

Interprétation combinatoire des coefficients dans les développement asymptotiques

Khaydar Nurligareev

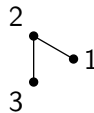
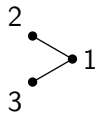
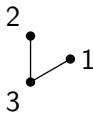
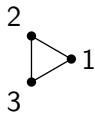
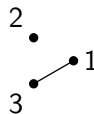
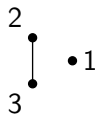
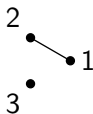
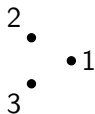
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9 avril, 2024

Simple labeled graphs

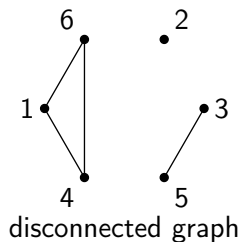
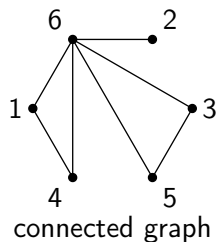
- $\mathfrak{g}_n = \#\{\text{graphs with } n \text{ vertices}\}$



$$\mathfrak{g}_n = 2^{\binom{n}{2}}$$

Connected graphs

- $cg_n = \#\{\text{connected graphs with } n \text{ vertices}\}$



$$(cg_n) = 1, 1, 4, 38, 728, 26704, 1866256, \dots$$

- Every graph is a disjoint union (SET) of connected graphs.

Probability of a graph to be connected

Question. What is the probability $p_n = \frac{c\mathfrak{g}_n}{\mathfrak{g}_n}$ that a random graph with n vertices is connected, as $n \rightarrow \infty$?

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1 folklore:

$$p_n = 1 + o(1)$$

2 Gilbert, 1959:

$$p_n = 1 - \frac{2n}{2^n} + O\left(\frac{n^2}{2^{3n/2}}\right)$$

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3 Wright, 1970:

$$p_n = 1 - \binom{n}{1} \frac{2}{2^n} - 2 \binom{n}{3} \frac{2^6}{2^{3n}} - 24 \binom{n}{4} \frac{2^{10}}{2^{4n}} + O\left(\frac{n^5}{2^{5n}}\right)$$

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4 Can we see the structure? What is the interpretation?

Asymptotics for p_n

Theorem (Monteil, N., 2021)

For every $r \geq 1$, the probability p_n that a random labeled graph of size n is connected satisfies

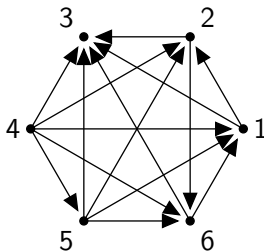
$$p_n = 1 - \sum_{k=1}^{r-1} \text{it}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where it_k is the number of *irreducible labeled tournaments* of size k .

$$(\text{it}_k) = 1, 0, 2, 24, 544, 22320, 1677488, \dots$$

Tournaments

A **tournament** is a complete directed graph.



The number of labeled tournaments with n vertices is

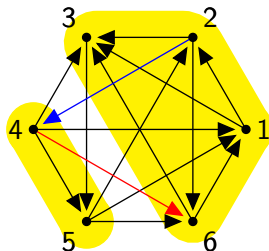
$$t_n = 2^{\binom{n}{2}}$$

Irreducible tournaments

A tournament is **irreducible**, if for every partition of vertices $V = A \sqcup B$

- 1 there exist an **edge from A to B** ,
- 2 there exist an **edge from B to A** .

$$V = \{1, 2, 3, 4, 5, 6\}$$



$$A = \{4, 5\}$$

$$B = \{1, 2, 3, 6\}$$

Irreducible tournaments

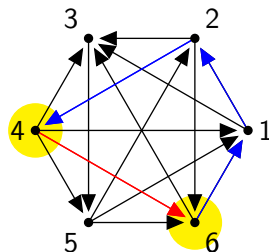
A tournament is **irreducible**, if for every partition of vertices $V = A \sqcup B$

- 1 there exist an **edge from A to B** ,
- 2 there exist an **edge from B to A** .

Equivalently, a tournament is **strongly connected**: for each two vertices u and v

- 1 there is a **path from u to v** ,
- 2 there is a **path from v to u** .

$$V = \{1, 2, 3, 4, 5, 6\}$$



$$u = 4$$

$$v = 6$$

Exponential generating functions and Bender's theorem

$$\text{EGF: } G(z) = \sum_{n=0}^{\infty} g_n \frac{z^n}{n!}$$

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Bender, 1975:

$$\mathbf{1} \quad A(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_n \neq 0$$

$$\mathbf{2} \quad F(x, y) \text{ is analytic in } U(0; 0)$$

$$\mathbf{3} \quad B(z) = \sum_{n=0}^{\infty} b_n z^n = F(z, A(z))$$

$$\mathbf{4} \quad C(z) = \sum_{n=0}^{\infty} c_n z^n = \left[\frac{\partial F}{\partial y}(z, y) \right]_{y=A(z)}$$

$$\mathbf{5} \quad \frac{a_{n-1}}{a_n} \rightarrow 0, \quad \text{as } n \rightarrow \infty$$

$$\mathbf{6} \quad \exists r \geq 1 : \sum_{k=r}^{n-r} |a_k a_{n-k}| = O(a_{n-r})$$

$$\text{Then } b_n = \sum_{k=0}^{r-1} c_k a_{n-k} + O(a_{n-r}).$$

Exponential generating functions and Bender's theorem

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$\mathbf{5}$ the sequence (a_n) is gargantuan

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$$\text{Then } b_n \approx \sum_{k \geq 0} c_k a_{n-k}.$$

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Exponential generating functions and Bender's theorem

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$$\frac{\partial F}{\partial y} = \frac{1}{y}$$

$$\frac{1}{1-y} = 1 + y + y^2 + \dots$$

$$\frac{cg_n}{g_n} \approx 1 - \sum_{k \geq 1} \text{it}_k \binom{n}{k} \frac{g_{n-k}}{g_n}$$

