Asymptotic probability of irreducibles III: Anti-SEQ

Thierry Monteil[*]   Khaydar Nurligareev[†]

Abstract

In this paper, we apply species theory to the study of large irreducible object asymptotics. We assume that the counting sequence of the combinatorial objects under consideration grows fast enough, while the notion of irreducibility is understood in terms of composition with the species of sets $\mathcal{E}$, cycles $\mathcal{C}$ or linear orders $\mathcal{L}$. Under these conditions, we establish the structure of the complete asymptotic expansion of the probability that a large object is irreducible or comprises a given number of irreducible components. In particular, we interpret the constants involved in the asymptotics as the counting sequence of derivative structures. These structures are described in terms of virtual species, which allow us to analyze a wide range of cases that were not accessible using previously known methods. Applications include the Erdős-Rényi $G(n,p)$ random graph model, as well as various surface and manifold models.

1 Introduction

Our research is devoted to the study of the probability that a large object is irreducible. More precisely, our interest is focused on the asymptotic probability that a combinatorial object of a certain kind is irreducible as its size tends to infinity. The notion of irreducibility is understood here in a broad sense: graphs can be connected, permutations can be indecomposable, polynomials can be irreducible, etc. Methods for establishing asymptotic expansions have been widely studied [3, 10, 31, 18]. However, in most cases, the combinatorial meaning of asymptotic coefficients was hidden until recently. The main goal of this paper is to provide a method to obtain a structure and combinatorial interpretation of asymptotics in cases that cannot be treated with the tools described in our previous work [27, 28]. To this end, we rely on species theory and, in particular, on the concept of virtual species, which provides a formal way to assign negative weights to combinatorial objects and admits inverses for multiplication and substitution.

In order to better perceive the need in this new approach, let us take a look at the obstacles that arose in the earlier steps of our investigation. In the first two papers [27, 28], we understand the notion of irreducibility within the symbolic method [18], in terms

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[*]LIPN, CNRS (UMR 7030), Université Paris 13, F-93430 Villetaneuse, France.
Email: `thierry.monteil@lipn.univ-paris13.fr`

[†]LIP6, CNRS (UMR 7606), Sorbonne Université, F-75005 Paris, France.
LIB, Université de Bourgogne, F-21078 Dijon, France.
LIPN, CNRS (UMR 7030), Université Paris 13, F-93430 Villetaneuse, France.
Email: `khaydar.nurligareev@lip6.fr`

of combinatorial constructions. We consider a labeled combinatorial class $\mathcal{A}$ that admits, for some other class $\mathcal{B}$, one of the following three decompositions: $\mathcal{A} = \mathrm{SET}(\mathcal{B})$, $\mathcal{A} = \mathrm{SEQ}(\mathcal{B})$, or $\mathcal{A} = \mathrm{CYC}(\mathcal{B})$. Assuming that the counting sequence $(\mathfrak{a}_n)$ of the class $\mathcal{A}$ grows fast enough (the exact meaning of these words is captured by the notion of *gargantuan sequence*, see Definition 2.3), we rely on exponential generating functions and Bender's theorem [4] as the key analytical tool. In particular, we show that the asymptotic probability that a random object $a \in \mathcal{A}$ belongs also to the class $\mathcal{B}$ satisfies

$$\mathbb{P}(a \in \mathcal{B}) \approx 1 - \sum_{k \geqslant 1} \mathfrak{d}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{a}_{n-k}}{\mathfrak{a}_n} \tag{1}$$

(see Notations 2.1 and 2.2 for the precise meaning of the symbol $\approx$ and *asymptotic probability* notion, respectively). Here, the sequence $(\mathfrak{d}_n)$ is an integer, and depending on the relation between $\mathcal{A}$ and $\mathcal{B}$, it can be interpreted as follows.

- If $\mathcal{A} = \mathrm{SET}(\mathcal{B})$ and the class $\mathcal{A}$ admits an additional decomposition $\mathcal{A} = \mathrm{SEQ}(\mathcal{D})$, then $(\mathfrak{d}_n)$ is the counting sequence of the class $\mathcal{D}$.

- If $\mathcal{A} = \mathrm{SEQ}(\mathcal{B})$, then $\mathfrak{d}_n = 2\mathfrak{b}_n - \mathfrak{b}_n^{(2)}$, where $(\mathfrak{b}_n)$ and $\left(\mathfrak{b}_n^{(2)}\right)$ are the counting sequences of the classes $\mathcal{B}$ and $\mathcal{B}^2$, respectively.

- If $\mathcal{A} = \mathrm{CYC}(\mathcal{B})$, then $(\mathfrak{d}_n) = (\mathfrak{b}_n)$ is the counting sequence of the class $\mathcal{B}$.

As an example of applications, we obtain the asymptotic probabilities for connected graphs and irreducible tournaments [26]. More examples are discussed in [27, 28].

The result presented by relation (1) has certain disadvantages. Thus, for the construction SEQ, a combinatorial interpretation is possible only in terms of linear combinations of counting sequences of some combinatorial classes. In fact, for the case where $\mathcal{B}$ is the class of labeled tournaments, the coefficient $\mathfrak{d}_2 = -2$ is negative, and therefore, the sequence $(\mathfrak{d}_n)$ cannot be interpreted as a counting sequence itself. Regarding the construction SET, to obtain an interpretation we need a double SET/SEQ decomposition as an additional necessary condition. Without an explicit SEQ decomposition, it is still possible to provide a complete asymptotic expansion of the probability $\mathbb{P}(a \in \mathcal{B})$ with the help of Bender's theorem, but the combinatorial meaning of the involved coefficients remains hidden. Thus, given a decomposition $\mathcal{A} = \mathrm{SET}(\mathcal{B})$, we would like to have an "anti-SEQ" operator that creates a class $\mathcal{D}$ satisfying $\mathcal{A} = \mathrm{SEQ}(\mathcal{D})$ and thus provides the desired combinatorial interpretation.

All of these observations show the limited applicability of the symbolic method in the case where the goal is to interpret asymptotic coefficients combinatorially. The aim of this paper is to cope with the above mentioned disadvantages using species theory. Within this theory, one replaces combinatorial constructions of the symbolic method with substitutions. In particular, the constructions SET, SEQ, and CYC are to be replaced with compositions with species $\mathcal{E}$ of sets, $\mathcal{L}$ of linear orders, and $\mathcal{C}$ of cyclic permutations, respectively. What allows us to reach our goals is the concept of virtual species. In a few words, virtual species is a way to construct combinatorial objects of negative weights, which is similar to extending natural numbers to integers. This concept is also helpful for

building inverses for multiplications and substitutions. Moreover, we can apply virtual species for restricted constructions as well.

All the said above allow us to obtain the following result.

Theorem 3.1. *Let $\mathcal{F} \in \{\mathcal{E}, \mathcal{L}, \mathcal{C}\}$, and let $\mathcal{A}$ be a gargantuan (weighted) species of structures that can be represented as $\mathcal{A} = \mathcal{F} \circ \mathcal{B}$ for some (weighted) species of structures $\mathcal{B}$. Assume that $m, n \in \mathbb{Z}_{\geqslant 0}$, and $s \in \mathcal{A}$ is a random $\mathcal{A}$-structure on $[n]$. In this case,*

$$\mathbb{P}(s \in \mathcal{F}_m \circ \mathcal{B}) \approx \sum_{k \geqslant 0} \mathfrak{d}_{k,m} \cdot \binom{n}{k} \cdot \frac{\mathfrak{a}_{n-k}}{\mathfrak{a}_n},$$

where $\mathfrak{a}_n$ and $\mathfrak{d}_{n,m}$ are the total weights on the set $[n]$ of the species of structures $\mathcal{A}$ and $\mathcal{D} = \mathcal{D}(m)$, respectively, and the (virtual) species $\mathcal{D}$ depends on $\mathcal{F}$ according to Table 1:

$\mathcal{F}$	$\mathcal{F}_m$	$\mathcal{D}(m)$	$\mathcal{D}(1)$
$\mathcal{E}$	$\mathcal{E}_m$	$(\mathcal{E}_{m-1} \circ \mathcal{B})(\mathcal{E}^{-1} \circ \mathcal{B})$	$\mathcal{E}^{-1} \circ \mathcal{B}$
$\mathcal{L}$	$\mathcal{L}_m$	$m\mathcal{B}^{m-1}(\mathbf{1} - \mathcal{B})^2$	$(\mathbf{1} - \mathcal{B})^2$
$\mathcal{C}$	$\mathcal{C}_m$	$\mathcal{B}^{m-1}(\mathbf{1} - \mathcal{B})$	$\mathbf{1} - \mathcal{B}$

Table 1: Correspondence between $\mathcal{F}$ and $\mathcal{D}$.

Theorem 3.1 can be employed in many settings. First of all, it is worth mentioning that it can be applied for all labeled structures discussed in the papers [27, 28]. Thus, we could apply this theorem to obtain a combinatorial interpretation of the complete asymptotic expansions of connected graphs, irreducible tournaments, indecomposable permutations, connected origamis, etc. In contrast, thanks to virtual species, negative coefficients are no longer a problem. One more advantage is that the relation $\mathcal{A} = \mathcal{L} \circ \mathcal{D}$ can be inverted as $\mathcal{D} = \mathcal{L}_+^{(-1)} \circ \mathcal{A}_+$, so that we obtain an "anti-SEQ" operator. We show (Lemma 3.6) that this operator satisfies

$$\mathcal{L}_+^{(-1)} \equiv \mathbf{1} - \mathcal{E}^{-1} \circ \mathcal{E}_+^{(-1)}$$

and thus gives indeed the combinatorial meaning provided by Theorem 3.1.

One of the new applications that cannot be treated using previously known methods is the Erdős–Rényi model $G(n, p)$ of graphs [14, 20]. According to this model, a graph with n vertices is constructed randomly: each possible edge is drawn independently with probability $p \in (0, 1)$. Gilbert [20] showed that the asymptotic probability to obtain a connected graph within $G(n, p)$ satisfies

$$\mathbb{P}(G \text{ is connected}) = 1 - nq^{n-1} + O\left(n^2 q^{3n/2}\right),$$

where $q = 1 - p$. Introducing the weight of a graph g as $w(g) = \rho^\ell$ where ℓ is the number of edges in g and $\rho = p/q$, with the help of Theorem 3.1 we establish the following more general result.

Theorem 4.2. *Let m be a fixed positive integer. The asymptotic probability that a random graph g in the Erdős–Rényi model $G(n, p)$ has m connected components satisfies*

$$\mathbb{P}(g \text{ has } m \text{ connected components}) \approx \sum_{k \geqslant 0} P_{k,m}(\rho) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}},$$

where

$$P_{k,m}(\rho) = \sum_{g' \in \mathcal{G}_k} (-1)^{\pi_0(g') - (m-1)} \binom{\pi_0(g')}{m-1} w(g')$$

and $\pi_0(g')$ is the number of connected components of the graph g'.

According to our previous results [26, 27], in the particular case $m = 1$ and $p = q = 1/2$ the coefficients $-P_{k,1}(1)$ can be interpreted as the counting sequence of irreducible tournaments. We show that this interpretation can be extended to the general case $p \in (0, 1)$, so that polynomials $-P_{k,1}(\rho)$ count generalized irreducible tournaments[1] (Theorem 4.7). Note that if $p < 1/2$, in the model $T(n, p)$ that we introduce for this purpose certain generalized tournaments are purely virtual. This fact certifies that our result cannot be obtained within the symbolic method and that the use of species theory is crucial. We also obtain the asymptotic probability that a random generalized tournament is irreducible or consists of a fixed number of irreducible parts (Theorem 4.9).

Other applications presented in this paper illustrate the use of the above constructed "anti-SEQ" operator for the cases where the symbolic method fails. In particular, we establish a combinatorial interpretation for asymptotic probabilities of quadratic square-tiled surfaces (Proposition 4.12), P-angulated surfaces (Proposition 4.14), and graph encoded manifolds (Proposition 4.16).

The paper is organized as follows. In Section 2, we introduce useful notations and recall the necessary concepts from species theory, as well as some properties of gargantuan sequences and Bender's theorem. In Section 3, we establish our main result, Theorem 3.1, and discuss its combinatorial interpretation and an adaptation for p-periodic sequences. The next part is devoted to applications. In particular, in Section 4.1 we discuss the asymptotic probability of connected graphs and irreducible generalized tournaments within the Erdős–Rényi model, while in Section 4.2 we study asymptotics of quadratic square-tiled surfaces, P-angulated surfaces and graph encoded manifolds. Finally, we conclude the paper with a number of open problems.

2 Tools

2.1 Notation

We use the standard O-notation (see [3]): for a sequence (a_n),

- $O(a_n)$ denotes the set of all sequences (b_n) satisfying $\limsup\limits_{n \to \infty} |b_n / a_n| < \infty$,

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[1]Caveat: in literature, the term *generalized tournament* is usually understood to refer to other objects; see Remark 4.6.

