Irreducibility of combinatorial objects: asymptotic probability and interpretation

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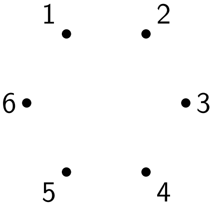
Graphs



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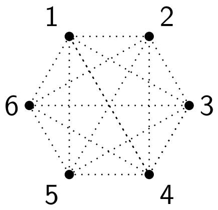

there are $\binom{n}{2}$ possible edges

(here, $n = 6$ and $\binom{n}{2} = 15$)

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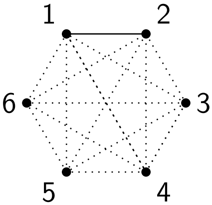

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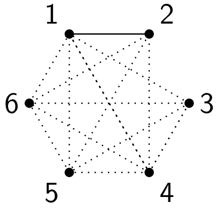

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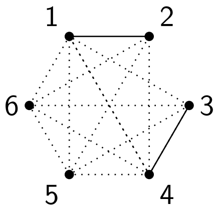

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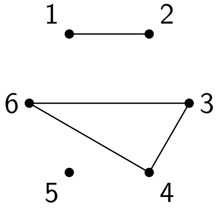
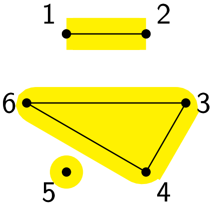


probability to pick this graph is $\frac{1}{2^{\binom{n}{2}}}$

(uniform probability)

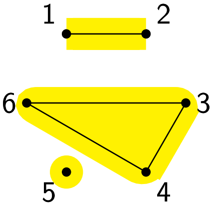
Graphs



every graph is a disjoint union (SET)

of connected graphs

Graphs



every graph is a disjoint union (SET)

of connected graphs

- $\mathfrak{g}_n = 2^{\binom{n}{2}}$: the number of labeled graphs with n vertices
- $\mathfrak{cg}_n$: the number of connected labeled graphs with n vertices

$$(\mathfrak{cg}_n)_{n \geqslant 0} = 1, 1, 4, 38, 728, 26704, 1866256, \ldots$$

Probability of a graph to be connected

Question. What is the probability $p_n = \frac{\mathfrak{cg}_n}{\mathfrak{g}_n}$ that a random graph with n vertices is connected, as $n \to \infty$?

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3. Wright, 1970:

$$p_n = 1 - \binom{n}{1}\frac{2}{2^n} - 2\binom{n}{3}\frac{2^6}{2^{3n}} - 24\binom{n}{4}\frac{2^{10}}{2^{4n}} + O\left(\frac{n^5}{2^{5n}}\right)$$

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4. Can we see the structure? What is the interpretation?

Asymptotics for p_n

Theorem

For every $r \geqslant 1$, the probability p_n that a random labeled graph of size n is connected satisfies

$$p_n = 1 - \sum_{k=1}^{r-1} \mathrm{it}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

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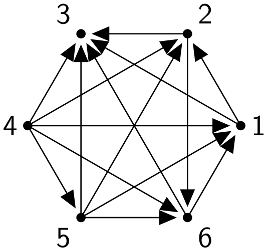
where it_k is the number of irreducible labeled tournaments of size k.

$$(\mathrm{it}_k) = 1, 0, 2, 24, 544, 22320, 1677488, \ldots$$

Tournaments

A **tournament** is a complete directed graph.



The number of labeled tournaments with n vertices is

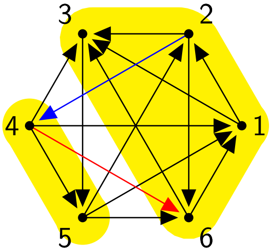
$$\mathfrak{t}_n = 2^{\binom{n}{2}}$$

Irreducible tournaments

A tournament is **irreducible**, if
for every partition of vertices $V = A \sqcup B$

1. there exist an edge from A to B,
2. there exist an edge from B to A.

$$V = \{1, 2, 3, 4, 5, 6\}$$



$$A = \{1, 2, 3, 6\}$$
$$B = \{4, 5\}$$

Irreducible tournaments

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1. there exist an edge from A to B,
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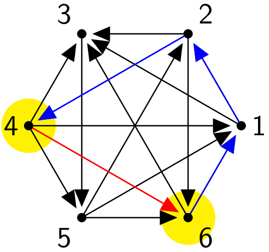
Equivalently, a tournament is **strongly connected**: for each two vertices u and v

1. there is a path from u to v,
2. there is a path from v to u.

