

# Optimal Approximations Made Easy

Mónika Csikós

Université Gustave Eiffel, LIGM, Equipe A3SI, ESIEE Paris,  
Cité Descartes, 2 boulevard Blaise Pascal, 93162 Noisy-le-Grand Cedex, France.  
Email: monika.csikos@esiee.fr

Nabil H. Mustafa

Université Sorbonne Paris Nord, Laboratoire LIPN CNRS 7030, Institut Galilée,  
99 avenue Jean-Baptiste Clément 93430 Villetaneuse, France.  
Email: nabil.mustafa@univ-paris13.fr

## Abstract

The fundamental result of Li, Long, and Srinivasan [LLS01] on approximations of set systems has become a key tool across several communities such as learning theory, algorithms, computational geometry, combinatorics, and data analysis.

The goal of this paper is to give a modular, self-contained, intuitive proof of this result for finite set systems. The only ingredient we assume is the standard Chernoff’s concentration bound. This makes the proof accessible to a wider audience, readers not familiar with techniques from statistical learning theory, and makes it possible to be covered in a single self-contained lecture in a geometry, algorithms or combinatorics course.

Keywords: relative approximations, VC theory, chaining, symmetrization.

Funding: The work of the authors has been supported by the grant ANR ADDS (ANR-19-CE48-0005).

## 1 Introduction

Given a finite set system  $(X, \mathcal{F})$ , our goal is to construct a small set  $A \subseteq X$  such that each set of  $\mathcal{F}$  is ‘well-approximated’ by  $A$ . Research on such approximations started in the 1950s, with random sampling being the key tool for showing their existence. A breakthrough in the study of approximations dates back to 1971 when Vapnik and Chervonenkis studied set systems with finite VC-dimension [VC71]. The *VC-dimension* of  $(X, \mathcal{F})$ , denoted by  $\text{VC-dim}(X, \mathcal{F})$ , is the size of the largest  $Y \subseteq X$  for which  $\mathcal{F}|_Y = 2^Y$ , where  $\mathcal{F}|_Y = \{Y \cap S : S \in \mathcal{F}\}$ . Since then, the notion of approximations has become a fundamental structure across several communities—learning theory, statistics, combinatorics and algorithms (see [Mus22]).

**Relative  $(\epsilon, \delta)$ -approximations.** Given a set system  $(X, \mathcal{F})$  with  $n = |X|$  and parameters  $0 < \epsilon, \delta < 1$ , a set  $A$  of size  $t$  is a *relative  $(\epsilon, \delta)$ -approximation* for  $(X, \mathcal{F})$  if for all  $S \in \mathcal{F}$ ,

$$\left| \frac{|S|}{n} - \frac{|A \cap S|}{t} \right| \leq \delta \cdot \max \left\{ \frac{|S|}{n}, \epsilon \right\}, \quad \text{or equivalently,} \quad |A \cap S| = \frac{|S|t}{n} \pm \delta t \max \left\{ \frac{|S|}{n}, \epsilon \right\}.$$

In this paper, we study guarantees for relative  $(\epsilon, \delta)$ -approximations obtained by random sampling. In particular, given a set  $X$  we say that  $A$  is a uniform random sample of  $X$  of size  $t$  if  $A$  is selected uniformly at random from the  $\binom{n}{t}$  subsets of  $X$  of size  $t$ .

A basic guarantee follows immediately from Chernoff’s bound (for completeness, we give the standard proof in the Appendix).

**Theorem A.** *Let  $X$  be a set of  $n$  elements and  $A$  be a uniform random sample of  $X$  of size  $t$ . Then for any  $S \subseteq X$  and  $\eta > 0$ ,*

$$\mathbb{P}\left[|A \cap S| \notin \left(\frac{|S|t}{n} - \eta, \frac{|S|t}{n} + \eta\right)\right] \leq 2 \exp\left(-\frac{\eta^2 n}{2|S|t + \eta n}\right).$$

*In particular, setting  $\eta = \delta t \max\left\{\frac{|S|}{n}, \epsilon\right\}$ , a uniform random sample  $A$  of size  $t$  fails to be a relative  $(\epsilon, \delta)$ -approximation for a fixed  $S \in \mathcal{F}$  with probability at most  $2 \exp\left(-\frac{\epsilon \delta^2 t}{3}\right)$ .*

Theorem A in conjunction with the union bound gives the following upper-bound on relative  $(\epsilon, \delta)$ -approximation sizes for *any* finite set system (the detailed proof is presented in the Appendix).