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$$G(z) = T(z) = \frac{1}{1 - IT(z)}$$

$$\frac{cg_n}{g_n} \approx 1 - \sum_{k \geq 1} it_k \binom{n}{k} \frac{g_{n-k}}{g_n}$$

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Exponential generating functions and Bender's theorem

$$1 \quad CG(z) = \log G(z)$$

$$2 \quad A(z) = G(z) - 1$$

$$3 \quad F(x, y) = \log(1 + y)$$

$$4 \quad \frac{\partial F}{\partial y} = \frac{1}{1 + y}$$

$$5 \quad C(z) = \frac{1}{G(z)} = \frac{1}{T(z)}$$

$$6 \quad \frac{1}{T(z)} = 1 - IT(z)$$

$$7 \quad \frac{cg_n}{g_n} \approx 1 - \sum_{k \geq 1} it_k \binom{n}{k} \frac{g_{n-k}}{g_n}$$

Bender, 1975:

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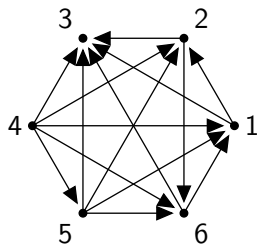
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$$b_n \approx \sum_{k \geq 0} c_k a_{n-k}.$$

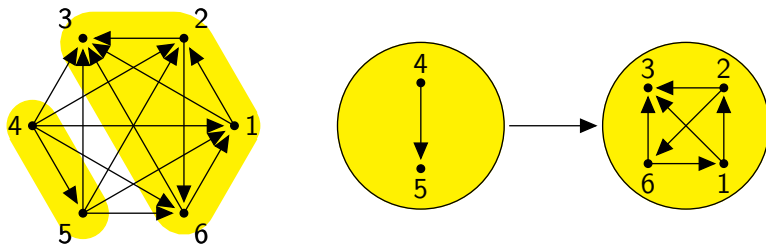
Tournament as a sequence

Folklore: Every labeled tournament can be uniquely decomposed into a sequence (SEQ) of irreducible labeled tournaments.



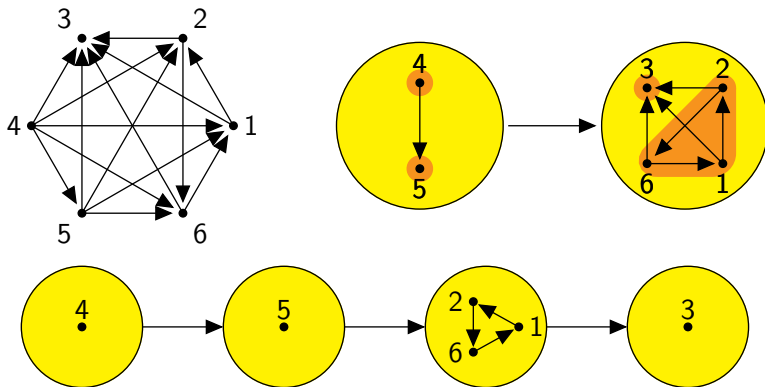
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Tournament as a sequence

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SET asymptotics

Theorem (Monteil, N., 2022)

If \mathcal{U} , \mathcal{V} and \mathcal{W} are such combinatorial classes that

- 1 \mathcal{U} is gargantuan with positive counting sequence,
- 2 $\mathcal{U} = \text{SET}(\mathcal{V})$ and $\mathcal{U} = \text{SEQ}(\mathcal{W})$,

then

$$p_n := \frac{v_n}{u_n} \approx 1 - \sum_{k \geq 1} w_k \cdot \binom{n}{k} \cdot \frac{u_{n-k}}{u_n}.$$

Combinatorial meaning: p_n is the probability that a random object of size n from \mathcal{U} is connected.

Double decomposition of permutations

$$P(z) = \sum_{n=0}^{\infty} n! \frac{z^n}{n!} = \frac{1}{1-z}$$

Double decomposition of permutations

permutations

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 5 & 4 & 1 & 2 \end{pmatrix}$$

counts



$$P(z) = \sum_{n=0}^{\infty} n! \frac{z^n}{n!} = \frac{1}{1-z} = L(z)$$

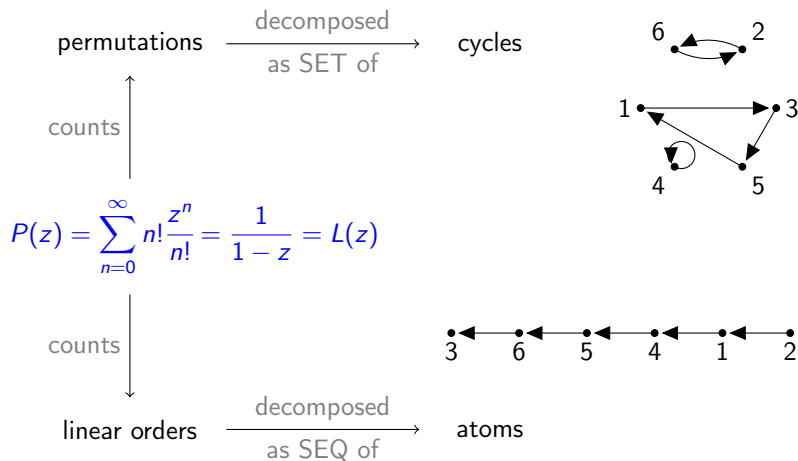
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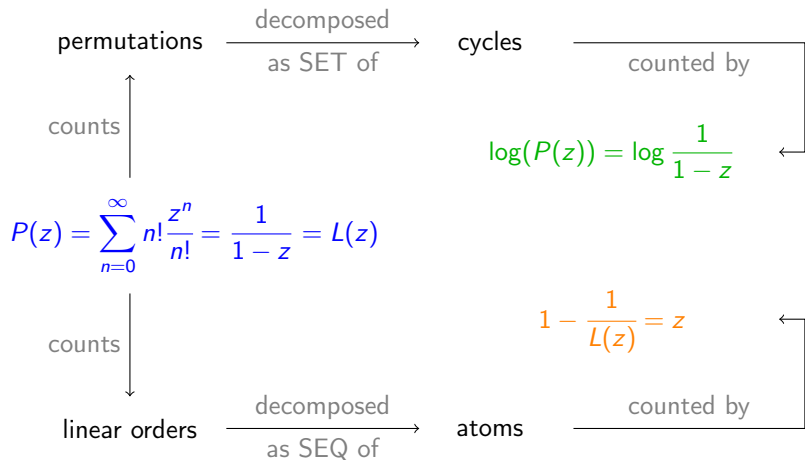
linear orders