$V = \{1, 2, 3, 4, 5, 6\}$



$u = 4$
$v = 6$

Exponential generating functions and Bender's theorem

EGF: $G(z) = \sum_{n=0}^{\infty} \mathfrak{g}_n \frac{z^n}{n!}$

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Bender, 1975:

1. $A(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_n \neq 0$
2. $F(x, y)$ is analytic in $U(0; 0)$
3. $B(z) = \sum_{n=0}^{\infty} b_n z^n = F(z, A(z))$
4. $C(z) = \sum_{n=0}^{\infty} c_n z^n = \left[\frac{\partial F}{\partial y}(z, y)\right]_{y=A(z)}$
5. $\frac{a_{n-1}}{a_n} \to 0, \quad \text{as } n \to \infty$
6. $\exists r \geqslant 1 : \sum_{k=r}^{n-r} |a_k a_{n-k}| = O(a_{n-r})$

Then $\quad b_n = \sum_{k=0}^{r-1} c_k a_{n-k} + O(a_{n-r}).$

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5. the sequence (a_n) is gargantuan

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Then $b_n \approx \sum_{k \geqslant 0} c_k a_{n-k}$.

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$\frac{\partial F}{\partial y} = \frac{1}{y}$

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$\frac{\partial F}{\partial y} = \frac{1}{y}$

$\frac{1}{1-y} = 1 + y + y^2 + \dots$

$\frac{\mathfrak{cg}_n}{\mathfrak{g}_n} \approx 1 - \sum_{k \geqslant 0} \mathfrak{it}_k \binom{n}{k} \frac{\mathfrak{g}_{n-k}}{\mathfrak{g}_n}$

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$$G(z) = T(z) = \frac{1}{1 - IT(z)}$$

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Exponential generating functions and Bender's theorem

1. $CG(z) = \log G(z)$
2. $A(z) = G(z) - 1$
3. $F(x,y) = \log(1+y)$
4. $\dfrac{\partial F}{\partial y} = \dfrac{1}{1+y}$
5. $C(z) = \dfrac{1}{G(z)} = \dfrac{1}{T(z)}$
6. $\dfrac{1}{T(z)} = 1 - IT(z)$
7. $\dfrac{\mathfrak{cg}_n}{\mathfrak{g}_n} \approx 1 - \sum\limits_{k \geqslant 0} \mathfrak{it}_k \binom{n}{k} \dfrac{\mathfrak{g}_{n-k}}{\mathfrak{g}_n}$

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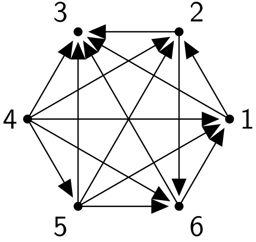
Tournament as a sequence

Folklore: Every labeled tournament can be uniquely decomposed into a sequence (SEQ) of irreducible labeled tournaments.



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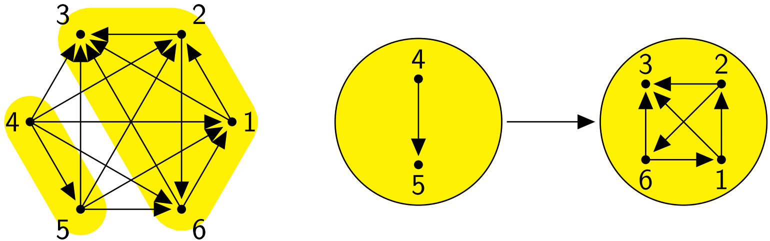

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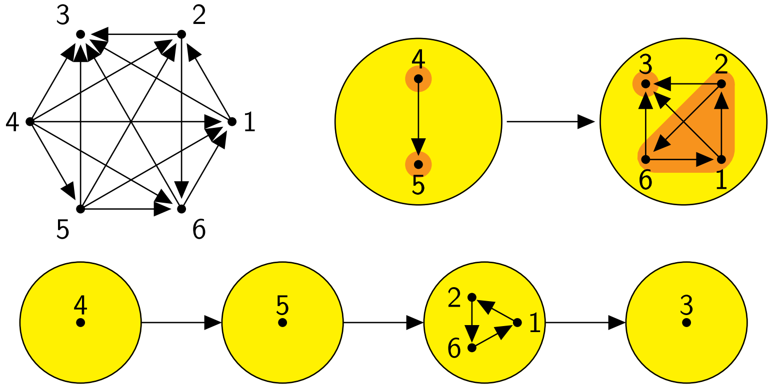


SET and SEQ decompositions

graphs —decomposed as SET of→ connected graphs

connected graphs —counted by→ $CG(z) = \sum_{n=0}^{\infty} \mathfrak{cg}_n \frac{z^n}{n!} = \log(G(z))$

graphs ←counts— $G(z) = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{z^n}{n!} = T(z)$ —counts→ tournaments

