-1}}{k!}$$

  as $n \to \infty$.

- This is a Poisson distribution $\mathrm{Pois}(1)$ with parameter $\lambda = 1$.

Asymptotics for $a_{n,k}(\tau)$ with $A_\tau(z) = 1$, $p > 2$

- Suppose that $\tau \in S_p$ cannot overlap, i.e. $A_\tau(z) = 1$.
- Generating function of permutations:

$$P(z) = \sum_{n=0}^{\infty} n!\, z^n$$

- Generating function (Claesson, 2022):

$$\sum_{n,k \geqslant 0} a_{n,k}(\tau)\, z^n u^k = P\Big(z + (u-1)z^p\Big)$$

- Asymptotics (Kirgizov, N., 2024+):

$$\frac{a_{n,k}(\tau)}{n!} \sim \frac{1}{k!} \cdot \frac{1}{n^{k(p-2)}}$$

as $n \to \infty$.

Asymptotics for $a_{n,k}(\tau)$ with $A_\tau(z) \neq 1$, $p > 2$

- Suppose that $\tau \in S_p$ can overlap, $A_\tau(z) = 1 + z^m + \dots$
- Generating function (Claesson, 2022):

$$\sum_{n,k \geqslant 0} a_{n,k}(\tau)\, z^n u^k = P\left(z + \frac{(u-1)z^p}{1-(u-1)(A_\tau(z)-1)}\right)$$

- Asymptotics (Kirgizov, N., 2024+): as $n \to \infty$,

$$\frac{a_{n,k}(\tau)}{n!} \sim \begin{cases} \dfrac{1}{k!} \cdot \dfrac{1}{n^{k(p-2)}} & \text{if} \quad m = p-1 \\ \dfrac{1}{n^{k(p-2)}} \cdot \displaystyle\sum_{s=1}^{k} \frac{1}{s!}\binom{k-1}{s-1} & \text{if} \quad m = p-2 \\ \dfrac{1}{n^{km+(p-2-m)}} & \text{if} \quad m < p-2 \end{cases}$$

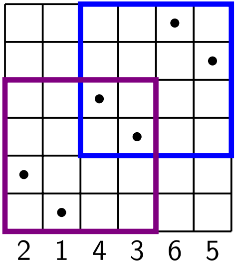
Interlude

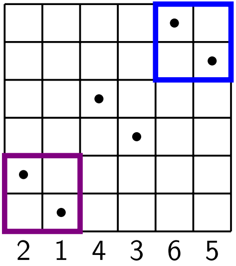
Self-overlapping permutations

Self-overlapping permutations

Permutation $\sigma \in S_n$ is **self-overlapping** if there is $k < n$:

1. $\{1, \ldots, k\}$ is invariant under σ,
2. $\{n-k+1, \ldots, n\}$ is invariant under σ,
3. $\sigma_1 \ldots \sigma_k$ and $\sigma_{n-k+1} \ldots \sigma_n$ are isomorphic.





It is always possible to choose $k \leqslant n/2$.

Structure of self-overlapping permutations

- Let $\sigma \in S_n$ and $\sigma_1 < \sigma_n$.

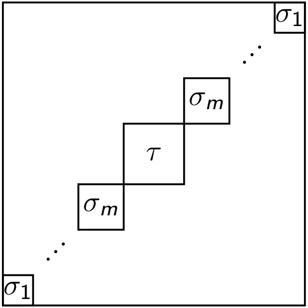
  Then σ is non-self-overlapping iff $A_\sigma(z) = 1$.

- Every permutation $\sigma \in S_n$ can be decomposed as

$$\sigma = \sigma_1 \oplus \ldots \oplus \sigma_m \oplus \tau \oplus \sigma_m \oplus \ldots \oplus \sigma_1$$

where

  - σ_i are non-self-overlapping,
  - τ is empty or non-self-overlapping.



Asymptotics of non-self-overlapping permutations

Generating functions (Kirgizov, N., 2023+):

$$P(z) = \frac{1 + N(z)}{1 - N(z^2)}\,,$$

where

- $P(z)$ is the OGF of permutations,
- $N(z)$ is the OGF of non-self-overlapping permutations.

Asymptotics (Kirgizov, N., 2023+):

$$\mathbb{P}(\sigma \text{ is non-self-overlapping}) = 1 - \sum_{k=1}^{r-1} \frac{\mathfrak{no}_k}{(n)_{2k}} + O\left(\frac{1}{n^{2r}}\right)\,,$$

where

- $\mathfrak{no}_k = \#\{\text{non-self-overlapping permutations of size } k\}$,
- $(n)_k = n(n-1)\ldots(n-k+1)$ are falling factorials.