- $o(a_n)$ denotes the set of all sequences (b_n) satisfying $\limsup\limits_{n\to\infty} |b_n/a_n| = 0$.

In particular, equations of the form $b_n = c_n + O(a_n)$ are interpreted as $(b_n - c_n) \in O(a_n)$.

Notation 2.1. For a sequence (a_n) and an integer m, we write

$$a_n \approx \sum_{k \geqslant m} c_k f_k(n),$$

if for every integer $r \geqslant m$,

$$a_n = \sum_{k=m}^{r} c_k f_k(n) + O\big(f_{r+1}(n)\big),$$

and for every integer $k \geqslant m$,

$$f_{k+1}(n) = o\big(f_k(n)\big).$$

Note that the sequence (c_k) of constants may contain zeros.

2.2   Species of structures

In this section, we recall the basic notions of the theory of combinatorial species introduced in 1981 by Joyal [25]. In order to avoid excessive abstraction, we do this in the spirit of the book written by Bergeron, Labelle, and Leroux [5]. Another introductory reference that the reader may find useful is [30, Chapter 4].

2.2.1   The concept of species of structures

A *(combinatorial) species of structures* is a rule F such that

1. for each finite set U, the rule F produces a finite set $F[U]$,

2. for each bijection $\sigma \colon U \to V$, the rule F produces a map

$$F[\sigma] \colon F[U] \to F[V],$$

   satisfying the following properties:

   (a) for all bijections $\sigma \colon U \to V$ and $\tau \colon V \to W$,

$$F[\tau \circ \sigma] = F[\tau] \circ F[\sigma];$$

   (b) for the identity map $\mathrm{Id}_U \colon U \to U$,

$$F[\mathrm{Id}_U] = \mathrm{Id}_{F[U]}.$$

The elements of the set $F[U]$ are called *F-structures on U*, or *structures of F on U*, while the map $F[\sigma]$ is called the *transport function of F-structures along σ*.

The transport of structures along bijections represents a functorial approach to combinatorics. Within this approach, structures can be thought of as labeled by elements of the set U. At the same time, in the case where the nature of these elements itself is not important, we can consider a structure $s \in F[U]$ and its image $F[\sigma](s) \in F[V]$ to be identical. In particular, unless otherwise specified, we assume that the structures considered in this paper are labeled by the elements of the sets $[n]$, where $[n] = \{1, \ldots, n\}$ for any positive integer n, and $[0] = \emptyset$.

Given a species of structures F, a species G is said to be a *subspecies* of F if

1. for each finite set U, we have $G[U] \subseteq F[U]$,

2. for each bijection $\sigma \colon U \to V$, we have $G[\sigma] = F[\sigma]|_{G[U]}$.

In particular, for any species F, the following two subspecies are used quite frequently:

- the subspecies F_+ consisting of F-structures on nonempty sets,

- the subspecies F_n consisting of F-structures on the set $[n]$, where $n \in \mathbb{Z}_{\geqslant 0}$.

For the purpose of enumeration, for a species of structure F, we use its *exponential generating series*

$$F(z) = \sum_{n=0}^{\infty} |F[n]| \frac{z^n}{n!},$$

where $F[n]$ is used as a short version of $F\big[[n]\big]$.

We also introduce the following notion of asymptotic probability.

Notation 2.2. Let F be a species of structures. For a nonnegative integer n, in the case where the subspecies F_n is nonempty, we endow it with the uniform probability $\mathbb{P}_n$: each object $s \in F[n]$ has probability $1/|F[n]|$. Now if Q is some property, we denote by

$$\mathbb{P}(s \text{ satisfies } Q)$$

the sequence of probabilities

$$\Big(\mathbb{P}_n\big\{s \in F[n] \mid s \text{ satisfies } Q\big\}\Big)_{|F[n]|>0}$$

and call it the *asymptotic probability that a random object $s \in F$ has the property Q*.

2.2.2   Particular examples of species

The following species are of particular interest in the frame of this paper (for details, see [25] and [5], as well as [30]). Here, we use the notation of [5], except for the species of permutations that we denote by $\mathcal{P}$ instead of $\mathcal{S}$.

- The *species of sets* $\mathcal{E}$, defined by $\mathcal{E}[U] = \{U\}$ for every finite set U, and equipped with the trivial transport function. The corresponding exponential generating series is
$$\mathcal{E}(z) = e^z.$$

- The species of *linear orders* $\mathcal{L}$, defined by
$$\mathcal{L}[U] = \{\phi_U \colon [n] \to U \mid \phi_U \text{ is bijective}\}$$
for every finite set U (for the appropriate integer n), and equipped with the transport function
$$\mathcal{L}[U] \to \mathcal{L}[V], \qquad \phi_U \mapsto \phi_V = \sigma\phi_U, \tag{2}$$
along the bijection $\sigma \colon U \to V$ of finite sets U and V. The exponential generating series of linear orders is
$$\mathcal{L}(z) = \frac{1}{1-z}.$$

- The species of *permutations* $\mathcal{P}$, defined by
$$\mathcal{P}[U] = \{\psi_U \colon U \to U \mid \psi_U \text{ is bijective}\}$$
for every finite set U, and equipped with the transport function
$$\mathcal{P}[U] \to \mathcal{P}[V], \qquad \psi_U \mapsto \psi_V = \sigma\psi_U\sigma^{-1}, \tag{3}$$
along the bijection $\sigma \colon U \to V$ of finite sets U and V. The exponential generating series $\mathcal{P}(z)$ coincides with that of linear orders:
$$\mathcal{P}(z) = \frac{1}{1-z}.$$

- The species of *cyclic permutations* $\mathcal{C}$, defined as a subspecies of $\mathcal{P}$ that are cycles. Their exponential generating series is
$$\mathcal{C}(z) = \log\frac{1}{1-z}.$$

Note that $\mathcal{Z} = \mathcal{E}_1 = \mathcal{L}_1 = \mathcal{P}_1 = \mathcal{C}_1$ is the species characteristic of singletons. Other particular restrictions are $\mathbf{1} = \mathcal{E}_0 = \mathcal{L}_0 = \mathcal{P}_0$, which represents the species characteristic of the empty set, and $\mathbf{0} = \mathcal{C}_0$, which is the empty species. The corresponding exponential generating series are $\mathcal{Z}(z) = z$, as well as $\mathbf{1}(z) = 1$ and $\mathbf{0}(z) = 0$, respectively. More generally,
$$\mathcal{E}_n(z) = \frac{z^n}{n!} \qquad \text{and} \qquad \mathcal{L}_n(z) = \mathcal{P}_n(z) = z^n.$$

2.2.3 Equipotent and isomorphic species

As we have seen, from the counting point of view the species of linear orders and permutations are similar: the number of structures on $[n]$ is $n!$ in both cases, and their exponential generating series coincide, $\mathcal{L}(z) = \mathcal{P}(z)$. For such type of species, it is said that they are *equipotent*, and one writes $\mathcal{L} \equiv \mathcal{P}$. Formally, two species F and G are *equiponent* if their exponential generating series coincide. In particular, this means that for each finite set U, there is a bijection $\alpha_U \colon F[U] \to G[U]$.

However, we cannot call species $\mathcal{L}$ and $\mathcal{P}$ equal, as the ways in which they behave are different. Let us explain it in the following way. Any two linear orders are *isomorphic* in the sense that one can be obtained from another by a simple relabeling. Formally, for any two linear orders $l_u \in \mathcal{L}[U]$ and $l_v \in \mathcal{L}[V]$ of the same size, there exists a bijection $\sigma \colon U \to V$ such that $l_v = \sigma \cdot l_u$, this is a consequence of (2). On the other hand, as follows from relation (3), two permutations are isomorphic if and only if they are of the same cycle type. Thus, there exist permutations of the same size that are not isomorphic.

In the general case, we say that two species of structures are *isomorphic* or *equal* if, for any finite set U, there is a bijection $\alpha_U \colon F[U] \to G[U]$ such that, for any F-structure $s \in F[U]$ and any bijection $\sigma \colon U \to V$, we have

$$\sigma \cdot \alpha_U(s) = \alpha_V(\sigma \cdot s).$$

That is, the following diagram is commutative:

$$
\begin{array}{ccc}
F[U] & \xrightarrow{\ \alpha_U\ } & G[U] \\
{\scriptstyle F[\sigma]}\downarrow & & \downarrow{\scriptstyle G[\sigma]} \\
F[V] & \xrightarrow{\ \alpha_V\ } & G[V]
\end{array}
$$

2.2.4 Species operations

Here, we briefly describe several operations that are useful for creating new species based on existing ones. We omit the definition of transport functions because it naturally follows from that of sets of structures. The reader can find all the necessary details in [5].

Let F and G be two species of structures.

- An $(F + G)$-structure is either an F-structure or a G-structure. In other words, for each finite set U, the *sum* $(F + G)$ produces a disjoint union

$$(F + G)[U] = F[U] + G[U].$$

Their exponential generating series satisfy

$$(F + G)(z) = F(z) + G(z).$$

- An $(F \cdot G)$-structure is an ordered pair (s, t) such that s is an F-structure and t is a G-structure. More precisely, for each finite set U, the *product* $F \cdot G$ produces a disjoint union of Cartesian products

$$(F \cdot G)[U] = \sum_{\substack{U_F \cup U_G = U \\ U_F \cap U_G = \emptyset}} F[U_F] \times G[U_G].$$

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The corresponding exponential generating series satisfies

$$(F \cdot G)(z) = F(z) \cdot G(z).$$

- Assuming that $G_0 = \emptyset$, we define an $(F \circ G)$-structure as an F-assembly of disjoint G-structures. Formally, for each finite set U, the *composition* $F \circ G$ produces a disjoint union of Cartesian products

$$(F \circ G)[U] = \sum_{\pi \in \text{Par}\,[U]} F[\pi] \times \prod_{p \in \pi} G[p],$$

where $\text{Par}\,[U]$ is the set of partitions of U. Here, we have the relation

$$(F \circ G)(z) = F\big(G(z)\big).$$

- Finally, for each finite set U, the *derivative* F' produces the set

$$F'[U] = F[U^+],$$

where $U^+ = U \cup \{*_U\}$ and $*_U \notin U$. Thus, an F'-structure is essentially an F-structure, but on a larger set enriched by an additional element. The exponential generating series of the corresponding species satisfies

$$F'(z) = \frac{d}{dz}F(z).$$

The species operations mentioned here possess natural properties that the reader would expect, such as $(F+G) \cdot H = F \cdot H + G \cdot H$ (distributive law), $(F \cdot G)' = F' \cdot G + F \cdot G'$ (Leibniz rule), $F \circ \mathcal{Z} = \mathcal{Z} \circ F = F$, etc. The particular species discussed above are also related to each other by means of some of these operations. Thus, we have

$$\mathcal{P} = \mathcal{E} \circ \mathcal{C}$$

and

$$\mathcal{L}_n = \mathcal{Z}^n.$$

We will also use the following relations concerning the derivatives of the species of sets, linear orders, and cyclic permutations, respectively:

$$\mathcal{E}' = \mathcal{E}, \qquad \mathcal{L}' = \mathcal{L}^2, \qquad \mathcal{C}' = \mathcal{L},$$

as well as the derivatives of their restrictions to a given cardinality:

$$(\mathcal{E}_n)' = \mathcal{E}, \qquad (\mathcal{L}_n)' = n\mathcal{L}_{n-1}, \qquad (\mathcal{C}_n)' = \mathcal{L}_{n-1}.$$

2.2.5 Virtual species

The class of species of structures forms a semi-ring with respect to addition $(+)$ and multiplication $(\cdot)$, where zero and one are the species $\mathbf{0}$ and $\mathbf{1}$. That is why it can be extended in a similar manner as the semi-ring $\mathbb{Z}_{\geqslant 0}$ of natural numbers is extended to build the ring $\mathbb{Z}$ of integers. The result of this extension is known as the class of *virtual species*, and is shown to be compatible with other operations, such as composition $(\circ)$ and derivation $(')$.

In addition to the subtraction operation, virtual species have inverses for multiplication and substitution.

- If a (virtual) species F satisfies $F_0 = \mathbf{1}$, then, in the class of virtual species, it admits the unique *multiplicative inverse*:

$$F^{-1} = \sum_{k=0}^{\infty} (-1)^k (F_+)^k.$$

  In the particular case where $F = \mathcal{L}$, this relation can be simplified:

$$\mathcal{L}^{-1} = \mathbf{1} - \mathcal{Z}.$$

  As expected, the exponential generating series of the multiplicative inverse satisfies $F^{-1}(z) = 1/F(z)$.

- If a (virtual) species F satisfies $F_0 = \mathbf{0}$ and $F_1 = \mathcal{Z}$, then, in the class of virtual species, it admits unique *inverse for substitution*:

$$F^{(-1)} = \sum_{k=0}^{\infty} (-1)^k \Delta_F^k(\mathcal{Z}),$$

  where the linear operator Δ_F is defined by the relation $\Delta_F(\Phi) = \Phi \circ F - \Phi$. Its exponential generating series satisfies $F\big(F^{(-1)}(z)\big) = F^{(-1)}\big(F(z)\big) = z$.