$CG(z)$ —derivative→ $IT(z)$

$IT(z) = \sum_{n=0}^{\infty} \mathfrak{it}_n \frac{z^n}{n!} = 1 - \frac{1}{T(z)}$

tournaments —decomposed as as SEQ of→ irreducible tournaments

irreducible tournaments —counted by→ $IT(z)$

Asymptotics for connected graphs

Theorem

The probability p_n that a random labeled graph of size n is connected, satisfies

$$p_n \approx 1 - \sum_{k=1} \mathrm{it}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}}$$

where it_k is the number of irreducible labeled tournaments of size k.

$$(\mathrm{it}_k) = 1, 0, 2, 24, 544, 22320, 1677488, \ldots$$

Combinatorial constructions

1. $\mathcal{U} = \mathrm{SET}(\mathcal{V})$, $\quad U(z) = \exp\big(V(z)\big)$.

2. $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$, $\quad U(z) = \dfrac{1}{1 - W(z)}$.

SET asymptotics

Theorem

If $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ are such combinatorial classes that

1. *$\mathcal{U}$ is gargantuan with positive counting sequence,*
2. $\mathcal{U} = \mathrm{SET}(\mathcal{V})$ *and* $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$,

then

$$p_n := \frac{\mathfrak{v}_n}{\mathfrak{u}_n} \approx 1 - \sum_{k \geqslant 1} \mathfrak{w}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}.$$

Combinatorial meaning: p_n is the probability that a random object of size n from $\mathcal{U}$ is irreducible in terms of SET-decomposition.

Random pair of permutations

Question. What is the probability p_n that a random pair of permutations $(\sigma, \tau) \in S_n^2$ generates a transitive group, as $n \to \infty$?

1. Dixon, 2005: $$p_n \approx 1 - \sum_{k \geqslant 1} \frac{\mathfrak{ip}_k}{(n)_k},$$

   where $(n)_k = n(n-1)\dots(n-k+1)$ are the falling factorials.

2. Cori, 2009: the sequence

   $$(\mathfrak{ip}_k) = 1, 1, 3, 13, 71, 461, 3447, 29093, \dots$$

   counts indecomposable permutations.

Square-tiled surfaces

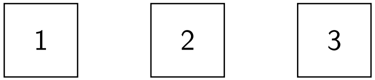
A pair $(h, v) \in S_n^2$ determines a square-tiled surface:

1. take n labeled squares,

$h = (13)(2)$

$v = (1)(23)$

$\leftrightarrow$



Square-tiled surfaces

A pair $(h, v) \in S_n^2$ determines a square-tiled surface:

1. take n labeled squares,
2. identify horizontal sides by the permutation h,

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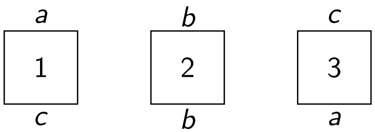

Square-tiled surfaces

A pair $(h, v) \in S_n^2$ determines a square-tiled surface:

1. take n labeled squares,
2. identify horizontal sides by the permutation h,
3. identify vertical sides by the permutation v,

$h = (13)(2)$

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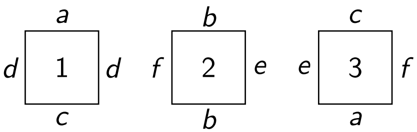
$\leftrightarrow$



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1. take n labeled squares,
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3. identify vertical sides by the permutation v,
4. glue together identified sides.

$$h = (13)(2)$$

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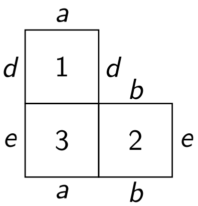

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Transitive action $\leftrightarrow$ connectedness of the square-tiled surface.

$$h = (13)(2)$$

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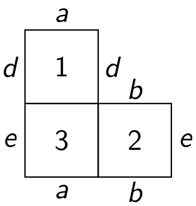

Indecomposable permutations

A permutation $\sigma \in S_n$ is

1. **decomposable**, if there is an index $p < n$ such that $\sigma\big(\{1,\dots,p\}\big) = \{1,\dots,p\}$.
2. **indecomposable** otherwise.

$$\left(\begin{array}{ccc|cc} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{array}\right) \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix}$$

decomposable $(p = 3)$ indecomposable

Obstacles. Not stable under relabeling, number of permutations is not $(n!)^2$, combinatorial class is not gargantuan.