$$(3 < 6 < 5 < 4 < 1 < 2)$$

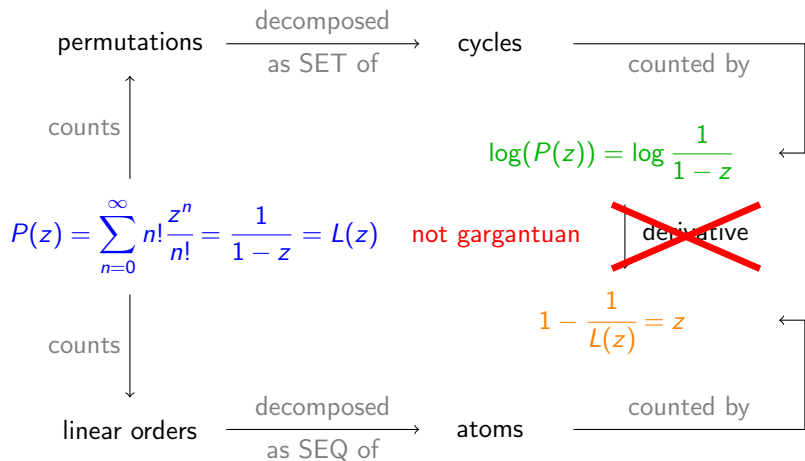
Double decomposition of permutations



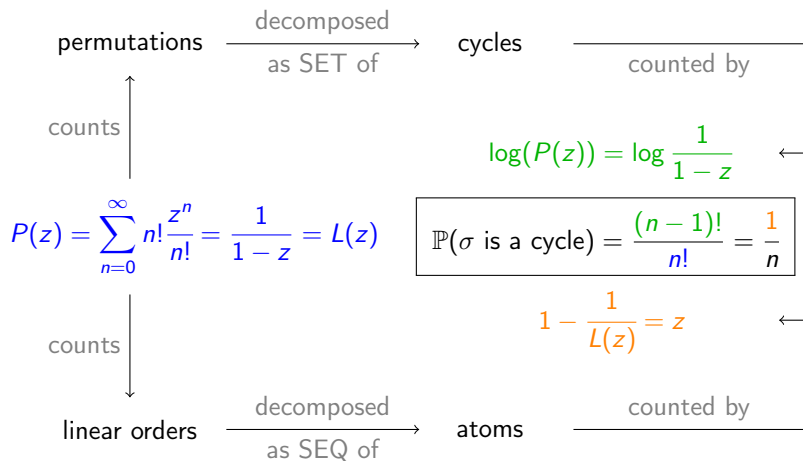
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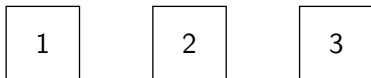
Double decomposition of permutations



Square-tiled surfaces

To obtain a **square-tiled surface** (determined by $(h, v) \in S_n^2$):

- 1 take n labeled squares,

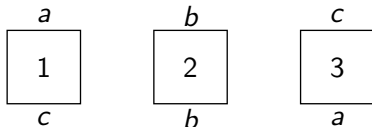


Square-tiled surfaces

To obtain a **square-tiled surface** (determined by $(h, \nu) \in S_n^2$):

- 1 take n labeled squares,
- 2 identify horizontal sides (corresponds to $h \in S_n$),

$$h = (13)(2)$$



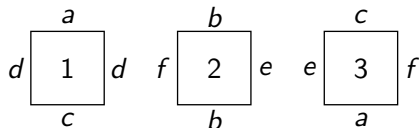
Square-tiled surfaces

To obtain a **square-tiled surface** (determined by $(h, \nu) \in S_n^2$):

- 1 take n labeled squares,
- 2 identify horizontal sides (corresponds to $h \in S_n$),
- 3 identify vertical sides (corresponds to $\nu \in S_n$),

$$h = (13)(2)$$

$$\nu = (1)(23)$$



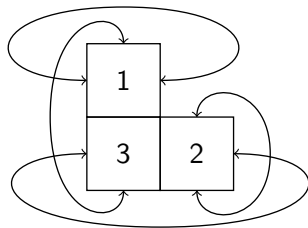
Square-tiled surfaces

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- 1 take n labeled squares,
- 2 identify horizontal sides (corresponds to $h \in S_n$),
- 3 identify vertical sides (corresponds to $v \in S_n$),
- 4 glue together identified sides.

$$h = (13)(2)$$

$$v = (1)(23)$$



Square-tiled surfaces

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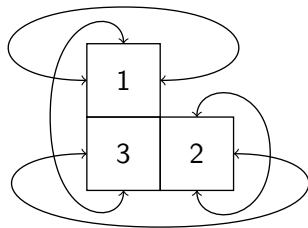
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- 4 glue together identified sides.

