An Application of the Universality Theorem for Tverberg Partitions to Data Depth and Hitting Convex Sets

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Abstract

We show that, as a consequence of a new result of Pór on universal Tverberg partitions, any large-enough set P of points in $\mathbb{R}^d$ has a $(d+2)$-sized subset whose Radon point has half-space depth at least $c_d \cdot |P|$, where $c_d \in (0,1)$ depends only on d. We then give two applications of this result. The first is to computing weak ϵ-nets by random sampling. The second is to show that given any set P of points in $\mathbb{R}^d$ and a parameter $\epsilon > 0$, there exists a set of $O\left(\epsilon^{-\lfloor \frac{d}{2} \rfloor + 1}\right)$ $\lfloor \frac{d}{2} \rfloor$-dimensional simplices (ignoring polylogarithmic factors) spanned by points of P such that they form a transversal for all convex objects containing at least $\epsilon \cdot |P|$ points of P.

Keywords: Tverberg's theorem, Radon's lemma, weak ϵ-nets, half-space depth, transversals.

1 Introduction

Radon's lemma states that, given any set Q of $(d+2)$ points in $\mathbb{R}^d$, there always exists a partition of Q into two sets, say Q_1 and Q_2, such that conv $Q_1 \cap$conv $Q_2 \neq \emptyset$. Further, if Q is in general position, then a dimension argument implies that such a partition $\{Q_1, Q_2\}$—called a Radon partition of Q—is unique and conv $Q_1 \cap$ conv Q_2 consists of a single point, called the *Radon point* of Q and denoted by Radon Q.

In this paper we present an application of the following statement, which is one consequence of a recent theorem of Pór (see [3]).

Lemma 1 (Proof in Section Section 2)**.** *For every $d \in \mathbb{N}$ there is $f(d) \in \mathbb{N}$ such that every set $P \subset \mathbb{R}^d$ of $f(d)$ points in general position contains two disjoint sets $A, B \subset P$ with $|A| = d+2$, $|B| = d+1$ such that the Radon point of A is contained in* conv B*. Furthermore, the Radon partition of A consists of two sets of sizes $\lfloor \frac{d}{2} \rfloor + 1$ and $\lceil \frac{d}{2} \rceil + 1$.*

For some background on Lemma 1, we refer the reader to [4].

We use Lemma 1 to prove the following theorem. Given a set P of points in $\mathbb{R}^d$, the *half-space depth* of a point $q \in \mathbb{R}^d$ with respect to P is defined to be the minimum number of points of P contained in any half-space containing q.

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Theorem 2 (Proof in Section 2)**.** *For every $d \in \mathbb{N}$ there is $h(d) \in \mathbb{N}$ such that every set P of at least $h(d)$ points in $\mathbb{R}^d$ in general position contains a set $P' \subseteq P$ of size $(d+2)$ with* Radon P' *being contained in at least $\frac{|P|}{h(d)}$ vertex-disjoint simplices spanned by the points of $P \setminus P'$. In particular,* Radon P' *has half-space depth at least $\frac{|P|}{h(d)}$ with respect to P.*

We expect that Theorem 2 will find further applications in algorithms, discrete and combinatorial geometry and data analysis. Here we give two applications related to the computation of weak ϵ-nets.

Definition 3. *Given a set P of points in $\mathbb{R}^d$ and a parameter $\epsilon > 0$, a set $N \subseteq \mathbb{R}^d$ is a weak ϵ-net with respect to convex sets for P if for every convex set K with $|K \cap P| \geq \epsilon \cdot |P|$, we have $K \cap N \neq \emptyset$.*