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where

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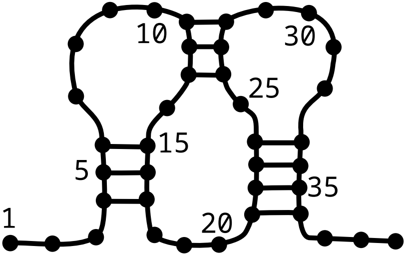
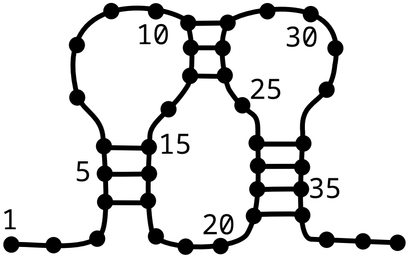
Part II

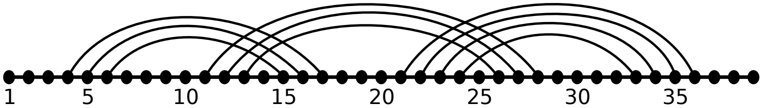
Patterns in matchings

RNA secondary structures and matchings



RNA secondary structures and matchings

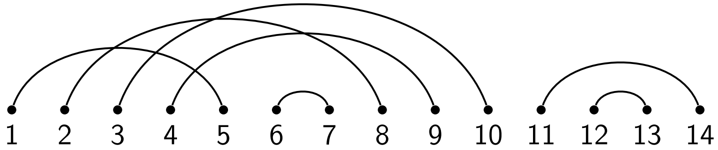




Matchings

- A (perfect) **matching** is an involution without fixed points.

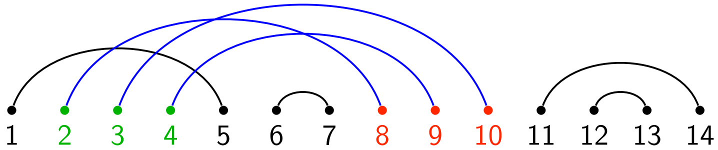
- A matching of size n consists of $2n$ points and n arcs:



- There are $(2n-1)!!$ matchings of size n.

Endhered patterns

- **Endhered pattern** in a matching:
  - starting points form an interval,
  - ending points form an interval.



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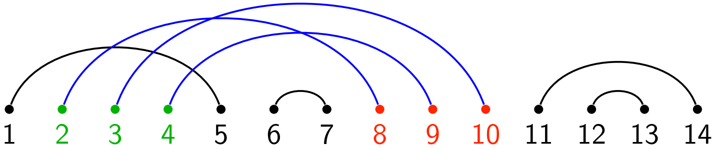
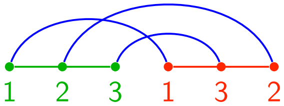


- Endhered patterns are encoded by permutations:

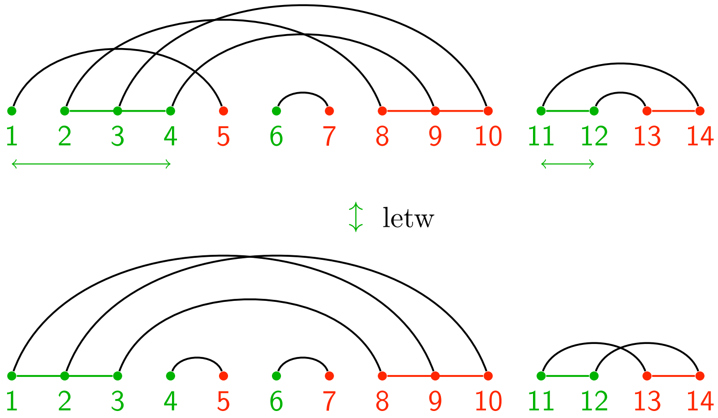


$\leftrightsquigarrow$ $\tau = 1\,3\,2$

- $b_{n,k}(\tau) = \#\{\text{matchings of size } n \text{ with } k \text{ patterns } \tau\}$.