Transitive action \leftrightarrow connectedness of the square-tiled surface.

$$h = (13)(2)$$

$$\leftrightarrow$$

$$v = (1)(23)$$



Pairs of linear orders

A pair of linear orders (\prec_1, \prec_2) of size n is

- 1 **reducible**, if there is a partition $\{1, \dots, n\} = A \sqcup B$ such that $\forall a \in A, b \in B: a \prec_1 b$ and $a \prec_2 b$.
- 2 **irreducible** otherwise.

$$\left(\begin{array}{cccccc} 3 & \prec_1 & 1 & \prec_1 & 4 & \prec_1 & 2 \\ 4 & \prec_2 & 3 & \prec_2 & 1 & \prec_2 & 2 \end{array} \right) \quad \left(\begin{array}{cccccc} 3 & \prec_1 & 1 & \prec_1 & 4 & \prec_1 & 2 \\ 4 & \prec_2 & 1 & \prec_2 & 2 & \prec_2 & 3 \end{array} \right)$$

reducible ($A = \{1, 3, 4\}, B = \{2\}$)

irreducible

Observation. Every pair of linear orders can be uniquely decomposed into a sequence of irreducible pairs of linear orders.

Indecomposable permutations

A permutation $\sigma \in S_n$ is

- 1 **decomposable**, if there is an index $p < n$ such that $\sigma(\{1, \dots, p\}) = \{1, \dots, p\}$.
- 2 **indecomposable** otherwise.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{array} \right)$$

decomposable ($p = 3$)

$$\left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{array} \right)$$

indecomposable

Observation.

$$\#\{\text{irreducible pairs of linear orders of size } n\} = n! \cdot \text{ip}_n.$$

Asymptotics for connected square-tiled surfaces

Theorem (reformulation of the results of Dixon and Cori)

The probability p_n that a random square-tiled surface of size n is *connected* satisfies

$$p_n \approx 1 - \sum_{k=1}^{\infty} \frac{ip_k}{n^k},$$

where $n^k = n(n-1)\dots(n-k+1)$ are the falling factorials and (ip_k) counts *indecomposable permutations*.

$$(ip_k) = 1, 1, 3, 13, 71, 461, 3447, 29093, \dots$$

More applications

- 1 Combinatorial maps and indecomposable perfect matchings.
- 2 Connected multigraphs and irreducible multitournaments.
- 3 Constellations and indecomposable multipermutations.
- 4 Colored tensor models and indecomposable multipermutations.

SEQ asymptotics

Theorem (Monteil, N., 2022)

If \mathcal{U} , \mathcal{W} and $\mathcal{W}^{(2)}$ are such combinatorial classes that

- \mathcal{U} is gargantuan with positive counting sequence,
- $\mathcal{U} = \text{SEQ}(\mathcal{W})$ and $\mathcal{W}^{(2)} = \mathcal{W} \star \mathcal{W} = \text{SEQ}_2(\mathcal{W})$,

then

$$q_n := \frac{w_n}{u_n} \approx 1 - \sum_{k \geq 1} (2w_k - w_k^{(2)}) \cdot \binom{n}{k} \cdot \frac{u_{n-k}}{u_n}.$$

Reasoning: $\frac{1}{y} \xrightarrow{\partial} -\frac{1}{y^2}, \quad (1 - W(z))^2 = 1 - 2W(z) + (W(z))^2.$

Probability of a permutation to be indecomposable

Theorem (Monteil, N., 2022)

The probability q_n that a random permutation of size n is *indecomposable*, satisfies

$$q_n \approx 1 - \sum_{k \geq 1} \frac{2ip_k - ip_k^{(2)}}{n^k},$$

where $(ip_k^{(2)})$ counts *permutations with two indecomposable parts*.

$$\begin{aligned} (ip_k) &= 1, 1, 3, 13, 71, 461, 3447, \dots \\ (ip_k^{(2)}) &= 0, 1, 2, 7, 32, 177, 1142, \dots \\ (2ip_k - ip_k^{(2)}) &= 2, 1, 4, 19, 110, 745, 5752, \dots \end{aligned}$$

Probability of a tournament to be irreducible

Theorem (Monteil, N., 2021)

The probability q_n that a random labeled tournament of size n is *irreducible*, satisfies

$$q_n \approx 1 - \sum_{k \geq 1} (2it_k - it_k^{(2)}) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}},$$

where $(it_k^{(2)})$ counts *labeled tournaments with two irreducible components*.

$$\begin{aligned} (it_k) &= 1, & 0, & 2, & 24, & 544, & 22320, & \dots \\ (it_k^{(2)}) &= 0, & 2, & 0, & 16, & 240, & 6608, & \dots \\ (2it_k - it_k^{(2)}) &= 2, & -2, & 4, & 32, & 848, & 38032, & \dots \end{aligned}$$

Combinatorial classes: limits of applicability

- 1 Coefficients can be negative (see tournaments).
- 2 In certain cases, there is a decomposition

$$\mathcal{U} = \text{SET}(\mathcal{V}),$$

but we have no class \mathcal{W} such that

$$\mathcal{U} = \text{SEQ}(\mathcal{W}),$$

and our theorem is not applicable. We would like to have an “anti-SEQ” operator to create this class.

Correspondance between combinatorial classes and species

combinatorial classes

$$\mathcal{A} = \text{SET}(\mathcal{B})$$

$$\mathcal{A} = \text{SEQ}(\mathcal{B})$$

$$\mathcal{A} = \text{CYC}(\mathcal{B})$$

$$\mathcal{A} = \text{SET}_m(\mathcal{B})$$

$$\mathcal{A} = \text{SEQ}_m(\mathcal{B})$$

$$\mathcal{A} = \text{CYC}_m(\mathcal{B})$$

 \Leftrightarrow

species of structures

$$\mathcal{A} = \mathcal{E} \circ \mathcal{B}$$

$$\mathcal{A} = \mathcal{L} \circ \mathcal{B}$$

$$\mathcal{A} = \mathcal{CP} \circ \mathcal{B}$$

$$\mathcal{A} = \mathcal{E}_m \circ \mathcal{B}$$

$$\mathcal{A} = \mathcal{L}_m \circ \mathcal{B}$$

$$\mathcal{A} = \mathcal{CP}_m \circ \mathcal{B}$$

“Anti-SEQ” operator

- 1** If a virtual species Φ satisfies $\Phi_0 = 1$, then there exists a unique inverse of Φ under multiplication:

$$\Phi^{-1} = 1 - \Phi_+ + \Phi_+^2 - \Phi_+^3 + \dots,$$

where $\Phi_+ = \Phi - 1$.