Definition 4. *Given positive integers d, p, q with $p \geq q > \left\lfloor \frac{d}{2} \right\rfloor$, let $\mathrm{CHS}(d, p, q)$ denote the smallest integer such that the following holds. For any compact convex object $K \subseteq \mathbb{R}^d$ and any set $P \subseteq \mathbb{R}^d \setminus K$ of points, if every subset of P of size p has a q-sized subset whose convex hull is disjoint from K, then P can be separated from K with $\mathrm{CHS}(d, p, q)$ half-spaces (that is, there exists a set $\mathcal{H}$ of $\mathrm{CHS}(d, p, q)$ half-spaces such that $K \subseteq \bigcap_{h \in \mathcal{H}} h$ and $\left(\bigcap_{h \in \mathcal{H}} h\right) \bigcap P = \emptyset$).*

It is known that $\mathrm{CHS}(d, p, q)$ is finite for large-enough values of q; in fact it is a special case of the more general so-called Hadwiger-Debrunner (p, q) problem for convex sets in $\mathbb{R}^d$ (see [12]). In particular,

1. ([8]) For $p \geq q = d + 1$ we have

$$\mathrm{CHS}(d, p, q) = O\left(p^{d^2} \log^{c' d^3 \log d} p\right),$$

   where c' is an absolute constant.

2. ([12]) For any real $\beta > 0$ and $p \geq q = (1 + \beta) \cdot \left\lfloor \frac{d}{2} \right\rfloor$ we have

$$\mathrm{CHS}(d, p, q) = O\left(q^2 p^{1+\frac{1}{\beta}} \log p\right).$$

Theorem 5 states our application of Theorem 2. The proof follows the method of Mustafa and Ray [11]; we present their proof modified appropriately to give a general explicit bound in terms of $\mathrm{CHS}\,(d, p, q)$ and $h(d)$.

Theorem 5 (Proof in Section 3)**.** *Let P be a set of n points in $\mathbb{R}^d$ and $\epsilon \in \left[0, \frac{1}{2}\right]$ a given parameter. Further let $q > \left\lfloor \frac{d}{2} \right\rfloor$ be an integer and define $p = q \cdot h(d)$, where $h(d)$ is the function from Theorem 2. Let R be a uniform random sample of P of size*

$$\frac{c_2 \cdot d \cdot \mathrm{CHS}\,(d, p, q) \cdot \log \mathrm{CHS}\,(d, p, q)}{\epsilon} \log \frac{1}{\epsilon},$$

where c_2 is a large-enough constant independent of d, ϵ and q. Then with probability at least $\frac{9}{10}$, the following holds.

1. *Let Q be the set of Radon points of all $(d+2)$-sized subsets of R. Then $Q \cup R$ is a weak ϵ-net for P, of size $O\left(|R|^{d+2}\right)$.*

2. *Let T be the set of convex-hulls of all $\left(\lfloor \frac{d}{2} \rfloor + 1\right)$-sized subsets of R. Then each convex object containing at least $\epsilon|P|$ points of P intersects at least one element of T. Note that $|T| = O\left(|R|^{\lfloor \frac{d}{2} \rfloor + 1}\right)$.*

In particular, one can set $q = (d+1)$ to get a random sample R satisfying the above, of size

$$|R| = O\left(d \cdot \mathrm{CHS}\left(d, (d+1) \cdot h\left(d\right), (d+1)\right) \cdot \log \mathrm{CHS}\left(d, (d+1) \cdot h\left(d\right), (d+1)\right) \frac{1}{\epsilon} \log \frac{1}{\epsilon}\right).$$

Remark 1. The first part of Theorem 5 gives a bound on the size of the ϵ-net that is weaker than the current best bound due to Matoušek and Wagner [10], which is of the order of $O\left(\frac{1}{\epsilon^d}\right)$ (ignoring polylogarithmic factors; see also [1, 5]). Yet our construction of a weak ϵ-net is novel and interesting as it uses certain Radon points of the underlying set P. It also shows that one can get close to the best-known bounds by using a single random sample of P.

