Asymptotics for graphically divergent series

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Motivation: strongly connected directed graphs

Question. What is the probability r_n that a random directed graph with n vertices is strongly connected, as $n \to \infty$?

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Wright, 1971:
$$r_n = \sum_{k=0}^{r-1} \frac{\omega_k(n)}{2^{kn}} \cdot \frac{n!}{(n+[k/2]-k)!} + O\left(\frac{n^r}{2^{rn}}\right),$$

where

$$\omega_k(n) = \sum_{\nu=0}^{[k/2]} \gamma_\nu \xi_{k-2\nu} \frac{2^{k(k+1)/2}}{2^{\nu(k-\nu)}} (n+[k/2]-k)\ldots(n+\nu+1-k),$$

$$\gamma_0 = 1,\ \gamma_\nu = \sum_{s=0}^{\nu-1} \frac{\gamma_s \eta_{\nu-s}}{(\nu-s)!},\ \ \sum_{\nu=0}^{\infty} \xi_\nu z^\nu = \left(1 - \sum_{s=0}^{\infty} \frac{\eta_s}{2^{s(s-1)/2}} \frac{z^s}{s!}\right)^2,$$

$$\eta_1 = 1,\ \ \eta_s = 2^{s(s-1)} - \sum_{t=1}^{s-1} \binom{s}{t} 2^{(s-1)(s-t)} \eta_t.$$

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Summary. The probability r_n has an expansion

$$r_n = \sum_{m=0}^{r-1} \frac{1}{2^{mn}} \sum_{\ell=0}^{\ell_m} n^{\underline{\ell}} a^{\circ}_{m,\ell} + O\left(\frac{n^r}{2^{rn}}\right),$$

where $n^{\underline{\ell}} = n(n-1)\dots(n-\ell+1)$ are falling factorials.

Observation. The array of coefficients $(a^{\circ}_{m,\ell})_{m,\ell=0}^{\infty}$ can be assembled into a bivariate generation function.

Question. Can we express this bivariate generating function explicitly in terms of other known generating functions?

Graphically divergent series

- $\alpha \in \mathbb{R}_{>1}$ and $\beta \in \mathbb{Z}_{>0}$,
- $\mathfrak{G}_\alpha^\beta$ is the set of **graphically divergent series**, i.e.

$$A(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$$

such that

$$a_n \approx \alpha^{\beta\binom{n}{2}} \left[\sum_{m \geqslant M} \frac{1}{\alpha^{mn}} \sum_{\ell=0}^{\infty} n^{\underline{\ell}} \, a^{\circ}_{m,\ell} \right],$$

where

- $M \in \mathbb{Z}$,
- $n^{\underline{\ell}} = n(n-1)\dots(n-\ell+1)$ are falling factorials,
- the support of $(a^{\circ}_{m,\ell})_{\ell=0}^{\infty}$ is finite for each $m \in \mathbb{Z}_{\geqslant M}$.

Coefficient generating function

- If $A \in \mathfrak{G}_\alpha^\beta$ with

$$a_n \approx \alpha^{\beta\binom{n}{2}} \left[\sum_{m \geqslant M} \frac{1}{\alpha^{mn}} \sum_{\ell=0}^{\infty} n^{\ell} a^\circ_{m,\ell} \right],$$

then the associated **coefficient generating function** of type (α, β) is

$$A^\circ(z, w) = \sum_{m=M}^{\infty} \sum_{\ell=0}^{\infty} a^\circ_{m,\ell} \frac{z^m}{\alpha^{\frac{1}{\beta}\binom{m}{2}}} w^\ell \, .$$

- $\mathfrak{C}_\alpha^\beta$ is the set of corresponding coefficient generating functions.
- $\mathcal{Q}_\alpha^\beta \colon \mathfrak{G}_\alpha^\beta \to \mathfrak{C}_\alpha^\beta$ is the mapping of the form

$$\mathcal{Q}_\alpha^\beta A = A^\circ.$$

First examples

- Labeled graphs:

$$G(z) = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{z^n}{n!}, \qquad \mathfrak{g}_n = 2^{\binom{n}{2}}.$$