Pairs of linear orders

A pair of linear orders $(\prec_1, \prec_2)$ of size n is

1. **reducible**, if there is a partition $\{1, \ldots, n\} = A \sqcup B$ such that $\forall\, a \in A,\ b \in B\colon \quad a \prec_1 b$ and $a \prec_2 b$.
2. **irreducible** otherwise.

$$\begin{pmatrix} 3 & \prec_1 & 1 & \prec_1 & 4 & \prec_1 & 2 \\ 4 & \prec_2 & 3 & \prec_2 & 1 & \prec_2 & 2 \end{pmatrix} \qquad \begin{pmatrix} 3 & \prec_1 & 1 & \prec_1 & 4 & \prec_1 & 2 \\ 4 & \prec_2 & 1 & \prec_2 & 2 & \prec_2 & 3 \end{pmatrix}$$

reducible $(A = \{1, 3, 4\}, B = \{2\})$ — irreducible

Observation.

$$\#\{\text{irreducible pairs of linear orders of size } n\} = n! \cdot \mathfrak{ip}_n.$$

Correspondence of classes

1. $\mathcal{U}$ = {square-tiled surfaces} = {pairs of linear orders of the same size}

2. $\mathcal{V}$ = {connected square-tiled surfaces}

3. $\mathcal{W}$ = {irreducible pairs of linear orders of the same size}

$$p_n = \mathfrak{w}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n} = k! \cdot \mathfrak{ip}_k \cdot \binom{n}{k} \cdot \frac{\left((n-k)!\right)^2}{(n!)^2} = \frac{\mathfrak{ip}_k}{(n)_k}$$

Asymptotics for connected square-tiled surfaces

Theorem (reformulation of the results of Dixon and Cori)

The probability p_n that a random square-tiled surface of size n is connected, satisfies

$$p_n \approx 1 - \sum_{k=1} \frac{\mathfrak{ip}_k}{(n)_k}$$

where $(n)_k = n(n-1)\dots(n-k+1)$ *are the falling factorials and* $\mathfrak{ip}_k$ *is the number of indecomposable permutations of size* k*.*

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More applications

1. Combinatorial maps and indecomposable perfect matchings.
2. Connected multigraphs and irreducible multitournaments.
3. Constellations and indecomposable multipermutations.
4. Colored tensor models and indecomposable multipermutations.

SEQ asymptotics

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If $\mathcal{U}$, $\mathcal{W}$ and $\mathcal{W}^{(2)}$ are such combinatorial classes that

- *$\mathcal{U}$ is gargantuan with positive counting sequence,*
- $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ *and* $\mathcal{W}^{(2)} = \mathcal{W} \star \mathcal{W} = \mathrm{SEQ}_2(\mathcal{W})$,

then

$$q_n := \frac{\mathfrak{w}_n}{\mathfrak{u}_n} \approx 1 - \sum_{k \geqslant 1} \left(2\mathfrak{w}_k - \mathfrak{w}_k^{(2)}\right) \cdot \binom{n}{k} \cdot \frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}.$$

Reasoning: $\frac{1}{y} \xrightarrow{\partial} -\frac{1}{y^2}$, $\left(1 - W(z)\right)^2 = 1 - 2W(z) + \left(W(z)\right)^2$.

Example: asymptotics for irreducible tournaments

Theorem

The probability q_n that a random labeled tournament of size n is irreducible, satisfies

$$q_n \approx 1 - \sum_{k \geqslant 1} \left(2\mathrm{it}_k - \mathrm{it}_k^{(2)}\right) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}},$$

where $\mathrm{it}_k^{(2)}$ is the number of labeled tournaments of size k with two irreducible components.

$$\begin{aligned}
(\mathrm{it}_k) &= 1, \quad 0, \quad 2, \quad 24, \quad 544, \quad 22320, \quad \ldots \\
(\mathrm{it}_k^{(2)}) &= 0, \quad 2, \quad 0, \quad 16, \quad 240, \quad 6608, \quad \ldots \\
(2\mathrm{it}_k - \mathrm{it}_k^{(2)}) &= 2, \quad -2, \quad 4, \quad 32, \quad 848, \quad 38032, \quad \ldots
\end{aligned}$$

Combinatorial classes: limits of applicability

1. Coefficients can be negative (see tournaments).

2. In certain cases, there is a decomposition

$$\mathcal{U} = \mathrm{SET}(\mathcal{V}),$$

but we have no class $\mathcal{W}$ such that

$$\mathcal{U} = \mathrm{SEQ}(\mathcal{W}),$$

and our theorem is not applicable. We would like to have an "anti-SEQ" operator to create this class.