- 2** If a virtual species Ψ satisfies $\Psi_0 = 0$ and $\Psi_1 = \mathcal{Z}$, then there exists a unique inverse of Ψ under substitution $\Psi^{(-1)}$.
- 3** “Anti-SEQ” operator:

$$\mathcal{L}_+^{(-1)} \equiv 1 - \mathcal{E}^{-1} \circ \mathcal{E}_+^{(-1)}.$$

Asymptotics in terms of species

Theorem (Monteil, N., 2022)

If \mathcal{A} , \mathcal{B} , \mathcal{C} and $\mathcal{B}(m)$, $m \in \mathbb{N}$, are such (weighted) species that

- 1 \mathcal{A} is gargantuan with positive total weights on $[n]$, $n \in \mathbb{N}$,
- 2 one of the following conditions holds:

| | | | |
|-----|--|---|--|
| (a) | $\mathcal{A} = \mathcal{E} \circ \mathcal{B}$ | $\mathcal{B}(m) = \mathcal{E}_m \circ \mathcal{B}$ | $\mathcal{C} = \mathcal{B}(m-1)(\mathcal{E}^{-1} \circ \mathcal{B})$ |
| (b) | $\mathcal{A} = \mathcal{L} \circ \mathcal{B}$ | $\mathcal{B}(m) = \mathcal{L}_m \circ \mathcal{B}$ | $\mathcal{C} = m\mathcal{B}^{m-1}(1 - \mathcal{B})^2$ |
| (c) | $\mathcal{A} = \mathcal{CP} \circ \mathcal{B}$ | $\mathcal{B}(m) = \mathcal{CP}_m \circ \mathcal{B}$ | $\mathcal{C} = \mathcal{B}^{m-1}(1 - \mathcal{B})$ |

then

$$p_n(m) := \frac{b_n(m)}{a_n} \approx \sum_{k \geq m-1} c_k \cdot \binom{n}{k} \cdot \frac{a_{n-k}}{a_n}.$$

Erdős-Rényi model $G(n, p)$

Consider a random labeled graph G :

- $p \in (0, 1)$ is the probability of edge presence;
- $q = 1 - p$ is the probability of edge absence.

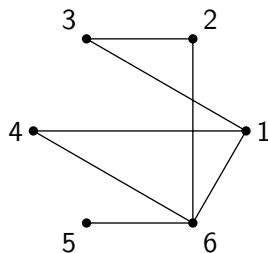
Weight of a graph:

$$w(G) = \rho^{|E(G)|},$$

where $\rho = p/q$.

Reason: if G_1 and G_2 are disjoint, then

$$w(G_1 \sqcup G_2) = w(G_1) \cdot w(G_2).$$



$$w = \rho^7$$

Asymptotics of the Erdős-Rényi model

Theorem (Monteil, N., 2022)

The probability p_n that a random graph with n vertices is connected satisfies

$$p_n \approx 1 - \sum_{k \geq 1} P_k(\rho) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}},$$

where

$$P_k(\rho) = \sum_{|V(G)|=k} (-1)^{\pi_0(G)-1} w(G)$$

and $\pi_0(G)$ is the number of connected components of G .

Meaning of the coefficients



$$w = 1$$

$$P_1(\rho) = 1$$



$$w = \rho$$

$$P_2(\rho) = \rho - 1$$



$$w = 1$$



$$w = \rho^3$$



$$w = \rho^2$$



$$w = \rho^1$$



$$w = 1$$

$$P_3(\rho) = \rho^3 + 3\rho^2 - 3\rho + 1$$

Meaning of the coefficients



$$w = 1$$

$$P_1(\rho) = 1$$

$$P_1(1) = 1 = it_1$$



$$w = \rho$$

$$P_2(\rho) = \rho - 1$$

$$P_2(1) = 0 = it_2$$



$$w = 1$$



$$w = \rho^3$$



$$w = \rho^2$$



$$w = \rho^1$$



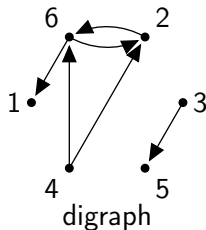
$$w = 1$$

$$P_3(\rho) = \rho^3 + 3\rho^2 - 3\rho + 1$$

$$P_3(1) = 2 = it_3$$

Digraphs (labeled directed graphs)

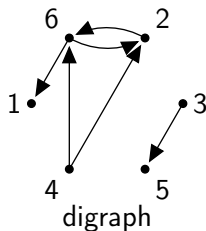
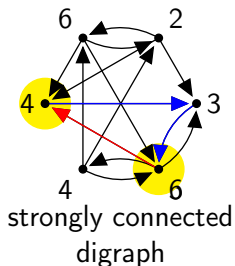
- $\mathfrak{d}_n = \#\{\text{digraphs with } n \text{ vertices}\}$



$$\mathfrak{d}_n = 2^{2^{\binom{n}{2}}}$$

Digraphs (labeled directed graphs)

- $\mathfrak{d}_n = \#\{\text{digraphs with } n \text{ vertices}\}$
- $\mathfrak{scd}_n = \#\{\text{strongly connected digraphs with } n \text{ vertices}\}$

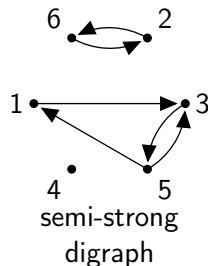
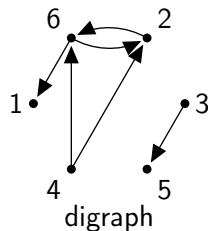
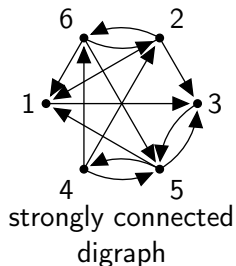


$$\mathfrak{d}_n = 2^{2\binom{n}{2}}$$

$$(\mathfrak{scd}_n) = 1, 1, 18, 1606, 565080, 734774776, \dots$$

Digraphs (labeled directed graphs)