Its coefficient generating function of type $(2,1)$ is

$$G^{\circ}(z,w) = (\mathcal{Q}_2^1 G)(z,w) = 1.$$

First examples

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- Labeled directed graphs:

$$D(z) = \sum_{n=0}^{\infty} 2^{2\binom{n}{2}} \frac{z^n}{n!}, \qquad \mathfrak{d}_n = 2^{2\binom{n}{2}} .$$

Its coefficient generating function of type $(2,2)$ is

$$D^{\circ}(z,w) = (\mathcal{Q}_2^2 D)(z,w) = 1 .$$

Properties, part I

1. The set $\mathfrak{G}_\alpha^\beta$ forms a ring with

$$\big(\mathcal{Q}_\alpha^\beta(A+B)\big)(z,w) = (\mathcal{Q}_\alpha^\beta A)(z,w) + (\mathcal{Q}_\alpha^\beta B)(z,w)$$

and

$$\begin{aligned}\big(\mathcal{Q}_\alpha^\beta(A\cdot B)\big)(z,w) =& A\Big(\alpha^{\frac{\beta+1}{2}} z^\beta w\Big)\cdot(\mathcal{Q}_\alpha^\beta B)(z,w) + \\ & B\Big(\alpha^{\frac{\beta+1}{2}} z^\beta w\Big)\cdot(\mathcal{Q}_\alpha^\beta A)(z,w)\,.\end{aligned}$$

2. Derivation:

$$(\mathcal{Q}_\alpha^\beta A')(z,w) = \alpha^{-\frac{\beta+1}{2}} z^{-\beta}\left((\mathcal{Q}_\alpha^\beta A)(z,w) + \frac{\partial}{\partial w}(\mathcal{Q}_\alpha^\beta A)(z,w)\right).$$

Properties, part II

3. Composition (interpretation of Bender's theorem): if
   - F is analytic in a neighbourhood of the origin,
   - $a_0 = 0$,
   - $H(z) = \frac{\partial}{\partial x} F(x)\Big|_{x=A(z)}$,

   then $\quad F \circ A \in \mathfrak{G}_\alpha^\beta \quad$ and

$$\big(\mathcal{Q}_\alpha^\beta(F \circ A)\big)(z,w) = H\big(\alpha^{\frac{\beta+1}{2}} z^\beta w\big) \cdot (\mathcal{Q}_\alpha^\beta A)(z,w).$$

4. Powers: if $\quad m \in \mathbb{Z}_{\geqslant 0} \quad$ (or $m \in \mathbb{Z}$ and $a_0 = 1$), then

$$\big(\mathcal{Q}_\alpha^\beta A^m\big)(z,w) = m \cdot A^{m-1}\big(\alpha^{\frac{\beta+1}{2}} z^\beta w\big) \cdot (\mathcal{Q}_\alpha^\beta A)(z,w)\,.$$

Connected graphs

Theorem (Monteil, N., 2021)

For every $r \geqslant 1$, the probability p_n that a random labeled graph of size n is connected satisfies

$$p_n = 1 - \sum_{k=1}^{r-1} \mathfrak{it}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where $\mathfrak{it}_k$ is the number of irreducible labeled tournaments of size k.

- A **tournament** is an orientation of a complete graph.
- A tournament is **irreducible** iff it is strongly connected.

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where it_k is the number of irreducible labeled tournaments of size k.

Theorem

The coefficient generating function of type $(2,1)$ of connected graphs satisfy

$$\mathrm{CG}^{\circ}(z,w) = 1 - \mathrm{IT}(2zw).$$

Irreducible tournaments

Theorem (Monteil, N., 2021)

For every $r \geqslant 1$, the probability q_n that a random irreducible tournament of size n is connected satisfies

$$q_n = 1 - \sum_{k=1}^{r-1} \left(2\mathfrak{it}_k - \mathfrak{it}_k^{(2)}\right) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where $\mathfrak{it}_k^{(2)}$ is the number of labeled tournaments of size k with exactly two irreducible components.

Theorem

The coefficient generating function of type $(2,1)$ of irreducible tournaments satisfy

$$\mathsf{IT}^{\circ}(z,w) = \big(1 - \mathsf{IT}(2zw)\big)^2.$$

Fixed number of components

Let $m \in \mathbb{Z}_{\geqslant 1}$.