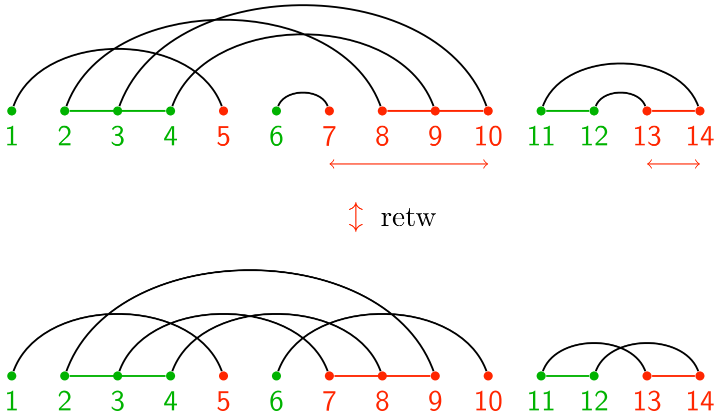
Endhered twists

Left endhered twist: reverse all runs of consecutive left points.

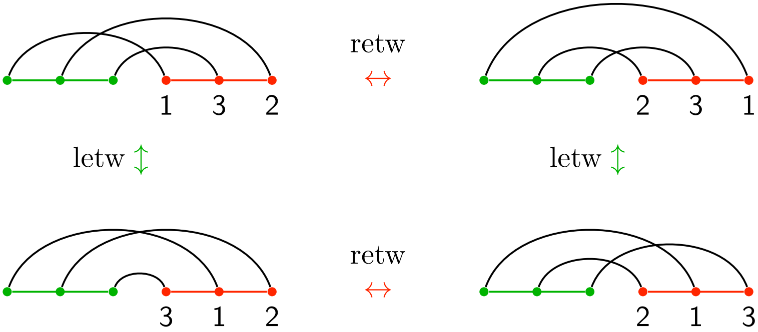


Endhered twists

Right endhered twist: reverse all runs of consecutive right points.



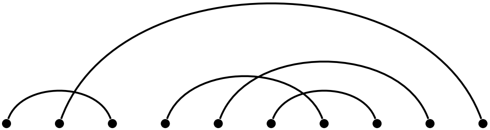
(Wilf) equivalent patterns



- Left twist: relabeling $1, \dots, p \ \rightarrow\ p, \dots, 1$ in a pattern.
- Right twist: reversing a pattern.
- $b_{n,k}(\tau) = b_{n,k}(\mathrm{letw}(\tau)) = b_{n,k}(\mathrm{retw}(\tau))$.

Pattern $\tau = 21$, recurrences

- Generating: $b_{n+1,k} =$



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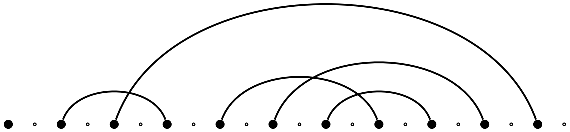
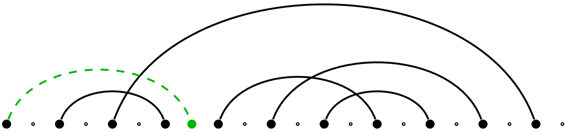


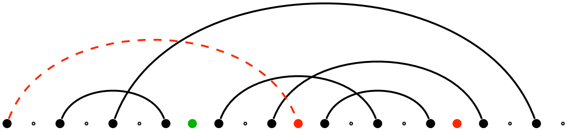
Pattern $\tau = 21$, recurrences

- Generating: $b_{n+1,k} = b_{n,k-1} +$



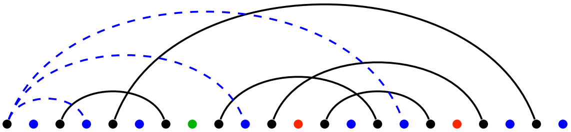
Pattern $\tau = 21$, recurrences

- Generating: $b_{n+1,k} = b_{n,k-1} + \qquad + 2(k+1)b_{n,k+1}$



Pattern $\tau = 21$, recurrences

- Generating: $b_{n+1,k} = b_{n,k-1} + 2(n-k)b_{n,k} + 2(k+1)b_{n,k+1}$



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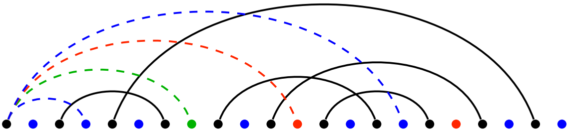
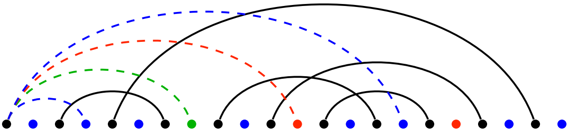