Again, these inverses behave exactly as the reader would naturally expect. For instance, taking a derivative of the identity

$$\Psi^{(-1)} \circ \Psi = \mathcal{Z}$$

leads us to the expression for the derivative of the inverse under substitution:

$$(\Psi^{(-1)})' = (\Psi')^{-1} \circ \Psi^{(-1)}.$$

For more detailed information on virtual species, we invite the interested reader to refer to [5, Section 2.5].

2.2.6 Weighted species

It is quite common in enumerative combinatorics to count certain objects according to specific parameters. To this end, marking variables are often used, and weights are

assigned to the objects under consideration. Applying this approach to species theory, we arrive at the concept of weighted species.

Formally, our set of all possible weights is a ring of formal power series $\mathbb{A}$ (in an arbitrary number of variables) with coefficients in an integral domain ($\mathbb{Z}$ or $\mathbb{R}$ in our case). We extend the definition of species discussed above to $\mathbb{A}$-*weighted species* as follows. The set $F[U]$, which is produced for a finite set U, is now $\mathbb{A}$-weighted, finite or summable. The latter means that $F[U]$ is equipped with a weight function $w\colon F[U] \to \mathbb{A}$ preserved by the transport function, and that there exists the sum

$$\big|F[U]\big|_w = \sum_{s \in F[U]} w(s).$$

called the *inventory* of the set of F-structures on U. The value $\big|F[n]\big|_w$ will be referred to as the *total weight* of F-structures on $[n]$.

All of the above concepts can be naturally extended to the weighted case. Thus, the exponential generating series of an $\mathbb{A}$-weighted species is now

$$F(z) = \sum_{n=0}^{\infty} \big|F[n]\big|_w \frac{z^n}{n!}.$$

The weight of a product of two species is defined as the product of their weights, the weight of a composition is the product of the weights of its components, etc. The concept of virtual species matches the weights as well. As usual, for all details, we refer the reader to the book [5].

Note that, in the case where all weights are positive, it is natural to endow $F[n]$ with a discrete probability: a structure $s \in F[n]$ appears with probability

$$\mathbb{P}(s) = \frac{w(s)}{\big|F[n]\big|_w}.$$

2.3   Gargantuan sequences and Bender's theorem

In this section, we introduce the concepts of gargantuan sequence and gargantuan species, which first appeared in [27] and [30, Chapter 2]. We also recall a simplified version of Bender's theorem [4], which serves as the main asymptotic tool in this work.

Definition 2.3. A sequence (a_n) is *gargantuan* if, as $n \to \infty$, the following two conditions hold:

(i) $\dfrac{a_{n-1}}{a_n} \to 0;$ (ii) $\displaystyle\sum_{k=r}^{n-r} |a_k a_{n-k}| = O\big(a_{n-r}\big)$ for each $r \in \mathbb{Z}_{\geqslant 0}$.

Furthermore, let F be a (virtual, weighted) species of structures. We call F *gargantuan* if the sequence $a_n = \big(\big|F[n]\big|/n!\big)$ is gargantuan. In particular, we assume that $a_n \neq 0$ for almost all $n \in \mathbb{Z}_{\geqslant 0}$.

The property of being gargantuan is closed with respect to the point-wise product and to point-wise multiplication with a geometric progression.

Lemma 2.4. *If two sequences (a_n) and (b_n) are gargantuan, then the sequence $(a_n b_n)$ is also gargantuan.*

Proof. Since the first condition of Definition 2.3 holds trivially for the sequence $(a_n b_n)$, it is sufficient to verify the second condition. The second condition is valid, too, because for any nonnegative integer r, we have

$$\sum_{k=r}^{n-r} |(a_k b_k) \cdot (a_{n-k} b_{n-k})| \leqslant \left(\sum_{k=r}^{n-r} |a_k a_{n-k}| \right) \left(\sum_{k=r}^{n-r} |a_k a_{n-k}| \right) \leqslant O(a_{n-r} b_{n-r}).$$

$\square$

Lemma 2.5. *If a sequence (a_n) is gargantuan and $b_n = b_0 c^n$ with $c \neq 0$, then the sequence $(a_n b_n)$ is gargantuan.*

Proof. The first condition of Definition 2.3 holds, since

$$\frac{a_{n-1} b_{n-1}}{a_n b_n} = \frac{1}{c} \cdot \frac{a_{n-1}}{a_n}.$$

As for the second condition, it holds as well, because

$$\sum_{k=r}^{n-r} \left| (a_k b_k) \cdot (a_{n-k} b_{n-k}) \right| = b_0^2 |c^n| \cdot \sum_{k=r}^{n-r} |a_k a_{n-k}| = |b_r b_{n-r}| \cdot O(a_{n-r}) = O(a_{n-r} b_{n-r}).$$

$\square$

Most gargantuan sequences that we will meet in this paper show the same typical behavior. In order to describe this behavior, for a given gargantuan sequence (a_n) and large n, let us introduce a finite sequence $x_k = |a_k a_{n-k}|$. Clearly, (x_k) is symmetric with respect to $k = n/2$. Taking into account the second condition of Definition 2.3, it is not surprising that $x_1 > x_2$, $x_2 > x_3$, etc. However, as we will see later, in most cases the behavior of (x_k) is even more strict, and the inequality $x_k > x_{k+1}$ holds for all $k < n/2$ (Figure 1). We will use this observation through the following lemma.

Lemma 2.6. *If a sequence (a_n) satisfies the following two conditions, as $n \to \infty$,*

(i)' $na_{n-1} = O(a_n)$,

(ii)' $x_k = |a_k a_{n-k}|$ is decreasing for $k < n/2$ and for all but finitely many n,

then (a_n) is gargantuan.

Proof. To verify the first condition of Definition 2.3, it suffices to apply (i)'. To check the second condition, note that the value of x_k decreases for $k < n/2$ and increases otherwise. Moreover, (x_k) is symmetric with respect to $k = n/2$. Hence, from (i)' and (ii)' it follows that

$$\sum_{k=r}^{n-r} |a_k a_{n-k}| = 2x_r + \sum_{k=r+1}^{n-r-1} x_k \leqslant 2x_r + (n-2r-1)x_{r+1} = O(x_r) = O(a_{n-r}).$$
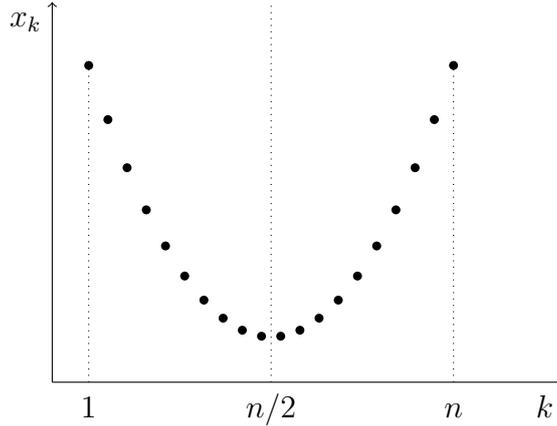
$\square$

Figure 1: Typical behavior of the sequence (x_k).

The following result is an adaptation of Bender's theorem [4], see also [31].

Theorem 2.7. *Consider a formal power series*

$$U(z) = \sum_{n=1}^{\infty} u_n z^n$$

and a function $F(x)$ analytic in a neighborhood of origin. Define

$$V(z) = \sum_{n=0}^{\infty} v_n z^n = F\big(U(z)\big) \qquad and \qquad W(z) = \sum_{n=0}^{\infty} w_n z^n = \frac{\partial}{\partial x} F(x) \bigg|_{x=U(z)}.$$

Assume that $u_n \neq 0$ for all sufficiently large n, and that the sequence (u_n) is gargantuan. In this case,

$$v_n \approx \sum_{k \geqslant 0} w_k u_{n-k}$$

and the sequence (v_n) is gargantuan.

3   Main result

This section is devoted to our main asymptotic result. Given a species of structures $\mathcal{A}$ that can be represented as a composite of two other species, $\mathcal{A} = \mathcal{F} \circ \mathcal{B}$, our objective is to describe the asymptotic behavior of its subspecies $\mathcal{F}_m \circ \mathcal{B}$ for any positive integer m. More precisely, the following three cases are of interest to us.

- The case $\mathcal{F} = \mathcal{E}$ corresponds to the concept of connectivity, which means that species of structures $\mathcal{B}$ can be interpreted as connected components of species of structures $\mathcal{A}$. Thus, $\mathcal{E}_m \circ \mathcal{B}$ represents subspecies with exactly m connected components. For simplicity, we will denote these subspecies by $\mathcal{B}^{\{m\}}$.

- If $\mathcal{F} = \mathcal{L}$, then the species of structures $\mathcal{A}$ are represented as sequences of the species of structures $\mathcal{B}$. Here, the target objects are $\mathcal{L}_m \circ \mathcal{B} = \mathcal{B}^m$, that is, sequences of $\mathcal{B}$-structures of length m.

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- Finally, the equality $\mathcal{F} = \mathcal{C}$ means that the species of structures $\mathcal{A}$ can be interpreted as cycles of the species of structures $\mathcal{B}$. In this case, we study the behavior of the subspecies of structures $\mathcal{C}_m \circ \mathcal{B}$ corresponding to cycles of $\mathcal{B}$-structures of length m.

In particular, in all the above cases, the equality $m = 1$ means that we look at the asymptotic behavior of the species of structures $\mathcal{B}$ as subspecies of $\mathcal{A}$.

3.1 Asymptotic theorem

Theorem 3.1. *Let $\mathcal{F} \in \{\mathcal{E}, \mathcal{L}, \mathcal{C}\}$, and let $\mathcal{A}$ be a gargantuan (weighted) species of structures that can be represented as $\mathcal{A} = \mathcal{F} \circ \mathcal{B}$ for some (weighted) species of structures $\mathcal{B}$. Assume that $m, n \in \mathbb{Z}_{\geqslant 0}$, and $s \in \mathcal{A}$ is a random $\mathcal{A}$-structure on $[n]$. In this case,*

$$\mathbb{P}(s \in \mathcal{F}_m \circ \mathcal{B}) \approx \sum_{k \geqslant 0} \mathfrak{d}_{k,m} \cdot \binom{n}{k} \cdot \frac{\mathfrak{a}_{n-k}}{\mathfrak{a}_n}, \tag{4}$$

where $\mathfrak{a}_n$ and $\mathfrak{d}_{n,m}$ are the total weights on the set $[n]$ of the species of structures $\mathcal{A}$ and $\mathcal{D} = \mathcal{D}(m)$, respectively, and the (virtual) species $\mathcal{D}$ depends on $\mathcal{F}$ according to Table 1:

$\mathcal{F}$	$\mathcal{F}_m$	$\mathcal{D}(m)$	$\mathcal{D}(1)$
$\mathcal{E}$	$\mathcal{E}_m$	$\mathcal{B}^{\{m-1\}}(\mathcal{E}^{-1} \circ \mathcal{B})$	$\mathcal{E}^{-1} \circ \mathcal{B}$
$\mathcal{L}$	$\mathcal{L}_m$	$m\mathcal{B}^{m-1}(\mathbf{1} - \mathcal{B})^2$	$(\mathbf{1} - \mathcal{B})^2$
$\mathcal{C}$	$\mathcal{C}_m$	$\mathcal{B}^{m-1}(\mathbf{1} - \mathcal{B})$	$\mathbf{1} - \mathcal{B}$

Table 1: Correspondence between $\mathcal{F}$ and $\mathcal{D}$.