Remark 2. The existence of weak ϵ-nets of size $o\left(\frac{1}{\epsilon^d}\right)$ is a long-standing open problem, and the case in $\mathbb{R}^d$ has seen no substantial progress since 1995 (recently the bound in two dimensions was improved in [15]). The second part of Theorem 5 shows that it is possible to improve the upper-bound if one is willing to consider hitting with higher-dimensional simplices instead of points.

Remark 3. The function $h(d)$ in Theorem 2 depends on $f(d)$ of Lemma 1, and is unlikely to be near-tight. We leave improving it as an open question; in particular, given $d \in \mathbb{N}$, the determination of the smallest $h\left(d\right)$ such that any set P of points in $\mathbb{R}^d$ has a set $Q \subseteq P$ of size $d+2$ with Radon Q having half-space depth at least $\frac{|P|}{h(d)}$.

2 Proof of Lemma 1 and Theorem 2.

We need some definitions. We set $m = (r-1)(d+1)+1$, and for $k \in [d+1]$ the block B_k is the set of integers $\{(r-1)(k-1)+1, (r-1)(k-1)+2, \ldots, (r-1)k+1\}$. The blocks are of size r each and they almost form a partition of $[m]$, only neighboring blocks have a common element, namely $(r-1)k+1 \in B_k \cap B_{k+1}$ for all $k \in [d]$. Call an r-partition $\{I_1, \ldots, I_r\}$ of $[m]$ *special* if $|I_j \cap B_k| = 1$ for every $j \in [r]$ and every $k \in [d+1]$.

Pór's result is about sequences $S = (a_1, \ldots, a_N)$ of vectors in $\mathbb{R}^d$. A sequence $(b_1, \ldots, b_t)$ is a subsequence of S of length t if $b_j = a_{i_j}$ for all $j \in [t]$ where $1 \le i_1 < i_2 < \ldots < i_t \le N$. Given a sequence $S = (a_1, \ldots, a_m)$, $a_i \in \mathbb{R}^d$, an r-partition $\{S_1, S_2, \ldots, S_r\}$ of S is in one-to-one correspondence with an r-partition $\{I_i, \ldots, I_r\}$ of $[m]$ via $a_i \in S_j$ if and only if $i \in I_j$. An r-partition of S is called *special* if the corresponding r-partition of $[m]$ is special.

Tverberg's theorem states that given a set P of $(r-1)(d+1)+1$ points in $\mathbb{R}^d$, there exists a partition of P into r sets whose convex-hulls contain a common point.

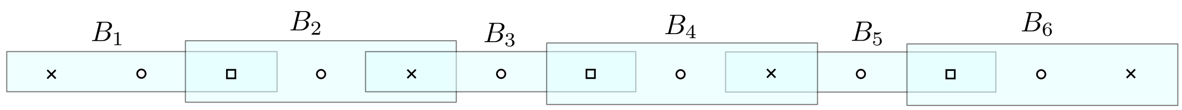
We can now state Pór's result [14].

Theorem A (Universality theorem for Tverberg partitions). *Assume $d, r, t \in \mathbb{N}$, $r \geq 2$, and $m = (r-1)(d+1)+1 \leq t$. Then there exists $N = N(d,r,t) \in \mathbb{N}$ such that every sequence $S = (a_1, \ldots, a_N)$ of vectors (in general position) in $\mathbb{R}^d$ contains a subsequence $S' = (b_1, \ldots, b_t)$ (of length t) such that the Tverberg partitions of every subsequence of length m of S' are exactly the special partitions.*

Remark. When the points of S (or P) come from the moment curve $\Gamma(x) = \{\gamma(x) \colon x \in \mathbb{R}^+\}$ where $\gamma(x) = (x, x^2, \ldots, x^d)$, then there is a natural ordering $S = (\gamma(x_1), \ldots, \gamma(x_n))$ with $x_1 < x_2 < \ldots < x_n$. Now let $0 < x_1 < \ldots < x_n$ a *rapidly increasing* sequence of real numbers, meaning that, for every $h \in [n-1]$, x_{h+1}/x_h is at least as large as some (large) constant $c_{d,r,h}$ depending only on d, r, h. It is not hard to check that in this case all Tverberg partitions of all $m = (r-1)(d+1)+1$ long subsequences of S are the special ones. This (and other examples as well) show that no other set of partitions can be universal, i.e., that exist as Tverberg partitions in a large-enough point set.