**Theorem 1.** *Let  $(X, \mathcal{F})$  be a finite set system and  $0 < \epsilon, \delta, \gamma < 1$  be given parameters. Then for any integer  $t \geq \frac{3}{\epsilon \delta^2} \ln \frac{2|\mathcal{F}|}{\gamma}$ , a uniform random sample  $A \subseteq X$  of size  $t$  is a relative  $(\epsilon, \delta)$ -approximation for  $\mathcal{F}$  with probability at least  $1 - \gamma$ .*

This paper addresses the following influential result of Li, Long, and Srinivasan [LLS01], described as ‘the pinnacle of a long sequence of papers’ in [HP11, Section 7.4].<sup>1</sup>

**Theorem 2** ([LLS01]). *There exists an absolute constant  $c$  such that the following holds. Let  $(X, \mathcal{F})$  be a set system such that  $|\mathcal{F}|_Y \leq (e|Y|/d)^d$  for all  $Y \subseteq X$  with  $|Y| \geq d$ , and let  $0 < \delta, \epsilon, \gamma < 1/2$  be given parameters. Then for any integer*

$$t \geq \frac{c}{\epsilon \delta^2} \cdot \left(d \ln \frac{1}{\epsilon} + \ln \frac{1}{\gamma}\right),$$

*a uniform random sample  $A \subseteq X$  of size  $t$  is a relative  $(\epsilon, \delta)$ -approximation for  $(X, \mathcal{F})$  with probability at least  $1 - \gamma$ .*

### Remarks.

1. Note that by the Sauer-Shelah lemma,  $\text{VC-dim}(X, \mathcal{F}) \leq d$  implies that  $|\mathcal{F}|_Y \leq (e|Y|/d)^d$  for any  $Y \subseteq X$  (see e.g., [Mat02, Lemma 10.2.5]). Thus, Theorem 2 also applies to set systems with  $\text{VC-dim}(X, \mathcal{F}) \leq d$ .
2. This bound is asymptotically tight [LLS01] and immediately implies many other approximation bounds such as  $\epsilon$ -approximations (Vapnik and Chervonenkis [VC71], Talagrand [Tal94]), sensitive  $\epsilon$ -approximations (Brönnimann et al. [BCM93]), and  $\epsilon$ -nets (Haussler and Welzl [HW87], Komlós et al. [KPW92]).

The original proof of Theorem 2 uses two techniques:

**Symmetrization.** To prove that a random sample  $A$  satisfies the required properties, one takes another random sample  $G$ , sometimes called a ‘ghost sample’. Properties of  $A$  are then proven by comparing it with  $G$ . Note that  $G$  is not used in the algorithm or its construction—it is solely a method of analysis, a ‘thought experiment’ of sorts.

**Chaining.** The idea is to analyze the interaction of the sets in  $\mathcal{F}$  with a random sample by partitioning each  $S \in \mathcal{F}$  into a logarithmic number of smaller sets, each belonging to a distinct ‘level’. The number of sets increase with increasing level while the size of each set decreases. The overall sum turns out to be a geometric series, which then gives the optimal bounds [KT59, Tal16].

<sup>1</sup>The original result was stated using the notion of  $(\epsilon, \delta)$ -samples, but they are asymptotically equivalent: an  $(\epsilon, \delta)$ -sample is a relative  $(\epsilon, 4\delta)$ -approximation and a relative  $(\epsilon, \delta)$ -approximation is an  $(\epsilon, \delta)$ -sample; see [HS11].

What makes the proof of Theorem 2 in [LLS01] difficult is that it combines chaining and symmetrization intricately. All the tail bounds are stated in their ‘symmetrized’ forms and symmetrization is carried through the entire proof. It is not an easy proof to explain to undergraduate or even graduate students in computer science, as it is difficult to see what is really going on in terms of the significance and intuition of these two ideas. In fact, even the proofs of simpler statements involving just symmetrization, as given in textbooks<sup>2</sup>—e.g., see [KV94, DGL96, Mat99, Cha00, Mat02, AB09, HP11, AS12, Mus22]—often come with the caveat that the idea is ingenious but difficult to understand intuitively (e.g., “one might be tempted to believe that it works by some magic” [Mat02, Section 10.2]).