Proof. The main idea of the proof is to apply Theorem 2.7. Let us show how this works in the case where $\mathcal{A} = \mathcal{L} \circ \mathcal{B}$. First, we express the species $\mathcal{B}^m = \mathcal{L}_m \circ \mathcal{B}$ via $\mathcal{A}_+ = \mathcal{A} - \mathbf{1}$:

$$\mathcal{A}_+ = (\mathcal{L}_+ + \mathbf{1}) \circ \mathcal{B} - \mathbf{1} = \mathcal{L}_+ \circ \mathcal{B},$$

and hence,

$$\mathcal{B} = \mathcal{L}_+^{(-1)} \circ \mathcal{A}_+ \qquad \text{and} \qquad \mathcal{B}^m = \mathcal{L}_m \circ \mathcal{L}_+^{(-1)} \circ \mathcal{A}_+.$$

Second, we observe that the exponential generating series

$$\left(\mathcal{L}_m \circ \mathcal{L}_+^{(-1)}\right)(z) = \left(\frac{z}{1+z}\right)^m$$

is analytic in some neighborhood of origin. Therefore, Theorem 2.7 is applicable to the exponential generating series of $\mathcal{U} = \mathcal{A}_+$ and $F = \mathcal{L}_m \circ \mathcal{L}_+^{(-1)}$ taken as the formal power series $U(z)$ and the analytic function $F(x)$, respectively. Taking into account that $F\big(U(z)\big)$ is the exponential generating series of the species $\mathcal{V} = \mathcal{B}^m$, this gives

$$\mathfrak{b}_n^{(m)} \approx \sum_{k \geqslant 0} \binom{n}{k} \mathfrak{d}_{k,m} \mathfrak{a}_{n-k}, \tag{5}$$

where $\mathfrak{b}_k^{(m)}$ and $\mathfrak{d}_{k,m}$ are the total weights on the set $[k]$ of the species of structures $\mathcal{B}^m$ and $\mathcal{D}(m) = F' \circ \mathcal{A}_+$, respectively. Since

$$F' = \left(\mathcal{L}_m \circ \mathcal{L}_+^{(-1)}\right)' = \left(m\mathcal{L}_{m-1} \circ \mathcal{L}_+^{(-1)}\right) \cdot \left(\left((\mathcal{L}_+)'\right)^{-1} \circ \mathcal{L}_+^{(-1)}\right),$$

and taking into account that $(\mathcal{L}_+)' = \mathcal{L}^2$ and $\mathcal{L}^{-1} = \mathbf{1} - \mathcal{Z}$, we have

$$\mathcal{D} = \left(\mathcal{L}_m \circ \mathcal{L}_+^{(-1)}\right)' \circ \mathcal{A}_+ = m\mathcal{B}^{m-1} \cdot (\mathbf{1} - \mathcal{B})^2.$$

Now, using the relation $\mathbb{P}(s \in \mathcal{B}) = \mathfrak{b}_n^{(m)}/\mathfrak{a}_n$, we divide both sides of formula (5) by $\mathfrak{a}_n$ to complete the proof.

In other two cases, the proof follows the same scheme. The key steps are reflected in Table 2. For details, we refer the reader to [30]. $\qquad\square$

$\mathcal{F}$	$\mathcal{A}_0$	F	$F(x)$	F'
$\mathcal{E}$	$\mathbf{1}$	$\mathcal{E}_m \circ \mathcal{E}_+^{-1}$	$\dfrac{\log^m(1+x)}{m!}$	$\left(\mathcal{E}_{m-1} \circ \mathcal{E}_+^{(-1)}\right) \cdot \left(\mathcal{E}^{-1} \circ \mathcal{E}_+^{(-1)}\right)$
$\mathcal{L}$	$\mathbf{1}$	$\mathcal{L}_m \circ \mathcal{L}_+^{-1}$	$\left(\dfrac{x}{1+x}\right)^m$	$\left(m\mathcal{L}_{m-1} \circ \mathcal{L}_+^{(-1)}\right) \cdot \left(\mathcal{L}^{-2} \circ \mathcal{L}_+^{(-1)}\right)$
$\mathcal{C}$	$\mathbf{0}$	$\mathcal{C}_m \circ \mathcal{C}^{(-1)}$	$\dfrac{(1-e^x)^m}{m}$	$\left(\mathcal{L}_{m-1} \circ \mathcal{C}^{(-1)}\right) \cdot \left(\mathcal{L}^{-1} \circ \mathcal{C}^{(-1)}\right)$

Table 2: Proof key steps summary.

Corollary 3.2. *If $\mathfrak{a}_1 \neq 0$, then the leading term of asymptotic expansion (4) satisfies*

$$\mathbb{P}(s \in \mathcal{F}_m \circ \mathcal{B}) = c(\mathcal{F}) \cdot (n)_{m-1} \cdot \frac{\mathfrak{a}_1^{m-1}\mathfrak{a}_{n-m+1}}{\mathfrak{a}_n} + O\left(n^m \cdot \frac{\mathfrak{a}_{n-m}}{\mathfrak{a}_n}\right), \qquad (6)$$

where $(n)_{m-1} = n(n-1)\ldots(n-m+2)$ are the falling factorials. In particular, for $\mathcal{F} \in \{\mathcal{E}, \mathcal{L}, \mathcal{C}\}$ the constants are $c(\mathcal{E}) = 1/(m-1)!$, $c(\mathcal{L}) = m$, and $c(\mathcal{C}) = 1$.

Proof. Given a fixed positive integer m, the leading term of (4) corresponds to the first nonzero element of the sequence $(\mathfrak{d}_{k,m})$. The behavior of this element is determined by the generation series of the species $\mathcal{D}(m)$ (according to Table 1). Relation (6) now follows from the fact that

$$\mathcal{B}^{m-1}(z) = (\mathfrak{a}_1 z + \ldots)^{m-1} = \mathfrak{a}_1^{m-1}(m-1)! \cdot \frac{z^{m-1}}{(m-1)!} + O(z^m),$$

and that the first nonzero element is $\mathfrak{d}_{m-1,m} = c(\mathcal{F}) \cdot \mathfrak{a}_1^{m-1}(m-1)!$. $\qquad\square$

Remark 3.3. Theorem 3.1 can be applied even if the species $\mathcal{A}$ is virtual. In this case, it is no longer possible to discuss probabilities, so the left side of asymptotic relation (4) should be replaced by the quotient $\mathfrak{b}_n^{(m)}/\mathfrak{a}_n$, where $\mathfrak{a}_n$ and $\mathfrak{b}_n^{(m)}$ are the total weights of, respectively, species $\mathcal{A}$ and $\mathcal{F}_m \circ \mathcal{B}$ on the set $[n]$.

3.2 Asymptotic theorem interpretation

The constants $\mathfrak{d}_{k,m}$ appearing in the statement of Theorem 3.1 possess a combinatorial interpretation in terms of the species of structures $\mathcal{B}$ that serves as a building block. In the cases where these blocks form sequences or cycles, that is, where $\mathcal{F} = \mathcal{L}$ or $\mathcal{F} = \mathcal{C}$, the interpretation is rather straightforward. In fact, it is seen from Table 1 that the species of structures $\mathcal{D}(m)$ are virtual, and the structures themselves are linear combinations of $\mathcal{B}$-structure sequences of certain lengths.

In order to provide such a meaning for sets, that is, for the case where $\mathcal{F} = \mathcal{E}$, we need to first interpret the species of structures $\mathcal{E}^{-1}$.

Lemma 3.4. *The virtual species of structures $\mathcal{E}^{-1}$ is equipotent to the alternating sum*

$$\tilde{\mathcal{E}} = \mathbf{1} - \mathcal{E}_1 + \mathcal{E}_2 - \mathcal{E}_3 + \dots, \tag{7}$$

meaning that their exponential generating series coincide.

Proof. The corresponding exponential generating series both are equal to

$$\frac{1}{e^z} = e^{-z} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \dots$$

$\square$

Remark 3.5. The species $\mathcal{E}^{-1}$ and $\tilde{\mathcal{E}}$ indicated in Lemma 3.4 are not equal. To see this, it suffices to consider their restrictions on cardinality 2. Here, we have $(\mathcal{E}^{-1})_1 = \mathcal{E}_1^2 - \mathcal{E}_2$ and $\tilde{\mathcal{E}}_1 = \mathcal{E}_2$, but $\mathcal{E}_1^2 \neq 2\mathcal{E}_2$.

Lemma 3.4 allows us to think of the coefficients of the species $\mathcal{D}(1) = \mathcal{E}^{-1} \circ \mathcal{B}$ as the coefficients of the alternating sum

$$\tilde{\mathcal{E}} \circ \mathcal{B} = \mathbf{1} - \mathcal{E}_1 \circ \mathcal{B} + \mathcal{E}_2 \circ \mathcal{B} - \mathcal{E}_3 \circ \mathcal{B} + \dots = \mathbf{1} - \mathcal{B}^{\{1\}} + \mathcal{B}^{\{2\}} - \mathcal{B}^{\{3\}} + \dots$$

The latter relation can be interpreted similarly to

$$\mathcal{A} = \mathcal{E} \circ \mathcal{B} = \mathbf{1} + \mathcal{B}^{\{1\}} + \mathcal{B}^{\{2\}} + \mathcal{B}^{\{3\}} + \dots,$$

where $\mathcal{B}$-structures are thought of as connected $\mathcal{A}$-structures. Indeed, the species $\tilde{\mathcal{E}} \circ \mathcal{B}$ consists of the same structures as $\mathcal{A}$, but a structure is considered negative if the number of its connected components is odd. This provides a meaning for the coefficients of the virtual species of structures $\mathcal{D}(1)$.

More generally, the species $\mathcal{B}^{\{m-1\}} \cdot (\mathcal{E}^{-1} \circ \mathcal{B})$ is equipotent to the alternate sum

$$\mathcal{B}^{\{m-1\}} - \mathcal{B}^{\{m-1\}} \cdot \mathcal{B}^{\{1\}} + \mathcal{B}^{\{m-1\}} \cdot \mathcal{B}^{\{2\}} - \mathcal{B}^{\{m-1\}} \cdot \mathcal{B}^{\{3\}} + \dots$$

Therefore, the coefficients $\mathfrak{d}_{k,m}$ can be thought of as the total weights on $[k]$ of the structures of $\mathcal{A}$ whose connected components are divided into two groups such that the first group contains exactly $(m-1)$ connected components. Note that a structure is considered negative here if the number of connected components in its second group is odd. In particular, from this interpretation it is seen that the dominant term of asymptotics (4) is determined by the weight of $\mathcal{B}^{\{m-1\}}$-structures of minimal size, that is, $\mathcal{A}$-structures with $(m-1)$ connected components whose size is minimal (compare with Corollary 3.2).

3.3 Building an "anti-SEQ" operator

The main result of the paper [27] allows us to obtain a combinatorial interpretation of the asymptotic probabilities of connected objects for gargantuan combinatorial classes that admit a double decomposition: $\mathcal{A} = \text{SET}(\mathcal{B}) = \text{SEQ}(\mathcal{D})$. In the case where the class $\mathcal{A} = \text{SET}(\mathcal{B})$ has no explicit SEQ decomposition, it would be desirable to generate one and thus obtain an interpretation. In other words, having a gargantuan combinatorial class $\mathcal{A}$, we would like to create an "anti-SEQ" operator that produces a class $\mathcal{D}$ such that $\mathcal{A} = \text{SEQ}(\mathcal{D})$. In this section, we reach this goal using species theory.

In the species language, combinatorial constructions SET and SEQ are translated, respectively, as species $\mathcal{E}$ of sets and $\mathcal{L}$ of linear orders. Thanks to virtual species, we have inverses for composition and, therefore, the relation $\mathcal{A} = \mathcal{L} \circ \mathcal{D}$ can be inverted as $\mathcal{D} = \mathcal{L}_+^{(-1)} \circ \mathcal{A}_+$. Now, the key step consists of the following lemma.

Lemma 3.6. *The virtual species $\mathcal{E}^{-1} \circ \mathcal{E}_+^{(-1)}$ and $\left(1 - \mathcal{L}_+^{(-1)}\right)$ are equipotent.*

Proof. Straightforward calculations show that

$$\mathcal{E}^{-1}(z) = e^{-z}, \qquad \mathcal{E}_+^{(-1)}(z) = \log(1 + z), \qquad \mathcal{L}_+^{(-1)}(z) = \frac{z}{1+z},$$

and hence,

$$\mathcal{E}^{-1}\big(\mathcal{E}_+^{(-1)}(z)\big) + \mathcal{L}_+^{(-1)}(z) = \frac{1}{1+z} + \frac{z}{1+z} = 1.$$

$\square$

Thus, the result of Lemma 3.6 can be interpreted as constructing an "anti-SEQ" operator of the form

$$\mathcal{L}_+^{(-1)} \equiv 1 - \mathcal{E}^{-1} \circ \mathcal{E}_+^{(-1)}.$$

In particular, in view of Theorem 3.1, if $\mathcal{B} = \mathcal{E} \circ \mathcal{A}$, then the asymptotic behavior of $\mathcal{B}$ is encoded by the species

$$\mathcal{D}(1) = \mathcal{E}^{-1} \circ \mathcal{B} \equiv \left(1 - \mathcal{L}_+^{(-1)}\right) \circ \mathcal{A}_+.$$

3.4 Adaptation for p-periodic sequences

In the settings of Theorem 3.1, the sequence (a_n) of the total weights of the species $\mathcal{A}$ is allowed to have finitely many zeros. In practice, however, it happens that (a_n) contains an infinite number of zeros, as, for example, in the case of perfect matchings or triangulated surfaces. Formally, Theorem 3.1 cannot be applied in these circumstances. Nevertheless, the underlying asymptotic approach can still be used due to the regularity of the zero distribution guaranteed by the other conditions of Theorem 3.1. More precisely, it can be shown [35] that if two species of structures $\mathcal{A}$ and $\mathcal{B}$ satisfy $\mathcal{A} = \mathcal{F} \circ \mathcal{B}$ where $\mathcal{F} \in \{\mathcal{E}, \mathcal{L}, \mathcal{C}\}$, then the sequence $(\mathfrak{a}_n)$ is a *p-periodic* for some positive integer p, *i.e.*

- $\mathfrak{a}_n \neq 0$ for $n = pk$, where k is sufficiently large,

- $\mathfrak{a}_n = 0$ in any other case.