We are going to apply the universality theorem in the special case $r = 3$ and $t = m = (r-1)(d+1)+1 = 2d+3$. In this case $N(d,r,t)$ depends on d only and thus we can set $f(d) = N(d,r,t) = N(d,3,2d+3)$.

Proof of Lemma 1. Order the elements of P arbitrarily to obtain a sequence $S = \big(p_1, \ldots, p_{f(d)}\big)$. Apply Theorem A to S with $r = 3$, $t = m = 2d+3$. We get a subsequence S' of length m all of whose Tverberg 3-partitions are exactly the special ones. Define $I_1 = \{z \in [m] : z \equiv 1 \bmod 4\}$ and $I_2 = \{z \in [m] : z \equiv 3 \bmod 4\}$ and $I_3 = \{z \in [m] : z \text{ is even}\}$. Note that $|I_1| = \lceil \frac{d}{2} \rceil + 1$, $|I_2| = \lfloor \frac{d}{2} \rfloor + 1$ and $|I_3| = d+1$. See figure for an example with $d = 5$ (elements of I_1, I_2 and I_3 are depicted with crosses, squares and circles, respectively).



It is easy to see that $\{I_1, I_2, I_3\}$ is a special partition of $[m]$: every block contains exactly one element of I_1, I_2, I_3. Let the corresponding partition of S' be $\{S_1, S_2, S_3\}$. Theorem A implies that $\bigcap_1^3 \operatorname{conv} S_i \neq \emptyset$. Set $A = S_1 \cup S_2$ and $B = S_3$. Then the Radon point of A, which is $\operatorname{conv} S_1 \cap \operatorname{conv} S_2$, is contained in $\operatorname{conv} B$. □

Proof of Theorem 2. Consider a $(2d+3)$-uniform hypergraph $\mathcal{H} = (P, E)$ on the vertex set P, where $e \in E$ if and only if the $(2d+3)$ points of e can be partitioned into two sets $e = e_1 \cup e_2$ such that $|e_1| = d+2$, and Radon $e_1 \in \operatorname{conv} e_2$. We will call the set e_1 the *Radon-base* of the edge e. By the result of de Caen [7], any r-uniform hypergraph on n vertices and m edges contains an independent set of size at least

$$\frac{r-1}{r^{\frac{r}{r-1}}} \cdot \frac{n^{\frac{r}{r-1}}}{m^{\frac{1}{r-1}}}.$$

On the other hand, Lemma 1 implies that any set Q of $f(d)$ points of P must contain two disjoint sets—$A_Q \subseteq Q$ of size $(d+2)$ and $B_Q \subseteq Q$ of size $(d+1)$—such that Radon $A_Q \in \operatorname{conv} B_Q$. Then the $(2d+3)$ points $A_Q \cup B_Q$ form an edge in $\mathcal{H}$. This implies that no subset of P of size $f(d)$ can be independent in $\mathcal{H}$. Thus, with $r = 2d+3$, we have

$$\frac{2d+2}{(2d+3)^{\frac{2d+3}{2d+2}}} \cdot \frac{|P|^{\frac{2d+3}{2d+2}}}{|E|^{\frac{1}{2d+2}}} \leq \text{ size of max. ind. set in } \mathcal{H} < f(d) \quad \Longrightarrow \quad |E| \geq \frac{|P|^{2d+3}}{2(2d+3)f(d)^{2d+2}}.$$