Correspondance between combinatorial classes and species

combinatorial classes		species of structures
$\mathcal{A} = \mathrm{SET}(\mathcal{B})$		$\mathcal{A} = \mathcal{E} \circ \mathcal{B}$
$\mathcal{A} = \mathrm{SEQ}(\mathcal{B})$		$\mathcal{A} = \mathcal{L} \circ \mathcal{B}$
$\mathcal{A} = \mathrm{CYC}(\mathcal{B})$	$\Leftrightarrow$	$\mathcal{A} = \mathcal{CP} \circ \mathcal{B}$
$\mathcal{A} = \mathrm{SET}_m(\mathcal{B})$		$\mathcal{A} = \mathcal{E}_m \circ \mathcal{B}$
$\mathcal{A} = \mathrm{SEQ}_m(\mathcal{B})$		$\mathcal{A} = \mathcal{L}_m \circ \mathcal{B}$
$\mathcal{A} = \mathrm{CYC}_m(\mathcal{B})$		$\mathcal{A} = \mathcal{CP}_m \circ \mathcal{B}$

“Anti-SEQ” operator

1. If a virtual species Φ satisfies $\Phi_0 = 1$, then there exists a unique inverse of Φ under multiplication:

$$\Phi^{-1} = 1 - \Phi_+ + \Phi_+^2 - \Phi_+^3 + \ldots,$$

where $\Phi_+ = \Phi - 1$.

2. If a virtual species Ψ satisfies $\Psi_0 = 0$ and $\Psi_1 = \mathcal{Z}$, then there exists a unique inverse of Ψ under substitution $\Psi^{(-1)}$.

3. “Anti-SEQ” operator:

$$\mathcal{L}_+^{(-1)} \equiv 1 - \mathcal{E}^{-1} \circ \mathcal{E}_+^{(-1)}.$$

SET_m asymptotics in terms of species

Theorem

If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{B}^{\{m\}}$, $m \in \mathbb{N}$, are such (weighted) species that

1. *$\mathcal{A}$ is gargantuan with positive total weights on $[n]$, $n \in \mathbb{N}$,*
2. *$\mathcal{A} = \mathcal{E} \circ \mathcal{B}$ and $\mathcal{B}^{\{m\}} = \mathcal{E}_m \circ \mathcal{B}$,*

then

$$p_n^{\{m\}} := \frac{\mathfrak{b}_n^{\{m\}}}{\mathfrak{a}_n} \approx \sum_{k \geqslant 0} \mathfrak{c}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{a}_{n-k}}{\mathfrak{a}_n}.$$

where $$\mathcal{C} = \mathcal{B}^{\{m-1\}}(\mathcal{E}^{-1} \circ \mathcal{B}) \equiv \mathcal{B}^{\{m-1\}}\left(\left(1 - \mathcal{L}_+^{(-1)}\right) \circ \mathcal{A}_+\right).$$

SEQ_m asymptotics in terms of species

Theorem

If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{B}^{(m)}$, $m \in \mathbb{N}$, are such (weighted) species that

1. $\mathcal{A}$ *is gargantuan with positive total weights on* $[n]$, $n \in \mathbb{N}$,
2. $\mathcal{A} = \mathcal{L} \circ \mathcal{B}$ *and* $\mathcal{B}^{(m)} = \mathcal{L}_m \circ \mathcal{B}$,

then

$$q_n^{(m)} := \frac{\mathfrak{b}_n^{(m)}}{\mathfrak{a}_n} \approx \sum_{k \geqslant 0} \mathfrak{c}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{a}_{n-k}}{\mathfrak{a}_n}.$$

where $\quad \mathcal{C} = m\mathcal{B}^{m-1}(1 - \mathcal{B})^2.$

CYC_m asymptotics in terms of species

Theorem

If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{B}^{[m]}$, $m \in \mathbb{N}$, are such (weighted) species that

1. *$\mathcal{A}$ is gargantuan with positive total weights on $[n]$, $n \in \mathbb{N}$,*
2. $\mathcal{A} = \mathcal{CP} \circ \mathcal{B}$ *and* $\mathcal{B}^{[m]} = \mathcal{CP}_m \circ \mathcal{B}$,

then

$$r_n^{[m]} := \frac{\mathfrak{b}_n^{[m]}}{\mathfrak{a}_n} \approx \sum_{k \geqslant 0} \mathfrak{c}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{a}_{n-k}}{\mathfrak{a}_n}.$$

where $\mathcal{C} = \mathcal{B}^{m-1}(1 - \mathcal{B})$.

Erdős-Rényi model $G(n, p)$

Consider a random labeled graph G:

1. $p \in (0, 1)$ is the probability of edge presence;
2. $q = 1 - p$ is the probability of edge absence;
3. the probability to pick this graph is

$$\mathbb{P}(G) = p^{|E(G)|} q^{\binom{n}{2} - |E(G)|} = \frac{\rho^{|E(G)|}}{(\rho + 1)^{\binom{n}{2}}},$$

where $\rho = \dfrac{p}{q} = q^{-1} - 1$.