Thus, for species whose sequences of total weight are p-periodic, Theorem 3.1 can be adapted in the following way.

Proposition 3.7. *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{F}$ be three (weighted) species of structures that satisfy $\mathcal{A} = \mathcal{F} \circ \mathcal{B}$. In addition, let the sequence $(\mathfrak{a}_n)$ of the total weights of the species of structures $\mathcal{A}$ be p-periodic for some positive integer p, and let the sequence $(\mathfrak{a}_{pn}/(pn)!)$ be gargantuan. Suppose that $m, n \in \mathbb{Z}_{\geqslant 0}$ and $s \in \mathcal{A}$ is a random $\mathcal{A}$-structure on $[n]$. In this case,*

$$\mathbb{P}(s \in \mathcal{F}_m \circ \mathcal{B}) \approx \sum_{k \geqslant 0} \mathfrak{d}_{pk,m} \cdot \binom{pn}{pk} \cdot \frac{\mathfrak{a}_{p(n-k)}}{\mathfrak{a}_{pn}}, \tag{8}$$

where the species of structures $\mathcal{D} = \mathcal{D}(m)$ depends on $\mathcal{F}$ according to Table 1, and the numbers $\mathfrak{d}_{n,m}$ are the total weights on $[n]$ of the species $\mathcal{D}$.

Proof. Here, we apply Theorem 2.7 to the formal power series

$$U(z) = \mathcal{A}_+(z^{1/p}) = \sum_{n=0}^{\infty} \mathfrak{a}_{np} \frac{z^n}{(np)!}$$

and the analytic function $F(x)$ indicated in Table 2. Similarly to the proof of Theorem 3.1, we ensure that the conditions of Theorem 2.7 hold and the same reasoning provides the desired result. $\qquad\square$

Corollary 3.8. *If $\mathfrak{a}_p \neq 0$, then the leading term of asymptotic expansion (8) satisfies*

$$\mathbb{P}(s \in \mathcal{F}_m \circ \mathcal{B}) = c(\mathcal{F}) \cdot \frac{(pn)_{p(m-1)}}{(p!)^{m-1}} \cdot \frac{\mathfrak{a}_p^{m-1} \mathfrak{a}_{p(n-m+1)}}{\mathfrak{a}_{pn}} + O\left(n^{pm} \cdot \frac{\mathfrak{a}_{p(n-m)}}{\mathfrak{a}_{pn}}\right). \tag{9}$$

where $(pn)_{p(m-1)} = pn(pn-1)\ldots\big(p(n-m+1)+1\big)$ are the falling factorials. In particular, for $\mathcal{F} \in \{\mathcal{E}, \mathcal{L}, \mathcal{C}\}$ the constants are $c(\mathcal{E}) = 1/(m-1)!$, $c(\mathcal{L}) = m$, and $c(\mathcal{C}) = 1$.

Proof. Since the sequence $(\mathfrak{a}_n)$ is p-periodic, we have

$$\mathcal{B}^{m-1}(z) = \left(\mathfrak{a}_p \frac{z^p}{p!} + \ldots\right)^{m-1} = \frac{\mathfrak{a}_p^{m-1}}{(p!)^{m-1}} \cdot \big(p(m-1)\big)! \cdot \frac{z^{p(m-1)}}{\big(p(m-1)\big)!} + O(z^{pm}).$$

Therefore, the first nonzero element of the sequence $(\mathfrak{d}_{pk,m})$ corresponds to $k = m - 1$ and equals

$$\mathfrak{d}_{p(m-1),m} = c(\mathcal{F}) \cdot \frac{\mathfrak{a}_p^{m-1}}{(p!)^{m-1}} \cdot \big(p(m-1)\big)!.$$

To complete the proof, it suffices to note that

$$\mathfrak{d}_{p(m-1),m} \cdot \binom{pn}{p(m-1)} \cdot \frac{\mathfrak{a}_{p(n-m+1)}}{\mathfrak{a}_{pn}} = c(\mathcal{F}) \cdot \frac{(pn)_{p(m-1)}}{(p!)^{m-1}} \cdot \frac{\mathfrak{a}_p^{m-1} \mathfrak{a}_{p(n-m+1)}}{\mathfrak{a}_{pn}}.$$

$\qquad\square$

4 Applications

Theorem 3.1 and Proposition 3.7 generalize the asymptotic results obtained for labeled combinatorial classes in our papers [27] and [28]. Therefore, in principle, all the applications discussed in these papers can be handled within species theory. In particular, we could obtain this way the complete asymptotic expansions for connected graphs, irreducible tournaments, indecomposable permutations, etc. We will not repeat these results here. Instead, the paper focus is on the applications that cannot be treated using combinatorial class approach. The first of them concerns the Erdős-Rényi graph model, while the second refers to various models of surfaces and manifolds.

4.1 The Erdős-Rényi model

Let $p \in (0,1)$ be a fixed real number, and let $n \in \mathbb{Z}_{\geqslant 0}$. Here we consider the *Erdős–Rényi model* $G(n,p)$ of a random labeled graph: its set of vertices is $[n]$, and each pair of vertices is joined by an edge independently with probability p. Thus, a particular graph with n vertices and k edges is picked with probability

$$p^k q^{\binom{n}{2}-k},$$

where $q = 1 - p$.

Take a random graph g in the Erdős–Rényi model $G(n,p)$. Our goal is to provide and interpret the asymptotic probability that this graph is connected as p is fixed and n tends to infinity. To this end, we introduce a weighted species of graphs compatible with the model and employ our asymptotic results described in Section 3.

4.1.1 Weighted species of graphs

Let, as before, $p \in (0,1)$, and $q = 1 - p$. We also define an additional parameter ρ as their quotient,

$$\rho = \frac{p}{q} = \frac{1}{q} - 1,$$

and consider the *weighted species $\mathcal{G}$ of graphs*, that is, the species $\mathcal{G}$ of graphs with the weight assigned to a graph g to be

$$w(g) = \rho^{|E(g)|},$$

where $|E(g)|$ is the number of edges in the graph g. For example, the weights of all graphs of size at most 3 are indicated in Fig. 2.

The following two facts will be useful for our investigation. First, the species $\mathcal{G}$ satisfies the relation

$$\mathcal{G} = \mathcal{E} \circ \mathcal{CG}, \tag{10}$$

where $\mathcal{CG} \subset \mathcal{G}$ is the subspecies of *connected graphs*. Second, the total weight of the species of graphs on $[n]$ is $(\rho+1)^{\binom{n}{2}} = q^{-\binom{n}{2}}$.

We will also need the following lemma.

Lemma 4.1. *The weighted species $\mathcal{G}$ of graphs is gargantuan.*
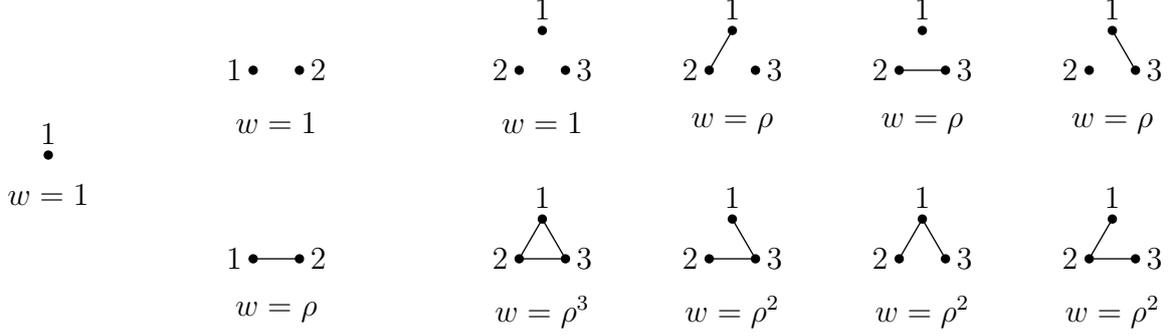
Figure 2: Weights of the labeled graphs whose size does not exceed 3.

Proof. According to Definition 2.3, our goal is to show that the sequence

$$a_n = \frac{|\mathcal{G}[n]|}{n!} = \frac{(\rho+1)^{\binom{n}{2}}}{n!}$$

is gargantuan. The proof idea is to apply Lemma 2.6 by checking its two conditions. The first of them holds, since

$$\frac{n a_{n-1}}{a_n} = \frac{n}{(\rho+1)^n} \to 0,$$

as $n \to \infty$. In order to verify the second one, let us show that the sequence

$$x_k = a_k a_{n-k}$$

decreases for $k < n/2$. For large n, we have

$$\frac{x_{k+1}}{x_k} \leqslant 1 \quad \Leftrightarrow \quad \frac{(\rho+1)^{k+1}}{(k+1)} \leqslant \frac{(\rho+1)^{n-k}}{(n-k)} \quad \Leftrightarrow \quad k+1 \leqslant n-k,$$

since the function

$$f(x) = \frac{(\rho+1)^x}{x}$$

is increasing for large x. Therefore, Lemma 2.6 is applicable and $\mathcal{G}$ is gargantuan. $\quad\square$

4.1.2 Asymptotic probability of connected graphs

Theorem 4.2. *Let m be a fixed positive integer. The asymptotic probability that a random graph g in the Erdős–Rényi model $G(n,p)$ has m connected components satisfies*

$$\mathbb{P}(g \text{ has } m \text{ connected components}) \approx \sum_{k \geqslant 0} P_{k,m}(\rho) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}}, \qquad (11)$$

where

$$P_{k,m}(\rho) = \sum_{g' \in \mathcal{G}_k} (-1)^{\pi_0(g') - (m-1)} \binom{\pi_0(g')}{m-1} w(g')$$

and $\pi_0(g')$ is the number of connected components of the graph g'.

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Proof. The main idea is to apply Theorem 3.1 to the case where $\mathcal{G} = \mathcal{E} \circ \mathcal{CG}$. This is possible, since the species $\mathcal{G}$ is gargantuan due to Lemma 4.1, and Theorem 3.1 gives us

$$\mathbb{P}(g \in \mathcal{E}_m \circ \mathcal{CG}) \approx \sum_{k \geqslant 0} P_{k,m}(\rho) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}},$$

where $P_{k,m}(\rho)$ is the total weight of the species $\mathcal{CG}^{\{m-1\}}(\mathcal{E}^{-1} \circ \mathcal{CG})$ on $[k]$. Now to finish the proof, we take into account that $\mathcal{E}_m \circ \mathcal{CG} = \mathcal{CG}^{\{m\}}$ is the species of graphs with exactly m connected components and that

$$\mathcal{E}^{-1} \circ \mathcal{CG} \equiv \mathbf{1} - \mathcal{CG} + \mathcal{CG}^{\{2\}} - \mathcal{CG}^{\{3\}} + \dots$$

(see Lemma 3.4). $\qquad \square$

Corollary 4.3. *The dominant term in asymptotics* (11) *is*

$$\mathbb{P}(g \text{ has } m \text{ connected components}) = \binom{n}{m-1} \cdot \frac{q^{n(m-1)}}{q^{m(m-1)/2}} + O(n^m q^{mn}).$$

Proof. This follows from Corollary 3.2 and the fact that $q^{-\binom{1}{2}} = 1$. $\qquad \square$

For small values of m and k, polynomials $P_{k,m}(\rho)$ are listed in Table 3. We can see that the sum of the coefficients in the kth column is zero for every $k > 0$, while the sum of the zeroth column is 1. This observation can be explained by the fact that the sum of probabilities of the form (11) taken over all positive integers m is equal to 1. That is to say, for any graph there is a unique $m \in \mathbb{Z}_{>0}$ such that m is the number of connected components of this graph.

k	0	1	2	3	4
$P_{k,1}(\rho)$	1	-1	$-\rho+1$	$-\rho^3 - 3\rho^2 + 3\rho - 1$	$-\rho^6 - 6\rho^5 - 15\rho^4 - 12\rho^3 + 15\rho^2 - 6\rho + 1$
$P_{k,2}(\rho)$	0	1	$\rho-2$	$\rho^3 + 3\rho^2 - 6\rho + 3$	$\rho^6 + 6\rho^5 + 15\rho^4 + 8\rho^3 - 30\rho^2 + 18\rho - 4$
$P_{k,3}(\rho)$	0	0	1	$3\rho - 3$	$4\rho^3 + 15\rho^2 - 18\rho + 6$
$P_{k,4}(\rho)$	0	0	0	1	$6\rho - 4$
$P_{k,5}(\rho)$	0	0	0	0	1

Table 3: Polynomials $P_{k,m}(\rho)$ for $k \leqslant 4$ and $m \leqslant 5$.