- $\mathfrak{d}_n = \#\{\text{digraphs with } n \text{ vertices}\}$
- $\mathfrak{scd}_n = \#\{\text{strongly connected digraphs with } n \text{ vertices}\}$
- $\mathfrak{ssd}_n = \#\{\text{semi-strong digraphs with } n \text{ vertices}\}$



$$\mathfrak{d}_n = 2^{2\binom{n}{2}}$$

$$(\mathfrak{scd}_n) = 1, 1, 18, 1606, 565080, 734774776, \dots$$

$$(\mathfrak{ssd}_n) = 1, 2, 22, 1688, 573496, 738218192, \dots$$

Strongly connected directed graphs

Question. What is the probability r_n that a random directed labeled graph with n vertices is strongly connected, $n \rightarrow \infty$?

Strongly connected directed graphs

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Wright, 1971:
$$r_n \approx \sum_{k \geq 0} \frac{\omega_k(n)}{2^{kn}} \cdot \frac{n!}{(n + [k/2] - k)!},$$

where

$$\omega_k(n) = \sum_{\nu=0}^{[k/2]} \gamma_\nu \xi_{k-2\nu} \frac{2^{k(k+1)/2}}{2^{\nu(k-\nu)}} (n + [k/2] - k) \dots (n + \nu + 1 - k),$$

$$\gamma_0 = 1, \quad \gamma_\nu = \sum_{s=0}^{\nu-1} \frac{\gamma_s \eta_{\nu-s}}{(\nu-s)!}, \quad \sum_{\nu=0}^{\infty} \xi_\nu z^\nu = \left(1 - \sum_{s=0}^{\infty} \frac{\eta_s}{2^{s(s-1)/2}} \frac{z^s}{s!} \right)^2,$$

$$\eta_1 = 1, \quad \eta_s = 2^{s(s-1)} - \sum_{t=1}^{s-1} \binom{s}{t} 2^{(s-1)(s-t)} \eta_t.$$

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$$\gamma_0 = 1, \quad \gamma_\nu = \sum_{s=0}^{\nu-1} \frac{\gamma_s \eta_{\nu-s}}{(\nu-s)!}, \quad \sum_{\nu=0}^{\infty} \xi_\nu z^\nu = \left(1 - \sum_{s=0}^{\infty} \frac{\eta_s}{2^{s(s-1)/2}} \frac{z^s}{s!} \right)^2,$$

$$\eta_1 = 1, \quad \eta_s = 2^{s(s-1)} - \sum_{t=1}^{s-1} \binom{s}{t} 2^{(s-1)(s-t)} \eta_t.$$

Need for a new approach

Summary. The probability r_n has an expansion

$$r_n \approx \sum_{m \geq 0} \frac{1}{2^{mn}} \sum_{\ell=0}^{L_m} n^\ell \cdot a_{m,\ell}^\circ,$$

where $n^\ell = n(n-1)\dots(n-\ell+1)$ are falling factorials.

Observation. The array of coefficients $(a_{m,\ell}^\circ)_{m,\ell=0}^\infty$ can be assembled into a bivariate generation function.

Question. Can we express this bivariate generating function explicitly in terms of other known generating functions?

Factorially divergent series (Borinsky)

$$a_n = \alpha^{n+\beta} \Gamma(n+\beta) \left(c_0 + \frac{c_1}{\alpha(n+\beta-1)} + \frac{c_2}{\alpha^2(n+\beta-1)(n+\beta-2)} + \dots \right)$$

$$\sum_{n=0}^{\infty} a_n z^n \xrightarrow{\mathcal{A}_\beta^\alpha} \sum_{n=0}^{\infty} c_n z^n$$

Properties:

- $(\mathcal{A}_\beta^\alpha(A \cdot B))(z) = A(z) \cdot (\mathcal{A}_\beta^\alpha B)(z) + B(z) \cdot (\mathcal{A}_\beta^\alpha A)(z),$
- $(\mathcal{A}_\beta^\alpha(A \circ B))(z) = A'(B(z)) \cdot (\mathcal{A}_\beta^\alpha B)(z) + \left(\frac{z}{B(z)}\right)^\beta \exp\left(\frac{1}{\alpha} \left(\frac{1}{z} - \frac{1}{B(z)}\right)\right) (\mathcal{A}_\beta^\alpha A)(B(z)).$

Graphically divergent series

$$a_n \approx \alpha^{\beta \binom{n}{2}} \sum_{m \geq 0} \frac{1}{\alpha^{mn}} \sum_{\ell=0}^{L_m} n^\ell a_{m,\ell}^\circ$$

- $\alpha \in \mathbb{R}_{>1}$ and $\beta \in \mathbb{Z}_{>0}$ are parameters,

Graphically divergent series

$$a_n \approx \alpha^{\beta \binom{n}{2}} \sum_{m \geq 0} \frac{1}{\alpha^{mn}} \sum_{\ell=0}^{L_m} n^\ell a_{m,\ell}^\circ$$

$$\sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \xrightarrow{Q_\alpha^\beta} \sum_{m=0}^{\infty} \sum_{\ell=0}^{L_m} a_{m,\ell}^\circ \frac{z^m}{\alpha^{\frac{1}{\beta} \binom{m}{2}}} w^\ell$$