Graph weight

1. **Weight** of a graph: $w(G) = \rho^{|E(G)|}$.
2. Reason: if G_1 and G_2 are disjoint, then

$$w(G_1 \sqcup G_2) = w(G_1) \cdot w(G_2).$$

3. The total weight of graphs of size n:

$$\sum_{|V(G)|=n} w(G) = q^{-\binom{n}{2}}.$$

4. The weight of connected graphs of size n:

$$\sum_{G \text{ is connected}} w(G).$$

Asymptotics of the Erdős-Rényi model

Theorem

The probability p_n that a random graph with n vertices is connected satisfies

$$p_n \approx 1 - \sum_{k \geqslant 1} P_k(\rho) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}},$$

where

$$P_k(\rho) = \sum_{|V(G)|=k} (-1)^{\pi_0(G)-1} w(G).$$

Meaning of the coefficients

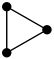
$w = 1$

$P_1(\rho) = 1$

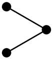
$w = \rho$

$w = 1$

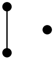
$P_2(\rho) = \rho - 1$



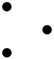
$w = \rho^3$



$w = \rho^2$



$w = \rho^1$



$w = 1$

$P_3(\rho) = \rho^3 + 3\rho^2 - 3\rho + 1$

Meaning of the coefficients

$w = 1$

$P_1(\rho) = 1$

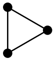
$P_1(1) = 1 = \mathfrak{it}_1$

$w = \rho$

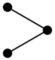
$w = 1$

$P_2(\rho) = \rho - 1$

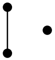
$P_2(1) = 0 = \mathfrak{it}_2$



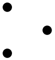
$w = \rho^3$



$w = \rho^2$



$w = \rho^1$



$w = 1$

$P_3(\rho) = \rho^3 + 3\rho^2 - 3\rho + 1$

$P_3(1) = 2 = \mathfrak{it}_3$

Asymptotics of the Erdős-Rényi model, continued

Theorem

The probability $p_n^{\{m\}}$ that a random graph with n vertices has exactly m connected components satisfies

$$p_n^{\{m\}} \approx \sum_{k \geqslant 0} P_k^{\{m\}}(\rho) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}},$$

where

$$P_k^{\{m\}}(\rho) = \sum_{|V(G)|=k} (-1)^{\pi_0(G)-m} \binom{\pi_0(G)}{m-1} w(G).$$

Probability of a directed graph to be strongly connected

Question. What is the probability r_n that a random directed graph with n vertices is strongly connected, as $n \to \infty$?

Probability of a directed graph to be strongly connected

Question. What is the probability r_n that a random directed graph with n vertices is strongly connected, as $n \to \infty$?

Wright, 1970: $$r_n = \sum_{k=0}^{r-1} \frac{\omega_k(n)}{2^{kn}} \cdot \frac{n!}{(n+[k/2]-k)!} + O\left(\frac{n^r}{2^{rn}}\right),$$

where

$$\omega_k(n) = \sum_{\nu=0}^{[k/2]} \gamma_\nu \xi_{k-2\nu} \frac{2^{k(k+1)/2}}{2^{\nu(k-\nu)}} (n+[k/2]-k)\ldots(n+\nu+1-k),$$

$$\gamma_0 = 1,\ \gamma_\nu = \sum_{s=0}^{\nu-1} \frac{\gamma_s \eta_{n-s}}{(\nu-s)!},\quad \sum_{\nu=0}^{\infty} \xi_\nu z^\nu = \left(1 - \sum_{n=0}^{\infty} \frac{\eta_n}{2^{n(n-1)/2}} \frac{z^n}{n!}\right)^2,$$

$$\eta_1 = 1, \quad \eta_n = 2^{n(n-1)} - \sum_{t=1}^{n-1} \binom{n}{t} 2^{(n-1)(n-t)} \eta_t.$$

Towards the asymptotics

1. Dovgal and de Panafieu, 2019:

$$SD(z) = -\log\left(G(z) \odot \frac{1}{G(z)}\right)$$

2. In terms of tournaments:

$$SD(z) = -\log\left(1 - T(z) \odot IT(z)\right)$$

3. Semi-strong directed graphs:

$$SSD(z) = \frac{1}{1 - T(z) \odot IT(z)}$$

Open problem: are there direct bijections?

Asymptotics for strongly connected graphs

Theorem

The probability r_n that a random directed graph with n vertices is strongly connected satisfies

$$r_n \approx \sum_{k \geqslant 0} \mathfrak{ssd}_k \binom{n}{k} \frac{2^{k(k+1)}}{2^{2nk}} \frac{\mathfrak{it}_{n-k}}{\mathfrak{t}_{n-k}},$$

where $\mathfrak{ssd}_k$, $\mathfrak{t}_k$ and $\mathfrak{it}_k$ are the numbers of semi-strong digraphs, tournaments and irreducible tournaments of size k, respectively.

Reasoning: $\log(1-y) \xrightarrow{\partial} -\frac{1}{1-y}$.