Example 4.4. Observe the outcome of Theorem 4.2 for connected graphs, that is, for $m = 1$. In this case, the statement reads: the asymptotic probability that a random graph g in the Erdős–Rényi model $G(n, p)$ is connected satisfies

$$\mathbb{P}(g \text{ is connected}) \approx 1 - \sum_{k \geqslant 1} P_k(\rho) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}}, \tag{12}$$

where

$$P_k(\rho) = -P_{k,1}(\rho) = \sum_{g' \in \mathcal{G}_k} (-1)^{\pi_0(g')-1} w(g'). \tag{13}$$

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In other words, the coefficient $P_k(\rho)$ is the sum of the weights of all (virtual) graphs of size k, where a weight is taken positive if and only if the number of connected components of the corresponding graph is odd (otherwise, the weight is negative). It can be seen from Fig. 2 that

$$P_1(\rho) = 1, \qquad P_2(\rho) = \rho - 1, \qquad P_3(\rho) = \rho^3 + 3\rho^2 - 3\rho + 1,$$

and hence,

$$\mathbb{P}(g \text{ is connected}) =$$
$$1 - \binom{n}{1} q^{n-1} - (\rho - 1) \binom{n}{2} q^{2n-3} - (\rho^3 + 3\rho^2 - 3\rho + 1) \binom{n}{3} q^{3n-6} + O(n^4 q^{4n}).$$

4.1.3 Weighted species of generalized tournaments

The asymptotic probability coefficients $P_k(\rho)$ described by relation (12) have a clear combinatorial meaning in the case where $\rho = 1$, that is, in the case where graphs of the same size are taken uniformly, with the same probability. Namely [26],

$$P_k(1) = \mathfrak{it}_k$$

is the number of irreducible tournaments[2] of size k. The goal of this section is to generalize the concept of irreducible tournament, so that it fits the interpretation of asymptotic coefficients within the Erdős-Rényi model for any positive ρ.

A clue to this issue consists in passing to a discrete generalization first. For any $d \geqslant 1$, we consider labeled d-multigraphs and d-multitournaments:

- in a *d-multigraph*, each pair of (distinct) vertices is joined by at most d indistinguishable edges,

- in a *d-multitournament*, each pair of vertices $i \neq j$ is joined by d directed edges (for some $l \in \{0, \ldots, d\}$ that varies from pair to pair, l of d edges are directed from i to j, the other $d - l$ edges are directed from j to i, and all co-directed edges are indistinguishable).

As we showed in our previous paper [27], the asymptotic probability that a random labeled d-multigraph g is connected satisfies

$$\mathbb{P}\big(g \text{ is connected}\big) \approx 1 - \sum_{k \geqslant 1} \mathfrak{it}_k(d) \cdot \binom{n}{k} \cdot \frac{(d+1)^{k(k+1)/2}}{(d+1)^{kn}}, \tag{14}$$

where $\mathfrak{it}_k(d)$ denotes the number of irreducible d-multitournaments of size k, that is, d-multitournaments that are strongly connected as directed graphs. In particular, a 1-multigraph is a (simple) graph, a 1-multitournament is a tournament, and $\mathfrak{it}_k(1) = \mathfrak{it}_k$.

---

[2]Reminder: a *tournament* is a directed graph in which each pair of (distinct) vertices is joined by exactly one directed edge. A tournament is *irreducible* if there is no partition of its vertices into two nonempty parts A and B such that every pair of vertices $(a, b) \in A \times B$ is joined by an edge directed from a to b. Equivalently, a tournament is *irreducible* if it is strongly connected.

On the other hand, in terms of connectivity, a d-multigraph of size n can be viewed as a graph within the Erdős-Rényi model $G(n,p)$ with

$$p = \frac{d}{d+1}.$$

This observation makes it natural to interpret d-multitournaments as directed graphs in which each pair of vertices $i \neq j$ is independently joined by an edge $\vec{ij}$, an edge $\vec{ji}$, both by edges $\vec{ij}$ and $\vec{ji}$, respectively, with probability $1/(d+1)$, $1/(d+1)$, and $(d-1)/(d+1)$.

Now we are ready to define generalized tournaments within the Erdős-Rényi model.

Definition 4.5. Let $p \in (1/2, 1)$ be a fixed real number, $q = 1 - p$, and $n \in \mathbb{Z}_{\geqslant 0}$. A *generalized tournament* within the *Erdős–Rényi model* $T(n,p)$ is a directed graph whose set of vertices is $[n]$ and each pair of vertices $i \neq j$ is joined independently by

- an edge directed from i to j with probability q,

- an edge directed from j to i with probability q,

- two edges, directed both from i to j and from j to i, with probability $p - q$.

Remark 4.6. In the literature, a generalized tournament is usually understood as an object that differs from that of Definition 4.5. Specifically, it is a directed graph in which an edge directed from i to j is taken with probability p_{ij}, where $p_{ij} + p_{ji} = 1$ for $i \neq j$ and $p_{ii} = 0$; see [29]. The particular case $p_{ij} = p$ for $i < j$ and $p_{ij} = q$ for $i > j$, where positive values p and q satisfy $p + q = 1$, concerns the counting by descents; see [1].

Clearly, we cannot extend Definition 4.5 to $p < 1/2$, since the probability value $p - q$ becomes negative. To avoid this kind of obstacle, we need to switch to virtual species of generalized tournaments. Thus, the *weighted species $\mathcal{T}$ of generalized tournaments* is the species of directed graphs in which every pair of vertices is joined by one directed edge or by two oppositely directed edges. For a tournament $t \in \mathcal{T}$, we assign a weight

$$w(t) = (\rho - 1)^k,$$

where $\rho = p/q$ and k is the number of pairs of vertices that are joined by two edges. Note that for $\rho < 1$, the weight $w(t)$ may be negative. In this case, the corresponding tournaments are virtual. Some examples of small-size generalized tournaments are presented in Fig. 3 (among them, the irreducible tournaments are depicted at the bottom of the illustration, while the reducible ones can be found at the top).

In view of the asymptotic probability of connected graphs, the following two observations are crucial. First, species $\mathcal{T}$ can be represented as a composition

$$\mathcal{T} = \mathcal{L} \circ \mathcal{IT}, \tag{15}$$

where $\mathcal{IT} \subset \mathcal{T}$ is the subspecies of irreducible (that is, strongly connected) generalized tournaments. The proof of this relation is essentially the same as for standard tournaments (see Lemma 1 in [26]): any generalized tournament can be viewed as a sequence of its strongly connected components. Second, the total weight of the species of generalized
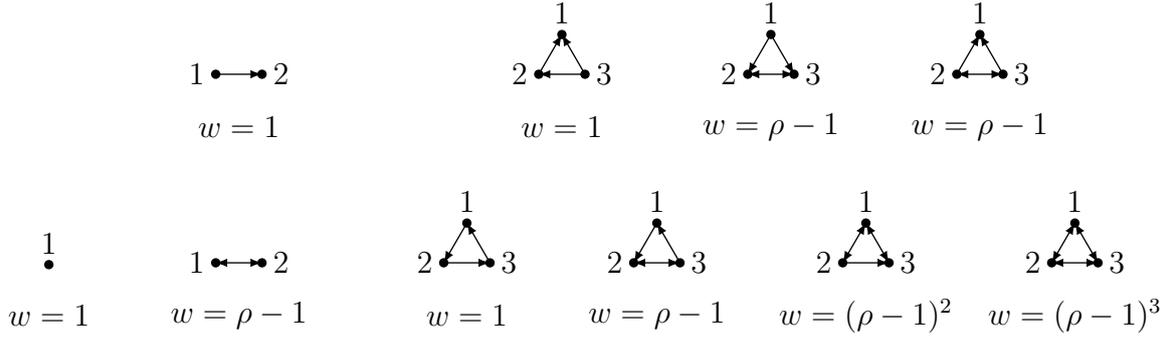
Figure 3: Weights of some labeled generalized tournaments.

tournaments on $[n]$ is the same as for species $\mathcal{G}$ and equals $(\rho + 1)^{\binom{n}{2}}$. This means that the species $\mathcal{G}$ of graphs and $\mathcal{T}$ of generalized tournaments are equipotent.

Taking into account the observations above, we can propose the following interpretation of relation (12).

Theorem 4.7. *The asymptotic probability that a random graph g in the Erdős–Rényi model $G(n,p)$ is connected satisfies*

$$\mathbb{P}(g \text{ is connected}) \approx 1 - \sum_{k \geqslant 1} P_k(\rho) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}},$$

where $P_k(\rho) = \big|\mathcal{IT}[k]\big|$ is the total weight of species $\mathcal{IT}$ on $[k]$. In particular, if $p \geqslant 1/2$ (that is, $\rho \geqslant 1$), then $P_k(\rho) \geqslant 0$ for any positive integer k.

Proof. According to Theorem 4.2, polynomials $P_k(\rho)$ represent the total weight of the species $\mathcal{E}^{-1} \circ \mathcal{CG}$ on $[k]$. Due to Lemma 3.6, this species is equipotent to the species $\big(\mathbf{1} - \mathcal{L}_+^{(-1)}\big) \circ \mathcal{G}_+$. The last one is equipotent to $\mathbf{1} - \mathcal{IT}$, since, as we have seen above, the species $\mathcal{G}_+$ and $\mathcal{T}_+$ are equipotent, and $\mathcal{T}_+ = \mathcal{L}_+ \circ \mathcal{IT}$. Therefore, $P_k(\rho)$ can be interpreted as the total weight of the species of irreducible generalized tournaments on $[k]$. $\qquad\square$

Example 4.8. As we discussed in Example 4.4, the first three polynomials $P_k(\rho)$ are

$$P_1(\rho) = 1, \qquad P_2(\rho) = \rho - 1, \qquad P_3(\rho) = \rho^3 + 3\rho^2 - 3\rho + 1.$$

The first two correspond to unique irreducible generalized tournaments of size 1 and 2, respectively (see Fig. 3). The third polynomial is equal to the sum of all irreducible generalized tournaments of size 3:

$$P_3(\rho) = (\rho - 1)^3 + 6(\rho - 1)^2 + 6(\rho - 1) + 2.$$

4.1.4 Asymptotic probability of irreducible generalized tournaments

Similarly to the main question stated for the Erdős-Rényi model $G(n,p)$ of random graphs, we could ask about the asymptotic probability that a random generalized tournament is irreducible within the Erdős-Rényi model $T(n,p)$ introduced in Definition 4.5 for

$p \geqslant 1/2$. In this section, we answer a more general question. Namely, we consider virtual weighted species $\mathcal{T}$ of generalized tournaments and establish the complete asymptotic expansion of the ratio $\mathfrak{it}_n^{(m)}/\mathfrak{it}_n$, where m is an arbitrary positive integer, while $\mathfrak{it}_n = \mathfrak{it}_n(\rho)$ and $\mathfrak{it}_n^{(m)} = \mathfrak{it}_n^{(m)}(\rho)$ are the total weights on $[n]$, respectively, of virtual species of all generalized tournaments and its subspecies of generalized tournaments consisting of m strongly connected components. This result is given by the following theorem.

Theorem 4.9. *Let m be a fixed positive integer. The virtual weighted species $\mathcal{T}$ satisfies*

$$\frac{\mathfrak{it}_n^{(m)}}{\mathfrak{it}_n} \approx m \sum_{k \geqslant 0} \left(\mathfrak{it}_k^{(m-1)} - 2\mathfrak{it}_k^{(m)} + \mathfrak{it}_k^{(m+1)} \right) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}}, \tag{16}$$

where, by convention, $\mathfrak{it}_0^{(0)} = 1$ and $\mathfrak{it}_k^{(0)} = 0$ for $k > 0$.

Proof. As follows from Lemma 4.1, the species $\mathcal{T}$ is gargantuan. Hence, we can apply Theorem 3.1 to the decomposition $\mathcal{T} = \mathcal{L} \circ \mathcal{IT}$ (see Remark 3.3). This gives us

$$\frac{\mathfrak{it}_n^{(m)}}{\mathfrak{it}_n} \approx \sum_{k \geqslant 0} \mathfrak{d}_{k,m}(\mathcal{T}) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}},$$

where $\mathfrak{d}_{k,m}(\mathcal{T})$ is the total weight of the species of structures $m \mathcal{IT}^{m-1}(1 - \mathcal{IT})^2$, that is,

$$\mathfrak{d}_{k,m}(\mathcal{T}) = m \left(\mathfrak{it}_k^{(m-1)} - 2\mathfrak{it}_k^{(m)} + \mathfrak{it}_k^{(m+1)} \right). \qquad \square$$

Corollary 4.10. *Let m be a fixed positive integer, and $p \in [1/2, 1)$. The asymptotic probability that a random generalized tournament t in the Erdős–Rényi model $T(n, p)$ has m strongly connected components satisfies*

$$\mathbb{P}(t \text{ has } m \text{ strongly connected components}) \approx \sum_{k \geqslant 0} Q_{k,m}(\rho) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}}, \tag{17}$$

where

$$Q_{k,m}(\rho) = m \left(\mathfrak{it}_k^{(m-1)} - 2\mathfrak{it}_k^{(m)} + \mathfrak{it}_k^{(m+1)} \right).$$

In particular,

$$\mathbb{P}(t \text{ is strongly connected}) \approx 1 - \sum_{k \geqslant 0} \left(2\mathfrak{it}_k - \mathfrak{it}_k^{(2)} \right) \cdot \binom{n}{k} \cdot \frac{q^{nk}}{q^{k(k+1)/2}}. \tag{18}$$

Corollary 4.11. *The dominant term in asymptotics (17) is*

$$\mathbb{P}(t \text{ has } m \text{ strongly connected components}) = m \cdot (n)_{m-1} \cdot \frac{q^{n(m-1)}}{q^{m(m-1)/2}} + O(n^m q^{mn}),$$

where $(n)_m = n(n-1)\ldots(n-m+1)$ are the falling factorials.