## Our Results.

This work is an attempt to improve this state of affairs. We show that in fact one can separate the roles of chaining and symmetrization, giving two separate statements which together immediately imply Theorem 2. The role of symmetrization is to get a bound on relative  $(\epsilon, \delta)$ -approximations that is independent of  $|\mathcal{F}|$  (but contains an additional factor of  $\log \frac{1}{\delta}$ ):

**Theorem 3.** *There exists an absolute constant  $c_1$  such that the following holds. Let  $(X, \mathcal{F})$  be a set system such that  $|\mathcal{F}|_Y \leq (e|Y|/d)^d$  for all  $Y \subseteq X$ ,  $|Y| \geq d$ , and let  $0 < \delta, \epsilon, \gamma < 1/2$  be given parameters. Then for any integer*

$$t \geq \frac{c_1}{\epsilon \delta^2} \cdot \left( d \ln \frac{1}{\epsilon \delta} + \ln \frac{1}{\gamma} \right),$$

*a uniform random sample  $A \subseteq X$  of size  $t$  is a relative  $(\epsilon, \delta)$ -approximation for  $(X, \mathcal{F})$  with probability at least  $1 - \gamma$ .*

**Remark.** The proof of Theorem 3 is standard using symmetrization. For completeness, we present a different, folklore proof at the end of the paper (Section 3), which in fact shows that symmetrization is not really necessary for finite set systems<sup>3</sup> and can be replaced by a more intuitive argument that makes it obvious, pedagogically, why the bound is independent of  $|\mathcal{F}|$ .

On the other hand, the role of chaining is to get rid of logarithmic factors that arise when applying union bound, by more carefully analyzing the failure probability for a collection of events. The key observation is that Theorem A provides a bound on the probability of failure for a set  $S \in \mathcal{F}$  which *decreases* as the size of  $S$  decreases. One can take advantage of this by partitioning each  $S \in \mathcal{F}$  into a logarithmic number of smaller sets, each belonging to a distinct level, such that the levels strike a proper balance—the number of sets (arising from partitioning every  $S \in \mathcal{F}$ ) increase each level, but their size across levels decreases geometrically. This way one gets an improved bound by applying the union bound separately to sets of different levels.

The resulting bound is captured in the next statement (it removes the factor of  $\log \frac{1}{\delta}$ , but depends on  $|\mathcal{F}|$ ):

**Theorem 4.** *There exists an absolute constant  $c_2$  such that the following holds. Let  $(X, \mathcal{F})$  be a set system such that  $|\mathcal{F}|_Y \leq (e|Y|/d)^d$  for all  $Y \subseteq X$ ,  $|Y| \geq d$ , and let  $0 < \delta, \epsilon, \gamma < 1/2$  be given parameters. Then for any integer*

$$t \geq c_2 \max \left\{ \frac{1}{\epsilon \delta} \ln \frac{|\mathcal{F}|}{\gamma}, \frac{1}{\epsilon \delta^2} \ln \left( \frac{1}{\epsilon^d \gamma} \right) \right\},$$

*a uniform random sample  $A \subseteq X$  of size  $t$  is a relative  $(\epsilon, \delta)$ -approximation for  $(X, \mathcal{F})$  with probability at least  $1 - \gamma$ .*

<sup>2</sup>Also used in teaching; to pick two arbitrary examples, see [here](#) for an example from the perspective of statistics/learning and [here](#) from the algorithmic side.

<sup>3</sup>This is typically the case in its use in algorithms, computational geometry, combinatorics. The infinite case can usually be reduced to the finite case by a sufficiently fine grid, see [MWW93].

The proof of Theorem 4 is given in Section 2.

The above two statements immediately imply a proof of Theorem 2: given  $(X, \mathcal{F})$ , apply Theorem 3 to get a set  $A_1 \subseteq X$  of size  $O\left(\frac{1}{\epsilon\delta^2} \ln \frac{1}{\epsilon^d \delta^d \gamma}\right)$ , which is a relative  $(\epsilon, \frac{\delta}{3})$ -approximation of  $\mathcal{F}$  with probability at least  $1 - \frac{\gamma}{2}$ . Now apply Theorem 4 to  $\mathcal{F}|_{A_1}$  to get  $A_2 \subseteq A_1$  of size

$$O\left(\max\left\{\frac{1}{\epsilon\delta} \ln \left(\frac{\frac{e}{d\epsilon\delta^2} \ln \frac{1}{\epsilon^d \delta^d \gamma}}{\gamma}\right)^d, \frac{1}{\epsilon\delta^2} \ln \left(\frac{1}{\epsilon^d \gamma}\right)\right\}\right) = O\left(\frac{1}{\epsilon\delta^2} \cdot \left(d \ln \frac{1}{\epsilon} + \ln \frac{1}{\gamma}\right)\right),$$

which is a relative  $(\epsilon, \frac{\delta}{3})$ -approximation of  $\mathcal{F}|_{A_1}$  with probability at least  $1 - \frac{\gamma}{2}$ . Thus  $A_2$  is a relative  $(\epsilon, \delta)$ -approximation of  $\mathcal{F}$  of the required size with probability at least  $1 - \gamma$ .