- Insertion:

$$b_{n+1,k} = \binom{n}{k} b_{n-k+1,0}$$

Pattern $\tau = 21$, recurrences

- Generating: $b_{n+1,k} = b_{n,k-1} + 2(n-k)b_{n,k} + 2(k+1)b_{n,k+1}$



- Insertion:

$$b_{n+1,k} = \binom{n}{k} b_{n-k+1,0}$$

- Inclusion-exclusion:

$$b_{n+1,0} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (2k+1)!!$$

Pattern $\tau = 21$, generating function and asymptotics

- Generating function:

$$B(z,u) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} b_{n,k} \frac{z^n}{n!} u^k$$

- Exact form:

$$\frac{\partial B}{\partial z}(z,u) = \frac{e^{z(u-1)}}{\sqrt{(1-2z)^3}}$$

- Asymptotics: as $n \to \infty$,

$$\frac{b_{n,k}}{(2n-1)!!} \sim \frac{e^{-1/2}}{2^k k!}$$

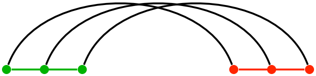
(Poisson distribution $\mathrm{Pois}(1/2)$ with parameter $\lambda = 1/2$)

Autocorrelation polynomials of patterns in matchings

Autocorrelation polynomial of $\tau \in S_p$ is $\quad A_\tau(z) = \sum_{j=0}^{|\tau|-1} c_j z^j,$

$$c_j = \begin{cases} 1 & \text{if } \tau \text{ matchs itself after shifting right by } j \text{ positions} \\ 0 & \text{otherwise} \end{cases}$$

(we suppose that $\tau_1 < \tau_p$)



$$A_{123}(z) = 1 +$$

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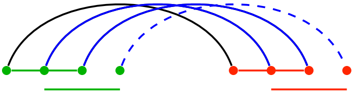

$$A_{123}(z) = 1 + z +$$

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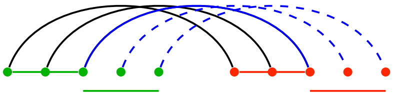

$$A_{123}(z) = 1 + z + z^2$$

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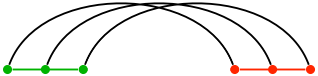

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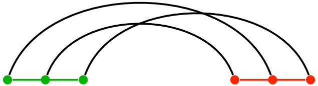

$$A_{213}(z) = 1 +$$

Autocorrelation polynomials of patterns in matchings

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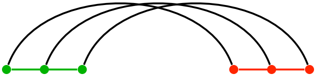

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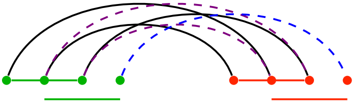

$$A_{213}(z) = 1 +$$

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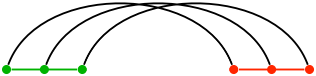

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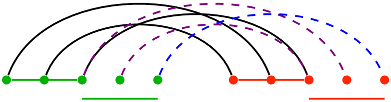

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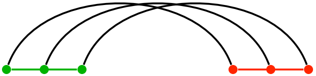
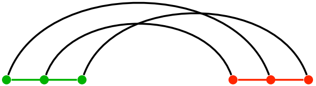
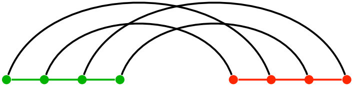


$$A_{123}(z) = 1 + z + z^2$$



$$A_{213}(z) = 1$$



$$A_{2143}(z) = 1 + z^2$$

Asymptotics for $b_{n,k}(\tau)$ with $A_\tau(z) = 1$, $p > 2$

- Let $\tau \in S_p$ be a non-self-overlapping pattern, i.e. $A_\tau(z) = 1$.
- Generating function of matchings:

$$M(z) = \sum_{n=0}^{\infty} (2n-1)!!\, z^n$$

- Generating function (Kirgizov, N., 2023+):

$$\sum_{n,k \geqslant 0} b_{n,k}(\tau)\, z^n u^k = M\Big(z + (u-1)z^p\Big)$$

- Asymptotics (Kirgizov, N., 2023+):

$$\frac{b_{n,k}(\tau)}{(2n-1)!!} \sim \frac{1}{k!\, 2^{k(p-1)}} \cdot \frac{1}{n^{k(p-2)}}$$

as $n \to \infty$.