Theorem

The coefficient generating function of type $(2,1)$ of graphs with m connected components satisfy

$$\left(\mathsf{CG}^{\{m\}}\right)^{\circ}(z,w) = \mathsf{CG}^{\{m-1\}}(2zw) \cdot \left(1 - \mathsf{IT}(2zw)\right).$$

Fixed number of components

Let $m \in \mathbb{Z}_{\geqslant 1}$.

Theorem

The coefficient generating function of type $(2,1)$ of graphs with m connected components satisfy

$$\left(\mathrm{CG}^{\{m\}}\right)^{\circ}(z,w) = \mathrm{CG}^{\{m-1\}}(2zw) \cdot \left(1 - \mathrm{IT}(2zw)\right).$$

Theorem

The coefficient generating function of type $(2,1)$ of tournaments with m irreducible parts satisfy

$$\left(\mathrm{IT}^{(m)}\right)^{\circ}(z,w) = m \cdot \mathrm{IT}^{(m-1)}(2zw) \cdot \left(1 - \mathrm{IT}(2zw)\right)^{2}.$$

Transitions, part I

Theorem (Dovgal, de Panafieu, 2019)

The exponential generating function of strongly connected digraphs satisfies

$$\mathrm{SCD}(z) = -\log\left(G(z) \odot \frac{1}{G(z)}\right).$$

- Hadamard product:

$$\left(\sum_{n=0} a_n \frac{z^n}{n!}\right) \odot \left(\sum_{n=0} b_n \frac{z^n}{n!}\right) = \left(\sum_{n=0} a_n b_n \frac{z^n}{n!}\right).$$

- Hadamard product (with $G(z)$) changes the rate of convergence (and hence, the type of coefficient generating function is changed too).

Transitions, part I

Theorem (Dovgal, de Panafieu, 2019)

The exponential generating function of strongly connected digraphs satisfies

$$\mathrm{SCD}(z) = -\log\left(G(z) \odot \frac{1}{G(z)}\right).$$

- If $\beta > 1$, then

$$\Delta_\alpha \colon \mathfrak{G}_\alpha^\beta \to \mathfrak{G}_\alpha^{\beta-1}$$

  is defined by

$$\Delta_\alpha\left(\sum_{n=0}^{\infty} f_n \frac{z^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{f_n}{\alpha^{\binom{n}{2}}} \frac{z^n}{n!}.$$

- $F(z) \odot G(z) = \Delta_2^{-1} F(z)$.

Transitions, part II

- If $\alpha \in \mathbb{R}_{>1}$ and $\beta_1, \beta_2 \in \mathbb{Z}_{>0}$, then

$$\Phi_\alpha^{\beta_1,\beta_2} \colon \mathfrak{C}_\alpha^{\beta_1} \to \mathfrak{C}_\alpha^{\beta_2}$$

is defined as

$$\Phi_\alpha^{\beta_1,\beta_2}\left(\sum_{m=M}^{\infty}\sum_{\ell=0}^{\infty} a^\circ_{m,\ell}\frac{z^m}{\alpha^{\frac{1}{\beta_1}\binom{m}{2}}}w^\ell\right) = \sum_{m=M}^{\infty}\sum_{\ell=0}^{\infty} a^\circ_{m,\ell}\frac{z^m}{\alpha^{\frac{1}{\beta_2}\binom{m}{2}}}w^\ell\,.$$

- The following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{G}_\alpha^{\beta_1} & \xrightarrow{\mathcal{Q}_\alpha^{\beta_1}} & \mathfrak{C}_\alpha^{\beta_1} \\ {\scriptstyle \Delta_\alpha^{\beta_1-\beta_2}}\big\downarrow & & \big\downarrow{\scriptstyle \Phi_\alpha^{\beta_1,\beta_2}} \\ \mathfrak{G}_\alpha^{\beta_2} & \xrightarrow{\mathcal{Q}_\alpha^{\beta_2}} & \mathfrak{C}_\alpha^{\beta_2} \end{array}$$