■ $\alpha \in \mathbb{R}_{>1}$ and $\beta \in \mathbb{Z}_{>0}$ are parameters,

Graphically divergent series

$$a_n \approx \alpha^{\beta \binom{n}{2}} \sum_{m \geq 0} \frac{1}{\alpha^{mn}} \sum_{\ell=0}^{L_m} n^\ell a_{m,\ell}^\circ$$

$$\boxed{Q_\alpha^\beta : \mathfrak{G}_\alpha^\beta \rightarrow \mathfrak{C}_\alpha^\beta}$$

coefficient generating function
of type (α, β)

$$\sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \xrightarrow{Q_\alpha^\beta} \sum_{m=0}^{\infty} \sum_{\ell=0}^{L_m} a_{m,\ell}^\circ \frac{z^m}{\alpha^{\frac{1}{\beta} \binom{m}{2}}} w^\ell$$

graphically divergent series

- $\alpha \in \mathbb{R}_{>1}$ and $\beta \in \mathbb{Z}_{>0}$ are parameters,
- $\mathfrak{G}_\alpha^\beta$ is the set of graphically divergent series,
- $\mathfrak{C}_\alpha^\beta$ is the set of bivariate power series.

Properties, part I

- 1** The set $\mathfrak{G}_\alpha^\beta$ forms a ring with

$$(Q_\alpha^\beta(A + B))(z, w) = (Q_\alpha^\beta A)(z, w) + (Q_\alpha^\beta B)(z, w)$$

and

$$\begin{aligned} (Q_\alpha^\beta(A \cdot B))(z, w) &= A(\alpha^{\frac{\beta+1}{2}} z^\beta w) \cdot (Q_\alpha^\beta B)(z, w) \\ &+ B(\alpha^{\frac{\beta+1}{2}} z^\beta w) \cdot (Q_\alpha^\beta A)(z, w). \end{aligned}$$

- 2** Derivation:

$$(Q_\alpha^\beta A')(z, w) = \alpha^{-\frac{\beta+1}{2}} z^{-\beta} \left((Q_\alpha^\beta A)(z, w) + \frac{\partial}{\partial w} (Q_\alpha^\beta A)(z, w) \right).$$

Properties, part II

3 Composition (interpretation of Bender's theorem): if

- F is analytic in a neighbourhood of the origin,
- $a_0 = 0$,
- $H(z) = \left. \frac{\partial}{\partial x} F(x) \right|_{x=A(z)}$,

then $F \circ A \in \mathfrak{G}_\alpha^\beta$ and

$$(Q_\alpha^\beta(F \circ A))(z, w) = H(\alpha^{\frac{\beta+1}{2}} z^\beta w) \cdot (Q_\alpha^\beta A)(z, w).$$

4 Powers: if $m \in \mathbb{Z}_{\geq 0}$ (or $m \in \mathbb{Z}$ and $a_0 = 1$), then

$$(Q_\alpha^\beta A^m)(z, w) = m \cdot A^{m-1}(\alpha^{\frac{\beta+1}{2}} z^\beta w) \cdot (Q_\alpha^\beta A)(z, w).$$

Connected graphs

Monteil, N., 2021: The probability p_n that a random graph of size n is connected satisfies

$$p_n \approx 1 - \sum_{k \geq 1} \mathit{it}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}},$$

where $\mathit{it}_k = \#\{\text{irreducible labeled tournaments of size } k\}$.

Theorem (Dovgal, N., 2023)

The coefficient generating function of type (2, 1) of connected graphs satisfies

$$(Q_2^1 CG)(z, w) = \frac{1}{G(2zw)} = 1 - IT(2zw).$$

Key ideas: $(Q_2^1 G)(z, w) = 1$, $CG(z) = \log(G(z))$, $\frac{1}{G(z)} = \frac{1}{T(z)} = 1 - IT(z)$.

Irreducible tournaments

Monteil, N., 2021: The probability q_n that a random tournament of size n is irreducible satisfies

$$q_n \approx 1 - \sum_{k \geq 1} (2it_k - it_k^{(2)}) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}},$$

where $it_k^{(2)} = \#\{\text{tournaments with two irreducible parts of size } k\}$.

Theorem (Dovgal, N., 2023)

The coefficient generating function of type (2, 1) of irreducible tournaments satisfies

$$(\mathcal{Q}_2^1 IT)(z, w) = (1 - IT(2zw))^2.$$

Key ideas: $(\mathcal{Q}_2^1 T)(z, w) = 1$, $IT(z) = 1 - \frac{1}{T(z)}$, $\frac{1}{T^2(z)} = (1 - IT(z))^2$.

Asymptotics of strongly connected directed graphs

Theorem (Dovgal, N., 2023)

The probability r_n that a random labeled digraph of size n is strongly connected satisfies

$$r_n \approx \sum_{m \geq 0} \frac{1}{2^{nm}} \sum_{\ell = \lceil m/2 \rceil}^m n^\ell \text{scd}_{m,\ell}^\circ + O\left(\frac{n^r}{2^{rn}}\right),$$

where

- $\text{scd}_{m,\ell}^\circ = \frac{2^{m(m+1)/2}}{2^{\ell(m-\ell)}} \frac{\text{ssd}_{m-\ell}}{(m-\ell)!} \frac{\mathbb{I}_{m=2\ell} - 2\text{it}_{2\ell-m} + \text{it}_{2\ell-m}^{(2)}}{(2\ell-m)!}$,
- ssd_k is the number of semi-strong digraphs of size k ,
- it_k is the number of irreducible tournaments of size k ,
- $\text{it}_k^{(2)}$ is the number of tournaments of size k with two irreducible components.