Proof. This follows from Corollary 3.2 and the fact that $q^{-\binom{1}{2}} = 1$. $\qquad \square$

It is easy to see that the results obtained in the sections are in accordance with the papers [26] and [28] that presented the particular case $p = 1/2$. In Table 4, the reader can find the polynomials $Q_{k,m}(\rho)$ for small values of m and k. Similarly to polynomials $P_{k,m}(\rho)$ (see Table 3), the sum of the coefficients in the kth column is zero for every $k > 0$, and the sum of the zeroth column is 1. This is due to the fact that the sum of probabilities of the form (17) taken over all positive integers m is equal to 1. In other words, for any generalized tournament, there is a unique $m \in \mathbb{Z}_{>0}$ such that m is the number of its strongly connected components.

k	0	1	2	3	4
$Q_{k,1}(\rho)$	1	-2	$-2\rho + 4$	$-2\rho^3 - 6\rho^2 + 12\rho - 8$	$-2\rho^6 - 12\rho^5 - 30\rho^4 - 16\rho^3 + 60\rho^2 - 48\rho + 16$
$Q_{k,2}(\rho)$	0	2	$2\rho - 10$	$2\rho^3 + 6\rho^2 - 30\rho + 38$	$2\rho^6 + 12\rho^5 + 30\rho^4 - 8\rho^3 - 150\rho^2 + 228\rho - 130$
$Q_{k,3}(\rho)$	0	0	6	$18\rho - 54$	$24\rho^3 + 90\rho^2 - 324\rho + 330$
$Q_{k,4}(\rho)$	0	0	0	24	$144\rho - 336$
$Q_{k,5}(\rho)$	0	0	0	0	120

Table 4: Polynomials $Q_{k,m}(\rho)$ for $k \leqslant 4$ and $m \leqslant 5$.

4.2 Surface and manifold models

This section is devoted to surface and manifold models whose asymptotic behavior cannot be treated using the approach described in our paper [27]. More precisely, our investigation is focused on the following models.

1. Quadratic square-tiled surfaces. First appearing in [12], they are widely used to resolve various problems related to the study of abelian and quadratic differentials on Riemann surfaces; see, for example, [15, 16, 24].

2. P-angulated surfaces. In case $P = 3$, they were independently introduced in [8] to study "typical" Riemann surfaces of high genus, and in [34] for the needs of quantum gravity. The general model first appeared in [19] as a continuation of the first of these papers, in connection with the study of Belyi surfaces.

3. Graph Encoded Manifolds (GEMs). They were introduced within "crystallization theory" in order to encode compact PL-manifolds [33, 32, 17]. They were also found to be useful for encoding color tensor models seen as quantum gravity theories; see, for instance, [6, 21, 22].

For each of the above models, our aim is to establish the asymptotic probability that a random surface (manifold) of size n is connected or consists of a prescribed number of connected components. In parallel, we would like to interpret the obtained result in a combinatorial manner. We will achieve these objectives using Theorem 3.1 and Proposition 3.7, both for the case where $\mathcal{F} = \mathcal{E}$.

4.2.1 Quadratic square-tiled surfaces

A *quadratic square-tiled surface* of size n is an oriented surface obtained by gluing n unit squares in such a way that the horizontal sides are glued to the horizontal sides, while the vertical sides are glued to the vertical sides. We assume that it is allowed to glue together the sides of the same square. We also assume that the squares are distinguishable, that is, they are labeled from 1 to n (or by elements of any other set of size n). The species of quadratic square-tiled surface will be denoted by $\mathcal{QSS}$, while the total number of quadratic square-tiled surfaces of size n will be denoted by $\mathfrak{qss}_n$.

From the definition of the model, it follows that $\mathfrak{qss}_n = \big((2n-1)!!\big)^2$. Another relation derived from this definition is the decomposition $\mathcal{QSS} = \mathcal{E} \circ \mathcal{CQSS}$, where $\mathcal{CQSS} \subset \mathcal{QSS}$ is the subspecies of connected quadratic square-tiled surfaces. We will shortly see that these two relations allow us to obtain a complete asymptotic expansion of the probability that a random quadratic square-tiled surface of size n is connected (and more).

The particular case where square-tiled surfaces are *abelian*, aka *origamis*, that is, the left and top sides of unit squares are glued to their right and bottom sides, respectively, was studied in our previous paper [27] (see also [11] and [9]). We showed that the probability that an origami o made up of n unit squares is connected admits the complete asymptotic expansion of the form

$$\mathbb{P}(o \text{ is connected}) \approx 1 - \sum_{k \geqslant 0} \frac{\mathfrak{ip}_k}{(n)_k},$$

where $(n)_k = n(n-1)\ldots(n-k+1)$ are falling factorials, while $\mathfrak{ip}_k$ is the number of indecomposable permutations of size k.

In the general case, for quadratic square-tiled surfaces, the following result holds.

Proposition 4.12. *Let m be a fixed positive integer. The asymptotic probability that a random quadratic square-tiled surface $s \in \mathcal{QSS}$ of size n has m connected components satisfies*

$$\mathbb{P}(s \text{ has } m \text{ connected components}) \approx \sum_{k \geqslant 0} \mathfrak{d}_{k,m}(\mathcal{QSS}) \cdot \binom{n}{k} \cdot \left(\frac{(2(n-k)-1)!!}{(2n-1)!!}\right)^2, \quad (19)$$

where

$$\mathfrak{d}_{k,m}(\mathcal{QSS}) = \sum_{s' \in \mathcal{QSS}_k} (-1)^{\pi_0(s')-(m-1)} \binom{\pi_0(s')}{m-1} \quad (20)$$

and $\pi_0(s')$ is the number of connected components of the surface s'.

Proof. Since $\mathcal{QSS} = \mathcal{E} \circ \mathcal{CQSS}$, the proof idea is to apply Theorem 3.1 to the species $\mathcal{A} = \mathcal{QSS}$ and $\mathcal{F} = \mathcal{E}$. To this end, it is sufficient to check that the sequence $a_n = \mathfrak{qss}_n/n!$ is gargantuan. This can be done using Lemma 2.6. As condition $na_{n-1} = O(a_n)$ is trivially satisfied, our goal is to verify that the sequence $x_n = a_k a_{n-k}$ is decreasing for $k < n/2$. Straightforward calculations show that

$$x_k = \frac{\left((2k-1)!!\big(2(n-k)-1\big)!!\right)^2}{k!(n-k)!},$$

and the ratio

$$\frac{x_{k+1}}{x_k} = \left(\frac{2k+1}{2(n-k)-1}\right)^2 \cdot \frac{n-k}{k+1}$$

is less than 1 if and only if

$$\frac{\left(2(n-k)-1\right)^2}{n-k} \geqslant \frac{\left(2(k+1)-1\right)^2}{k+1}.$$

Since the function $f(x) = (2x-1)^2/x$ is increasing for $x > 1$, the latter inequality is equivalent to $(k+1) \leqslant (n-k)$. Hence, the sequence (x_k) is decreasing for $k < n/2$, and the sequence (a_n) is gargantuan. Thus, Theorem 3.1 is applicable in our case, and relation (4) converts into (19), which completes our proof. $\qquad\square$

Corollary 4.13. *Let m be a fixed positive integer. The dominant term of the asymptotic probability* (19) *that a random quadratic square-tiled surface $s \in \mathcal{QSS}$ of size n has m connected components is*

$$\mathbb{P}(s \text{ has } m \text{ connected components}) = \frac{1}{(4n)^{m-1}} + O\left(\frac{1}{n^m}\right). \qquad (21)$$

Proof. Since $\mathfrak{qss}_1 = 1$, it follows from Corollary 3.2 that the dominant term of (19) is

$$\binom{n}{m-1} \cdot \left(\frac{(2(n-m+1)-1)!!}{(2n-1)!!}\right)^2 \sim \frac{1}{(4n)^{m-1}}.$$

$$\square$$

4.2.2 *P*-angulated surfaces

Fix an integer $P \geqslant 3$. A *P-angulation* of size n is an oriented surface obtained by gluing n unit P-gons along their sides. Here, we assume that P-gons are labeled from 1 to n, and that their sides are distinguishable. At the same time, there is no restriction on the adhesion of the sides: any side can be glued to any of the $(Pn-1)$ other sides. The only restriction is combinatorial: a gluing is a partition of polygon sides in pairs, and so the number Pn must be even. We will denote the species of P-angulated surfaces and the total number of P-angulated surfaces of size n by $\mathcal{PS}(P)$ and $\mathfrak{ps}_n(P)$, respectively. From the definition, it follows immediately that $\mathfrak{ps}_n(P) = (Pn-1)!!$ for even values of P, while for odd values of P we have

$$\mathfrak{ps}_n(P) = \begin{cases} (2Pk-1)!! & \text{if } n = 2k \\ 0 & \text{if } n = 2k+1. \end{cases}$$

Clearly, the quadratic and abelian square-tiled surfaces discussed in the previous section are particular cases of P-angulated surfaces for $P = 4$. Similarly to these particular models, each P-angulated surface is a union of its connected components, which gives us the decomposition $\mathcal{PS}(P) = \mathcal{E} \circ \mathcal{CPS}(P)$, where by $\mathcal{CPS}(P)$ we denote the species of connected P-angulated surfaces. In this section, we establish the asymptotic probability that a random P-angulated surface consists of a given number of connected components. As we shall see, the form of the formulas slightly depends on the parity of P.

Proposition 4.14. *Let $m \geqslant 1$ and $P \geqslant 3$ be two fixed integers. Depending on the parity of the parameter P, the asymptotic probability that a random surface $s \in \mathcal{PS} = \mathcal{PS}(P)$ has m connected components satisfies*

$$\mathbb{P}(s \text{ has } m \text{ connected components}) \approx$$

$$\begin{cases} \displaystyle\sum_{k \geqslant 0} \mathfrak{d}_{k,m}(\mathcal{PS}) \cdot \binom{n}{k} \cdot \frac{\big(P(n-k)-1\big)!!}{(Pn-1)!!} & \text{if } P \text{ is even} \\[2em] \displaystyle\sum_{k \geqslant 0} \mathfrak{d}_{2k,m}(\mathcal{PS}) \cdot \binom{2n}{2k} \cdot \frac{\big(2P(n-k)-1\big)!!}{(2Pn-1)!!} & \text{if } P \text{ is odd}, \end{cases} \tag{22}$$

where

$$\mathfrak{d}_{k,m}(\mathcal{PS}) = \sum_{s' \in \mathcal{PS}_k(P)} (-1)^{\pi_0(s')-(m-1)} \binom{\pi_0(s')}{m-1} \tag{23}$$

and $\pi_0(s')$ is the number of connected components of the surface s'.

Proof. The relation $\mathcal{PS}(P) = \mathcal{E} \circ \mathcal{CPS}(P)$ suggests that we apply Theorem 3.1 to species $\mathcal{A} = \mathcal{PS}(P)$ and $\mathcal{F} = \mathcal{E}$. To verify that the theorem is applicable, it would suffice to check that the species $\mathcal{PS}(P)$ is gargantuan. However, this fact is formally correct only for the case where P is even, since for odd values of P the sequence $\big(\mathfrak{ps}_n(P)\big)$ is 2-periodic. Thus, depending on the parity of the parameter P, we reason slightly differently.

For the case where P is even, we show that the sequence $a_n = \mathfrak{ps}_n(P)/n!$ is gargantuan using Lemma 2.6. Since condition $na_{n-1} = O(a_n)$ holds trivially, the only thing to check is that the sequence $x_n = a_k a_{n-k}$ is decreasing for $k < n/2$. The case in the hands reads

$$x_k = \frac{(Pk-1)!!\big(P(n-k)-1\big)!!}{k!(n-k)!},$$

and the ratio

$$\frac{x_{k+1}}{x_k} = \frac{\big(P(k+1)-1\big)\ldots\big(Pk+1\big)}{\big(P(n-k)-1\big)\ldots\big(P(n-k-1)+1\big)} \cdot \frac{n-k}{k+1}$$

is less than 1 if and only if

$$\frac{\big(P(n-k)-1\big)\big(P(n-k)-3\big)\ldots\big(P(n-k)-(P-1)\big)}{\big(P(k+1)-1\big)\big(P(k+1)-3\big)\ldots\big(P(k+1)-(P-1)\big)} \geqslant \frac{n-k}{k+1}.$$

Since the function $\big(Px-1\big)\big(Px-3\big)\ldots\big(Px-(P-1)\big)/x$ is increasing for $x > 1$, the latter inequality is equivalent to $(k+1) \leqslant (n-k)$. Hence, the sequence (x_k) is decreasing for $k < n/2$, and the sequence (a_n) is gargantuan. Theorem 3.1 can therefore be applied to species $\mathcal{A} = \mathcal{PS}(P)$ and $\mathcal{F} = \mathcal{E}$, and this gives relation (22).