Asymptotics for $b_{n,k}(\tau)$ with $A_\tau(z) \neq 1$, $p > 2$

- Let $\tau \in S_p$ be self-overlapping, $A_\tau(z) = 1 + z^m + \dots$
- Generating function (Kirgizov, N., 2023+):

$$\sum_{n,k \geqslant 0} b_{n,k}(\tau)\, z^n u^k = M\left(z + \frac{(u-1)z^p}{1-(u-1)(A_\tau(z)-1)}\right)$$

- Asymptotics (Kirgizov, N., 2023+): as $n \to \infty$,

$$\frac{b_{n,k}(\tau)}{(2n-1)!!} \sim \begin{cases} \dfrac{1}{k!\,2^{k(p-1)}} \cdot \dfrac{1}{n^{k(p-2)}} & \text{if} \quad m = p-1 \\ \dfrac{1}{(2n)^{k(p-2)}} \displaystyle\sum_{s=1}^{k} \frac{1}{s!\,2^s} \binom{k-1}{s-1} & \text{if} \quad m = p-2 \\ \dfrac{1}{2(2n)^{km+(p-2-m)}} & \text{if} \quad m < p-2 \end{cases}$$

Part III

Ideas of proofs

Asymptotics of factorially divergent series (Borinsky)

$$a_n = \alpha^{n+\beta}\Gamma(n+\beta)\left(c_0 + \frac{c_1}{\alpha(n+\beta-1)} + \frac{c_2}{\alpha^2(n+\beta-1)(n+\beta-2)} + \dots\right)$$

$$\sum_{n=0}^{\infty} a_n z^n \quad \xmapsto{\mathcal{A}_\beta^\alpha} \quad \sum_{n=0}^{\infty} c_n z^n$$

Properties:

- $\big(\mathcal{A}_\beta^\alpha(A \cdot B)\big)(z) = A(z) \cdot (\mathcal{A}_\beta^\alpha B)(z) + B(z) \cdot (\mathcal{A}_\beta^\alpha A)(z),$
- $$\big(\mathcal{A}_\beta^\alpha(A \circ B)\big)(z) = A'\big(B(z)\big) \cdot (\mathcal{A}_\beta^\alpha B)(z) + \left(\frac{z}{B(z)}\right)^\beta \exp\left(\frac{1}{\alpha}\left(\frac{1}{z} - \frac{1}{B(z)}\right)\right)(\mathcal{A}_\beta^\alpha A)\big(B(z)\big).$$

Extracting asymptotics for permutation patterns

- $P(z) = \sum_{n=0}^{\infty} n!\, z^n \quad \Rightarrow \quad \left(\mathcal{A}_1^1 P\right)(z) = 1$

- $G(z) = z + \dfrac{(u-1)z^p}{1-(u-1)(A_\tau(z)-1)} \quad \Rightarrow \quad \left(\mathcal{A}_1^1 G\right)(z) = 0$

- Composition:

$$\left(\mathcal{A}_1^1 (P \circ G)\right)(z) = \frac{1-(u-1)z^{p-1}}{1-(u-1)(A_\tau(z)-1-z^{p-1})} \times \exp\left(\frac{(u-1)z^{p-2}}{1-(u-1)(A_\tau(z)-1-z^{p-1})}\right)$$

Extracting asymptotics for matching patterns

- $M(z) = \sum_{n=0}^{\infty} (2n-1)!!\, z^n \qquad \Rightarrow \qquad \left(\mathcal{A}^2_{1/2} M\right)(z) = \frac{1}{\sqrt{2\pi}}$

- $G(z) = z + \frac{(u-1)z^p}{1-(u-1)(A_\tau(z)-1)} \quad \Rightarrow \quad \left(\mathcal{A}^2_{1/2} G\right)(z) = 0$

- Composition:

$$\left(\mathcal{A}^2_{1/2}(M \circ G)\right)(z) = \frac{1}{\sqrt{2\pi}} \left(1 + \frac{(u-1)z^{p-1}}{1-(u-1)(A_\tau(z)-1)}\right)^{-1/2} \times \exp\left(\frac{(u-1)z^{p-2}}{2\left(1-(u-1)(A_\tau(z)-1-z^{p-1})\right)}\right)$$

Conclusion

1. Studied objects:
   - consecutive patterns in permutations and matchings,
   - self-overlapping permutations.

2. Tools:
   - the symbolic method,
   - singularity analysis,
   - Goulden-Jackson cluster method,
   - Borinsky's approach.

3. Results:
   - asymptotics for any very tight pattern in permutations,
   - enumeration and asymptotics for any endhered pattern,
   - enumeration and asymptotics of non-self-overlapping permutations.

Thank you for your attention!