Asymptotics for strongly connected graphs, continued

Theorem

The probability r_n that a random directed graph with n vertices is strongly connected satisfies

$$r_n \approx 1 - \sum_{k \geqslant 1} \frac{R_k(n)}{2^{nk}},$$

where a $R_k(n)$ is a polynomial of degree k.

Explanation of terms involved in Wright's asymptotics:

$$\eta_n = \mathfrak{t}_n \mathfrak{i}\mathfrak{t}_n, \qquad \gamma_n = \frac{\mathfrak{s}\mathfrak{s}\mathfrak{d}_n}{n!}, \qquad \xi_0 = 1, \;\; \xi_n = -\frac{2\mathfrak{i}\mathfrak{t}_n + \mathfrak{i}\mathfrak{t}_n^{(2)}}{n!}.$$

Explicit form of $R_k(n)$

For any positive integer k,

$$R_k(n) = 2^{k(k+1)/2} \sum_{\nu=0}^{[k/2]} \binom{n}{\nu, k-2\nu} \frac{\mathfrak{ssd}_\nu \beta_{k-2\nu}}{2^{\nu(k-\nu)}},$$

and

$$\beta_k = \begin{cases} 1, & \text{if } k = 0, \\ -2\mathfrak{it}_k + \mathfrak{it}_k^{(2)}, & \text{if } k \neq 0. \end{cases}$$

- $\mathfrak{ssd}_k$ is the number of semi-strong digraphs of size k,
- $\mathfrak{it}_k$ is the number of irreducible tournaments of size k,
- $\mathfrak{it}_k^{(2)}$ is the number of tournaments of size k with two irreducible parts.

Another type of convergence rate or irreducibles

1. Some classes are not gargantuan (forests, polynomials).

2. The notion of irreducibility can be understood broader. For instance, ordinary generating functions of "noncrossing compositions" satisfy

$$A(z) = 1 + I\big(zA(z)\big).$$

Question. Can we have any combinatorial interpretation for the coefficients arising in the asymptotic expansions of the probabilities in the above cases?

Algorithmic aspects

For the asymptotic expansion for connected graphs,

$$p_n = 1 - \binom{n}{1}\frac{2\mathfrak{it}_1}{2^n} - \binom{n}{2}\frac{2^3\mathfrak{it}_2}{2^{2n}} - \binom{n}{3}\frac{2^6\mathfrak{it}_3}{2^{3n}} - \ldots,$$

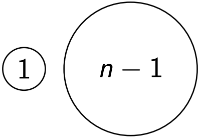
the inclusion-exclusion principle shows the origin of terms:

Algorithmic aspects

For the asymptotic expansion for connected graphs,

$$p_n = 1 - \binom{n}{1}\frac{2\mathfrak{it}_1}{2^n} - \binom{n}{2}\frac{2^3\mathfrak{it}_2}{2^{2n}} - \binom{n}{3}\frac{2^6\mathfrak{it}_3}{2^{3n}} - \dots,$$

the inclusion-exclusion principle shows the origin of terms:



Algorithmic aspects

For the asymptotic expansion for connected graphs,

$$p_n = 1 - \binom{n}{1}\frac{2\mathfrak{it}_1}{2^n} - \binom{n}{2}\frac{2^3\mathfrak{it}_2}{2^{2n}} - \binom{n}{3}\frac{2^6\mathfrak{it}_3}{2^{3n}} - \ldots,$$

the inclusion-exclusion principle shows the origin of terms:

Algorithmic aspects

For the asymptotic expansion for connected graphs,

$$p_n = 1 - \binom{n}{1}\frac{2\mathfrak{it}_1}{2^n} - \binom{n}{2}\frac{2^3\mathfrak{it}_2}{2^{2n}} - \binom{n}{3}\frac{2^6\mathfrak{it}_3}{2^{3n}} - \ldots,$$

the inclusion-exclusion principle shows the origin of terms:

Question. Can we create a rejection algorithm for producing connected graphs randomly, so that we reject with a probability of a smaller order?

Erdős-Rényi model

The form of the asymptotic expansion is

$$p_n = 1 - \binom{n}{1}\frac{q^n P_1(\rho)}{q} - \binom{n}{2}\frac{q^{2n} P_2(\rho)}{q^2} - \binom{n}{3}\frac{q^{3n} P_3(\rho)}{q^3} - \ldots$$

Question Can we interpret the coefficients $P_k(\rho)$ as a generalization of irreducible tournaments?

The straightforward generalization fails. Archer, Gessel, Graves and Liang showed that enumeration of tournaments counted by descents uses Eulerian generating functions (instead of exponential ones).