By the pigeonhole principle, there exists a $(d+2)$-sized set $P' \subseteq P$ that is the Radon-base of a set E' of edges of E, where

$$|E'| \geq \frac{|E|}{\binom{|P|}{d+2}} \geq \frac{\frac{|P|^{2d+3}}{2(2d+3)f(d)^{2d+2}}}{\binom{|P|}{d+2}}.$$

The $(d+1)$-uniform hypergraph consisting of the sets $E'' = \{e' \setminus P' \colon e' \in E'\}$ has the property that the convex hull of the elements of each set contains Radon P'. It suffices to show that it contains a matching of size $\Omega(|P|)$—and this follows from known lower-bounds on matchings in uniform hypergraphs (see [2]). For simplicity, we instead present a direct argument, though with worse constants.

Iteratively construct a matching by adding a $(d+1)$-sized set from E'' to the matching, and deleting all sets from E'' whose intersection with this added set is non-empty. Each set added to the matching can cause the deletion of at most $(d+1) \cdot \binom{|P|}{d}$ sets of E'', as a vertex of $P \setminus P'$ can belong to at most $\binom{n}{d}$ sets of E'' (each set in E'' has size $(d+1)$). The size of the final matching is the number of iterations, which, by the above discussion, is lower-bounded by

$$\frac{\frac{|P|^{2d+3}}{2(2d+3)f(d)^{2d+2}}}{\binom{|P|}{d+2}} \Bigg/ \binom{|P|}{d} (d+1).$$

A calculation then shows that

$$\frac{\frac{|P|^{2d+3}}{2(2d+3)f(d)^{2d+2}}}{\binom{|P|}{d+2}} \Bigg/ \binom{|P|}{d} (d+1) \geq \frac{|P|}{h(d)}, \quad \text{where } h(d) = \frac{2(2d+3)(d+1)f(d)^{2d+2}}{(d+2)!d!}.$$

□

3 Proof of Theorem 5

Proof of Theorem 5. Consider the system system induced on P by the intersection of CHS (d,p,q) half-spaces in $\mathbb{R}^d$. It has VC-dimension $\Theta\left(d \cdot \text{CHS}\,(d,p,q) \cdot \log \text{CHS}\,(d,p,q)\right)$ [6] and so the ϵ-net theorem ([9]; see also [13]) implies that R is an ϵ-net for this set system with probability at least $\frac{9}{10}$. Assume this is the case and let K be any convex set containing at least $\epsilon|P|$ points of P.

Claim 6. *There exists $R_K \subseteq R$ of size p such that the convex hull of every subset of R_K of size q intersects K.*

Proof. If for every subset of R of size p there exists a q-sized subset whose convex hull is *disjoint* from K, then by the definition of CHS (d,p,q), all points of R can be separated from K by a set $\mathcal{H}$ of CHS (d,p,q) half-spaces. The common intersection of these half-spaces contains K and hence at least $\epsilon|P|$ points of P and no point of R, a contradiction to the assumption that R is an ϵ-net for the set system induced on P by the intersection of CHS (d,p,q) half-spaces. □

By Theorem 2, R_K has a $(d+2)$-sized subset, say R'_K, such that Radon $R'_K \in Q$ is contained in at least $\frac{|R_K|}{h(d)}$ vertex-disjoint simplices spanned by points of $R_K \setminus R'_K$. Now Radon R'_K must lie inside K: otherwise the half-space separating it from K must contain at least one point from each simplex containing Radon R'_K—namely it must contain at least $\frac{|R_K|}{h(d)} = \frac{p}{h(d)} = q$ points of R_K. But then the convex hull of these q points does not intersect K, a contradiction to Claim 6. Thus $R \cup Q$ is a weak ϵ-net for P.

The proof of *2.* follows from the fact that one of the two Radon partitions of R'_K has size at most $\lfloor \frac{d}{2} \rfloor + 1$, and its convex-hull must intersect K.

This completes the proof. □

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