For the case where P is odd, we verify that the sequence $b_n = \mathfrak{ps}_{2n}(P)/(2n)!$ is gargantuan. Again, we use Lemma 2.6, and the reasoning is similar: trivially, we have $nb_{n-1} = O(b_n)$ as $n \to \infty$, and the sequence

$$y_n = b_k b_{n-k} = \frac{(2Pk-1)!!\big(2P(n-k)-1\big)!!}{(2k)!\big(2(n-k)\big)!}$$

is decreasing for $k < n/2$. The latter arises from the fact that the inequality $y_{k+1} < y_k$ is equivalent to $(k + 1) \leqslant (n - k)$ for large n, which, in turn, follows from the fact that the function $\big(2Px - 1\big)\big(2Px - 3\big) \ldots \big(2Px - (2P - 1)\big)/\big(x(2x - 1)\big)$ is increasing for $x > 1$.

Since the sequence (b_n) is gargantuan, we can apply Proposition 3.7 with $p = 2$ to the species $\mathcal{A} = \mathcal{PS}(P)$ and $\mathcal{F} = \mathcal{E}$. As a consequence, we obtain relation (22), which completes the proof. $\qquad\square$

Corollary 4.15. *Let $m \geqslant 1$ and $P \geqslant 3$ be two fixed integers. Depending on the parity of the parameter P, the dominant term of the asymptotic probability* (22) *that a random surface $s \in \mathcal{PS} = \mathcal{PS}(P)$ has m connected components satisfies*

$\mathbb{P}(s \ \text{has} \ m \ \text{connected components}) =$

$$
\begin{cases}
\dfrac{1}{(m-1)!} \cdot \left(\dfrac{(P-1)!!}{P^{P/2}} \cdot \dfrac{1}{n^{P/2-1}} \right)^{m-1} + O\left(\dfrac{1}{n^{(P/2-1)m}} \right) & \text{if } P \text{ is even} \\[3mm]
\dfrac{1}{(m-1)!} \cdot \left(\dfrac{(2P-1)!!}{P^P \cdot 2^{P-1}} \cdot \dfrac{1}{n^{P-2}} \right)^{m-1} + O\left(\dfrac{1}{n^{(P-2)m}} \right) & \text{if } P \text{ is odd.}
\end{cases}
\tag{24}
$$

Proof. If P is even, then the total number of a single P-gon gluings is $\mathfrak{ps}_1(P) = (P-1)!!$. Hence, according to Corollary 3.2, the leading term of (22) behaves as

$$
\binom{n}{m-1} \cdot \frac{\big((P-1)!!\big)^{m-1} \cdot \big(P(n-m+1) - 1\big)!!}{(Pn - 1)!!} \sim \frac{1}{(m-1)!} \cdot \left(\frac{(P-1)!!}{P^{P/2}} \cdot \frac{1}{n^{P/2-1}} \right)^{m-1}.
$$

If P is odd, then we have $\mathfrak{ps}_2(P) = (2P-1)!!$ gluings of two P-gons, and therefore, due to Corollary 3.8 with $p = 2$, the leading term of (22) is equivalent to

$$
\frac{(2n)! \cdot \big((2P-1)!!\big)^{m-1} \cdot \big(2P(n-m+1) - 1\big)!!}{(m-1)! \cdot 2^{m-1} \cdot \big(2(n-m+1)\big)!! \cdot (2Pn - 1)!!} \sim \frac{1}{(m-1)!} \cdot \left(\frac{(2P-1)!!}{P^P \cdot 2^{P-1}} \cdot \frac{1}{n^{P-2}} \right)^{m-1}.
$$

$\qquad\square$

4.2.3 Graph Encoded Manifolds

Fix an integer $D \geqslant 2$, which we will refer to as the dimension. To obtain a *graph encoded manifold (GEM)* of dimension D of size n, we take n simplexes of dimension D and glue them together along their hyperfaces. Here, we assume that the simplexes are labeled from 1 to n, and that the vertices of each simplex are labeled from 1 to $D + 1$. The gluing is carried out according to the following rule: the hyperfaces identified with each other must be opposed to vertices with identical labels. Formally, we choose a collection of perfect matchings $\alpha_1, \ldots, \alpha_{D+1} \in S_n$ and identify the hyperface that is opposite to the kth vertex of the ith simplex with the hyperface opposite to the kth vertex of the $\alpha_k(i)$ simplex. In an equivalent manner, one can think of GEMs of dimension D and size n as $(D+1)$-regular graphs on n vertices labeled from 1 to n, where every edge carries a color from the set $[D + 1]$ and every vertex is incident to exactly one edge of each color.

We denote the species of D-dimensional GEMs and its connected subspecies, respectively, by $\mathcal{GEM}(D)$ and $\mathcal{CGEM}(D)$. From the definition of the model it follows that these species are linked by the relation

$$
\mathcal{GEM}(D) = \mathcal{E} \circ \mathcal{CGEM}(D).
$$

Another implication of the definition is the formula for the number $\mathfrak{gem}_{2n}(D)$ of GEMs of dimension D and size $2n$:

$$\mathfrak{gem}_{2n}(D) = \left((2n - 1)!!\right)^{D+1}.$$

Evidently, there are no GEMs of any odd size. Our goal is to derive from these two relations the asymptotic probability that a GEM of dimension D consists of a given number of connected components.

Proposition 4.16. *Let m be a fixed positive integer. The asymptotic probability that a random GEM $g \in \mathcal{GEM} = \mathcal{GEM}(D)$ of size $2n$ has m connected components satisfies*

$$\mathbb{P}(g \text{ has } m \text{ connected components}) \approx \sum_{k \geqslant 0} \mathfrak{d}_{2k,m}(\mathcal{GEM}) \cdot \binom{2n}{2k} \cdot \left(\frac{(2(n-k)-1)!!}{(2n-1)!!}\right)^{D+1},$$

where

$$\mathfrak{d}_{2k,m}(\mathcal{GEM}) = \sum_{g' \in \mathcal{GEM}_{2k}(D)} (-1)^{\pi_0(g')-(m-1)} \binom{\pi_0(g')}{m-1} \tag{25}$$

and $\pi_0(g')$ is the number of connected components of the GEM g'.

Proof. Due to the relation $\mathcal{GEM}(D) = \mathcal{E} \circ \mathcal{CGEM}(D)$, the result comes from Proposition 3.7 applied to species $\mathcal{A} = \mathcal{GEM}(D)$ and $\mathcal{F} = \mathcal{E}$ with $p = 2$. Thus, the only thing to verify is that the 2-periodic sequence $a_n(D) = \mathfrak{gem}_{2n}(D)/(2n)!$ is gargantuan. This can be done in several steps.

First, we prove that $\left(a_n(D)\right)$ is gargantuan for $D = 2$. In this case, we have

$$a_n(2) = \frac{\left((2n-1)!!\right)^3}{(2n)!} = \frac{\left((2n-1)!!\right)^2}{n!} \cdot \frac{1}{2^n}.$$

As we have seen in the proof of Proposition 4.12, the sequence $\mathfrak{qss}_n/n! = \left((2n-1)!!\right)^2/n!$ is gargantuan. Therefore, due to Lemma 2.5, the sequence $\left(a_n(2)\right)$ is gargantuan too.

Second, we make sure that the sequence $b_n = (2n-1)!!$ is gargantuan. To this end, we apply Lemma 2.6. Indeed, its first condition holds, since

$$nb_{n-1} = n(2n-3)!! = O(b_n).$$

For the second condition, the sequence $x_k = a_k a_{n-k}$ is decreasing for $k < n/2$, because

$$\frac{x_{k+1}}{x_k} = \frac{2k+1}{2(n-k)-1} \leqslant 1$$

in this case. Thus, the sequence (b_n) is truly gargantuan.

Finally, the general case comes from the relation $a_n(D) = a_n(2) \cdot b_n^{D-2}$. Here, according to Lemma 2.4, we can claim that $\left(a_n(D)\right)$ is gargantuan as a piecewise product of several gargantuan sequences. $\qquad\square$

Corollary 4.17. *Let m be a fixed positive integer. The dominant term of the asymptotic probability that a random GEM $g \in \mathcal{GEM} = \mathcal{GEM}(D)$ of size $2n$ has m connected components satisfies*

$$\mathbb{P}(g \text{ has } m \text{ connected components}) = \frac{1}{(m-1)!} \cdot \left(\frac{1}{2^D n^{D-1}}\right)^{m-1} + O\left(\frac{1}{n^{(D-1)m}}\right). \quad (26)$$

Proof. Since $\mathfrak{gem}_2(D) = 1$, this relation follows from Corollary 3.8:

$$\frac{1}{(m-1)!} \cdot \frac{(2n)_{2(m-1)}}{2^{m-1}} \cdot \left(\frac{(2(n-m+1)-1)!!}{(2n-1)!!}\right)^{D+1} \sim \frac{1}{(m-1)!} \cdot \left(\frac{1}{2^D n^{D-1}}\right)^{m-1}.$$

$\square$

5 Conclusion

We have seen that the structure of an asymptotic expansion of irreducibles determined by a composition $\mathcal{A} = \mathcal{F}(\mathcal{B})$ arises from the derivative structure $\mathcal{D}$ of this composition. Theorem 3.1 provides an expression for $\mathcal{D}$ in terms of the initial species of structures $\mathcal{A}$ in the case where $\mathcal{A}$ is gargantuan and $\mathcal{F} \in \{\mathcal{E}, \mathcal{L}, \mathcal{C}\}$. In general, the derivative structure $\mathcal{D}$ is virtual, which is well-illustrated by connected graphs within the Erdős-Rényi model $G(n, p)$: the involved coefficients represent generalized tournaments that are purely virtual for $p < 1/2$. On the other hand, in some cases, the asymptotic coefficients are nonnegative, and hence, can be potentially interpreted as total weights of species that are not virtual (and even as counting sequences of some combinatorial classes). For instance, this is the case for the cycle composition $\mathcal{A} = \mathcal{C}(\mathcal{B})$, as well as for the double decomposition $\mathcal{A} = \mathcal{E}(\mathcal{B}) = \mathcal{L}(\mathcal{D})$ where irreducibility is understood as connectivity. Establishing general conditions for the coefficients to be nonnegative is an open problem.

The condition of being gargantuan, imposed on the species $\mathcal{A}$, is necessary to apply Theorem 3.1. In the case where this condition does not hold, we cannot ensure that a random object $s \in \mathcal{A}$ is irreducible, *i.e.* belongs to $\mathcal{F}_1(\mathcal{B})$, with high probability. Extending our structural approach to the case where the sequence $(\mathfrak{a}_n)$ of weights is not gargantuan and the limiting probability differs from 1 is of great interest. Potential applications include, for example, the asymptotic probability of various families of trees.

Another setting that we cannot seize by the presented approach concerns such combinatorial structures as biconnected graphs or noncrossing partitions whose generating series are defined implicitly. Thus, the generating series $C(z)$ and $B(z)$ of connected and biconnected graphs, respectively, are linked by the relation [23, formula (1.3.3)]

$$\log C(z) = B'(zC'(z)),$$

while the (ordinary) generating functions $A(z)$ and $I(z)$ of, respectively, all and noncrossing partitions satisfy [2]

$$A(z) = 1 + I(zA(z)).$$

To establish asymptotic expansions in these settings, Bender's theorem (Theorem 2.7) is not applicable anymore. A promising approach was proposed by Borinsky [7] who assembled the asymptotic coefficients into a new generating function and discovered general

rules it obeys, including rules for compositions and inverses. Unfortunately, Borinsky's method works only for factorially divergent series. Some ideas of Borinsky were extended for graphically divergent series by Dovgal and one of the authors of this paper [13]. However, the latter approach does not allow one to work with compositions and inverses, and therefore the asymptotics of biconnected graphs and other similar structures cannot be obtained by this way. Further advances in this direction are still needed.

One final remark we would like to make concerns unlabeled structures. In general, species theory is well-adapted for a unified study of both labeled and unlabeled combinatorial objects. In this paper, we discuss only labeled structures, but in principle, in certain cases, our method also works for unlabeled structures; see [28, Section 7]. Creating a unified approach that embraces labeled and unlabeled cases under the same framework is another intriguing open problem.

Acknowledgments

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