Erdős-Rényi model, continued

The form of the asymptotic expansion is

$$p_n = 1 - \binom{n}{1}\frac{q^n P_1(\rho)}{q} - \binom{n}{2}\frac{q^{2n} P_2(\rho)}{q^2} - \binom{n}{3}\frac{q^{3n} P_3(\rho)}{q^3} - \dots$$

When the parameter p approaches the threshold for connectedness,

$$p = \frac{(1+\varepsilon)\ln n}{n},$$

all terms become equivalent:

$$P_k(\rho)\binom{n}{k}\frac{q^{nk}}{q^{k(k+1)/2}} \sim n^{-\varepsilon k}.$$

Question. Can we build a fruitful theory of phase transition for asymptotic expansions?

Summary

We obtained asymptotic expansions and combinatorial interpretation of the involved constants for probabilities

1. related to constructions SET, SEQ and CYC;
2. of particular combinatorial classes:
    1. connected graphs and irreducible tournaments,
    2. connected square-tiled surfaces and indecomposable permutations,
    3. combinatorial maps and indecomposable perfect matchings,
    4. ...
3. related to virtual species;
4. within the Erdős-Rényi model;
5. of strongly connected directed graphs.

Also, we stated several open problems.

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CYC asymptotics

Theorem

If $\mathcal{V}$ and $\mathcal{W}$ are such combinatorial classes that

- *$\mathcal{V}$ is gargantuan with positive counting sequence,*
- *$\mathcal{V} = \mathrm{CYC}(\mathcal{W})$,*

then

$$r_n := \frac{\mathfrak{w}_n}{\mathfrak{v}_n} \approx 1 - \sum_{k \geqslant 1} \mathfrak{w}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{v}_{n-k}}{\mathfrak{v}_n}.$$

Reasoning: $e^{-y} \xrightarrow{\partial} -e^{-y}$.

SET_m asymptotics

Theorem

If $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{V}^{\{m\}}$, $m \in \mathbb{N}$, are such combinatorial classes that

- *$\mathcal{U}$ is gargantuan with positive counting sequence,*
- $\mathcal{U} = \mathrm{SET}(\mathcal{V})$ *and* $\mathcal{V}^{\{m\}} = \mathrm{SET}_m(\mathcal{V})$,

then

$$p_n^{\{m\}} := \frac{\mathfrak{v}_n^{\{m\}}}{\mathfrak{u}_n} \approx \sum_{k\geqslant 0} \alpha_k^{\{m\}} \cdot \binom{n}{k} \cdot \frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}.$$

where $\alpha_k^{\{m\}}$ are the coefficients of

$$\sum_{n=0}^{\infty} \alpha_k^{\{m\}} \frac{z^n}{n!} = \sum_{s=m-1}^{\infty} (-1)^{s+m-1} \binom{s}{m-1} V^{\{s\}}(z).$$

SEQ_m asymptotics

Theorem

If $\mathcal{U}$, $\mathcal{W}$ and $\mathcal{W}^{(m)}$, $m \in \mathbb{N}$, are such combinatorial classes that

- *$\mathcal{U}$ is gargantuan with positive counting sequence,*
- $\mathcal{U} = \mathrm{SEQ}(\mathcal{W})$ *and* $\mathcal{W}^{(m)} = \mathrm{SEQ}_m(\mathcal{W})$,

then

$$q_n^{(m)} := \frac{\mathfrak{w}_n^{(m)}}{\mathfrak{u}_n} \approx \sum_{k \geqslant 0} \beta_k^{(m)} \cdot \binom{n}{k} \cdot \frac{\mathfrak{u}_{n-k}}{\mathfrak{u}_n}.$$

where $$\beta_k^{(m)} = m\Big(\mathfrak{w}_k^{(m-1)} - 2\mathfrak{w}_k^{(m)} + \mathfrak{w}_k^{(m+1)}\Big).$$

CYC_m asymptotics

Theorem

If $\mathcal{V}$, $\mathcal{W}$, $\mathcal{W}^{[m]}$ and $\mathcal{W}^{(m)}$, $m \in \mathbb{N}$, are such combinatorial classes that

- *$\mathcal{V}$ is gargantuan with positive counting sequence,*
- $\mathcal{V} = \mathrm{CYC}(\mathcal{W})$, $\mathcal{W}^{[m]} = \mathrm{CYC}_m(\mathcal{W})$, $\mathcal{W}^{(m)} = \mathrm{SEQ}_m(\mathcal{W})$,

then

$$r_n^{[m]} := \frac{\mathfrak{w}_n^{[m]}}{\mathfrak{v}_n} \approx \sum_{k \geqslant 0} \gamma_k^{[m]} \cdot \binom{n}{k} \cdot \frac{\mathfrak{v}_{n-k}}{\mathfrak{v}_n}.$$

where $\gamma_k^{[m]} = \mathfrak{w}_k^{(m-1)} - \mathfrak{w}_k^{(m)}$.