Asymptotics of strongly connected directed graphs

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where

$$\text{scd}_{m,\ell}^\circ = \frac{2^{m(m+1)/2} \text{ssd}_{m-\ell}}{2^{\ell(m-\ell)} (m-\ell)!} \frac{\mathbb{I}_{m=2\ell} - 2\text{it}_{2\ell-m} + \text{it}_{2\ell-m}^{(2)}}{(2\ell-m)!},$$

■ Interpretation of Wright's coefficients:

$$\eta_k = 2^{\binom{k}{2}} \text{it}_k, \quad \gamma_k = \frac{\text{ssd}_k}{k!}, \quad \xi_k = \frac{\mathbb{I}_{k=0} - 2\text{it}_k + \text{it}_k^{(2)}}{k!}.$$

Conclusion

- 1 General methods for combinatorial interpretation of coefficients in asymptotic expansions of irreducibles:
 - in terms of combinatorial classes (SET, SEQ, CYC),
 - in terms of species (\mathcal{E} , \mathcal{L} , \mathcal{CP}),
 - for graphically divergent series.
- 2 Applications:
 - connected graphs and irreducible tournaments,
 - square-tiled surfaces and indecomposable permutations,
 - strongly connected digraphs,
 - the Erdős-Rényi model,
 - ...

Thank you for your attention!

Literature I



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An asymptotic expansion for some coefficients of some formal power series

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Cori R.

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The Electronic Journal of Combinatorics, 12 (2005), #R56.

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Gilbert E.N.

Random graphs

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Asymptotic relations between enumerative functions in graph theory

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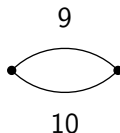
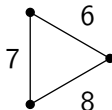
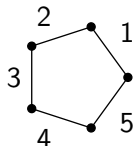
Wright E.M.

The number of strong digraphs

Bull. Lond. Math. Soc., Ser. 3 (1971), pp. 348-350.

Combinatorial maps

- Take several labeled polygons of total perimeter $N = 2n$ (1-gons and 2-gons are allowed).
- Identify their sides randomly to obtain a surface.
- Each surface is determined by a pair $(\phi, \alpha) \in S_N^2$, where α is a perfect matching.



$$\phi = (12345)(678)(910), \quad \alpha = (13)(26)(410)(59)(78)$$

Asymptotics for combinatorial maps

Theorem

The probability p_n that a random combinatorial map is *connected* satisfies

$$p_n \approx 1 - \sum_{k \geq 1} \text{im}_{2k} \cdot \frac{(2(n-k)-1)!!}{(2n-1)!!},$$

where (im_{2k}) counts *indecomposable perfect matchings*.

$$(\text{im}_{2k}) = 1, 2, 10, 74, 706, 8162, 110410, 1708394, \dots$$

Transitions, part I

Theorem (Dovgal, de Panafieu, 2019)

The exponential generating function of strongly connected digraphs satisfies

$$SCD(z) = -\log \left(G(z) \odot \frac{1}{G(z)} \right).$$

- Exponential Hadamard product:

$$\left(\sum_{n=0} a_n \frac{z^n}{n!} \right) \odot \left(\sum_{n=0} b_n \frac{z^n}{n!} \right) = \left(\sum_{n=0} a_n b_n \frac{z^n}{n!} \right).$$

- Exponential Hadamard product (with $G(z)$) changes:
 - the rate of growth,
 - the type of coefficient generating function.

Transitions, part I

Theorem (Dovgal, de Panafieu, 2019)

The exponential generating function of strongly connected digraphs satisfies

$$SCD(z) = -\log \left(G(z) \odot \frac{1}{G(z)} \right).$$

- If $\beta > 1$, then

$$\Delta_\alpha : \mathfrak{G}_\alpha^\beta \rightarrow \mathfrak{G}_\alpha^{\beta-1}$$

is defined by

$$\Delta_\alpha \left(\sum_{n=0}^{\infty} f_n \frac{z^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{f_n}{\alpha \binom{n}{2}} \frac{z^n}{n!}.$$

- $F(z) \odot G(z) = \Delta_2^{-1} F(z).$

Transitions, part II

- If $\alpha \in \mathbb{R}_{>1}$ and $\beta_1, \beta_2 \in \mathbb{Z}_{>0}$, then

$$\Phi_{\alpha}^{\beta_1, \beta_2} : \mathfrak{C}_{\alpha}^{\beta_1} \rightarrow \mathfrak{C}_{\alpha}^{\beta_2}$$

is defined as

$$\sum_{m=0}^{\infty} \sum_{\ell=0}^{L_m} a_{m,\ell}^{\circ} \frac{z^m w^{\ell}}{\alpha^{\frac{1}{\beta_1} \binom{m}{2}}} \xrightarrow{\Phi_{\alpha}^{\beta_1, \beta_2}} \sum_{m=0}^{\infty} \sum_{\ell=0}^{L_m} a_{m,\ell}^{\circ} \frac{z^m w^{\ell}}{\alpha^{\frac{1}{\beta_2} \binom{m}{2}}}.$$

- The following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{G}_{\alpha}^{\beta_1} & \xrightarrow{Q_{\alpha}^{\beta_1}} & \mathfrak{C}_{\alpha}^{\beta_1} \\ \Delta_{\alpha}^{\beta_1 - \beta_2} \downarrow & & \downarrow \Phi_{\alpha}^{\beta_1, \beta_2} \\ \mathfrak{G}_{\alpha}^{\beta_2} & \xrightarrow{Q_{\alpha}^{\beta_2}} & \mathfrak{C}_{\alpha}^{\beta_2} \end{array}$$

Asymptotics of strongly connected directed graphs

Theorem (Dovgal, N., 2023)

The coefficient generating function of type (2, 2) of strongly connected digraphs satisfies

$$(Q_2^2 \text{SCD})(z, w) = \text{SSD}(2^{3/2} z^2 w) \cdot \Phi_2^{1,2}(1 - IT(2zw))^2.$$

where $\text{SSD}(z)$ is the exponential generating function of semi-strong digraphs.

Key ideas (Dovgal, de Panafieu, 2019; Monteil, N., 2021):

- $\text{SCD}(z) = -\log \left(G(z) \odot \frac{1}{G(z)} \right) = -\log \left(1 - \Delta_2^{-1} IT(z) \right),$
- $\text{SSD}(z) = \left(G(z) \odot \frac{1}{G(z)} \right)^{-1} = \frac{1}{1 - \Delta_2^{-1} IT(z)}.$