Strongly connected directed graphs, part I

Theorem

The coefficient generating function of type $(2,2)$ *of strongly connected digraphs satisfies*

$$\mathrm{SCD}^{\circ}(z,w) = \mathrm{SSD}(2^{3/2}z^2w)\cdot \Phi_2^{1,2}\big(1-\mathrm{IT}(2zw)\big)^2 .$$

where $\mathrm{SSD}(z)$ *is the exponential generating function of semi-strong digraphs (a* **semi-strong digraph** *is a disjoint union of strongly connected digraphs).*

Key ideas (Dovgal, de Panafieu, 2019; Monteil, N., 2021):

- $\mathrm{SCD}(z) = -\log\left(G(z)\odot\frac{1}{G(z)}\right) = \log\frac{1}{1-\Delta_2^{-1}\mathrm{IT}(z)}$,
- $\mathrm{SSD}(z) = \left(G(z)\odot\frac{1}{G(z)}\right)^{-1} = \frac{1}{1-\mathrm{IT}(z)\odot\mathrm{G}(z)}$.

Strongly connected directed graphs, part II

Corollary

For every $r \geqslant 1$, the probability r_n that a random labeled digraph of size n is strongly connected satisfies

$$r_n = \sum_{m=0}^{r-1} \frac{1}{2^{nm}} \sum_{\ell=\lceil m/2 \rceil}^{m} n^{\ell} \mathfrak{scd}^{\circ}_{m,\ell} + O\left(\frac{n^r}{2^{rn}}\right),$$

where

- $\mathfrak{scd}^{\circ}_{m,\ell} = 2^{m(m+1)/2+\ell(\ell-m)} \frac{\mathfrak{ssd}_{m-\ell}}{(m-\ell)!} \frac{\mathbb{I}_{m=2\ell} - 2\mathfrak{it}_{2\ell-m} + \mathfrak{it}^{(2)}_{2\ell-m}}{(2\ell-m)!}$,
- $\mathfrak{ssd}_k$ *is the number of semi-strong digraphs of size k,*
- $\mathfrak{it}_k$ *is the number of irreducible tournaments of size k,*
- $\mathfrak{it}^{(2)}_k$ *is the number of tournaments of size k with two irreducible components.*

Asymptotics of 2-SAT formulae

Let

$$\ddot{\mathrm{SAT}}(z) = \sum_{n=0}^{\infty} \mathfrak{sat}_n \frac{z^n}{2^{n^2} n!}$$

be implication generating function of 2-SAT formulae.

Theorem

The coefficient generating function of type $(2,1)$ *of 2-SAT formulae satisfies*

$$(\mathcal{Q}_2^1 \ddot{\mathrm{SAT}})(z,w) = \frac{\ddot{\mathrm{SAT}}(2zw)}{\mathrm{G}(2zw)} = \ddot{\mathrm{SAT}}(2zw)\big(1 - \mathrm{IT}(2zw)\big)\,.$$

Asymptotics of contradictory components

Theorem

The coefficient generating function of type $(2,4)$ *of contradictory strongly connected implication digraphs satisfy*

$$\mathrm{CSC}^{\circ}(z,w) = \exp\left(\frac{1}{2}\mathrm{SCD}(2^{7/2}z^4w) - \mathrm{CSC}(2^{5/2}z^4w)\right)\cdot$$
$$\Phi_2^{2,4}\big(1 - \mathrm{IT}(2^{5/2}z^2zw)\big).$$

Key ideas (Dovgal, de Panafieu, Ravelomanana, 2021):

- $\mathrm{S\ddot{A}T}(z) = G(z)\cdot\Delta_2^2\left(G(z)\odot\dfrac{1}{G(z)}\right)^{1/2},$
- $\mathrm{CSC}(z) = \dfrac{1}{2}\mathrm{SCD}(2z) + \log\left(D(z)\odot\dfrac{D(z)}{G(2z)}\right).$

Conclusion

1. We have constructed a tool for manipulating coefficients of asymptotic expansions.

2. Transfers extend to graphic families with marked patterns: any family with a fixed number of components:
   - strongly connected components in digraphs, contradictory components in 2-sat,
   - source-like, sink-like, isolated components, ...
   - any graphically divergent series with marking variables.

3. Bonus: combinatorial explanations of the expansion coefficients.

Thank you for your attention!

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