## 2 Proof of Theorem 4

Let  $n = |X|$  and  $t = |A|$ . We use the following consequence of Theorem 3 (better bounds exist [Hau95, Mus16]; however the one derived below suffices for our needs):

**Lemma 5.** *There is an absolute constant  $c_3$  such that the following holds. Let  $\alpha \geq 2$  and let  $\mathcal{P} \subseteq \mathcal{F}$  be an  $\alpha$ -packing of  $\mathcal{F}$ ; that is, for any  $S, S' \in \mathcal{P}$ , the symmetric difference of  $S$  and  $S'$ , denoted by  $\Delta(S, S')$ , has size at least  $\alpha$ . Then  $|\mathcal{P}| \leq \left(\frac{c_3 n}{\alpha}\right)^{2d}$ .*

*Proof.* Let  $\mathcal{G} = \{\Delta(S, S') : S, S' \in \mathcal{P}\}$ . By Theorem 3 there exists a relative  $(\frac{\alpha}{n}, \frac{1}{2})$ -approximation  $A'$  for  $\mathcal{G}$  of size

$$|A'| = \frac{c_1}{\frac{\alpha}{n} \cdot \frac{1}{4}} \left(d \ln \frac{2n}{\alpha} + \ln \frac{2n}{\alpha}\right) \leq \frac{8c_1 dn}{\alpha} \cdot \ln \frac{2n}{\alpha} \leq \frac{8c_1 dn^2}{\alpha^2},$$

where we set  $\gamma = \frac{\alpha}{2n}$  (note that we could set any positive value for  $\gamma$  as we only use the *existence* of such approximations). Then for any  $S, S' \in \mathcal{P}$ , we get

$$|\Delta(S, S') \cap A'| \geq \frac{|\Delta(S, S')| |A'|}{n} - \frac{|A'|}{2} \cdot \max\left\{\frac{|\Delta(S, S')|}{n}, \frac{\alpha}{n}\right\} = \frac{1}{2} \cdot \frac{|\Delta(S, S')| |A'|}{n} > 0.$$

This implies that  $A' \cap S \neq A' \cap S'$  for any  $S, S' \in \mathcal{P}$ , and so we have that  $|\mathcal{P}| = |\mathcal{P}|_{A'}$ . Finally, we use that  $\mathcal{P} \subset \mathcal{F}$  and thus  $|\mathcal{P}| = |\mathcal{P}|_{A'} \leq |\mathcal{F}|_{A'} \leq \left(\frac{8ec_1 n^2}{\alpha^2}\right)^d = \left(\frac{\sqrt{8ec_1} n}{\alpha}\right)^{2d}$ . Setting  $c_3 = \sqrt{8ec_1}$  concludes the proof.  $\square$

Set  $k = \lceil \log \frac{1}{\delta} \rceil$  and for  $i \in [0, k]$ , let  $\mathcal{P}_i$  be a *maximal*  $\frac{en}{2^i}$ -packing of  $\mathcal{F}$  and set  $\mathcal{P}_{k+1} = \mathcal{F}$ . For any  $S \in \mathcal{P}_{i+1} \setminus \mathcal{P}_i$  there exists a set  $F_S \in \mathcal{P}_i$  such that  $|\Delta(S, F_S)| < \frac{en}{2^i}$ . Define

$$\mathcal{A}_i = \{S \setminus F_S : S \in \mathcal{P}_{i+1} \setminus \mathcal{P}_i\} \quad \text{and} \quad \mathcal{B}_i = \{F_S \setminus S : S \in \mathcal{P}_{i+1} \setminus \mathcal{P}_i\}.$$

Lemma 5 implies that

$$|\mathcal{A}_i|, |\mathcal{B}_i| \leq |\mathcal{P}_{i+1}| \leq \left(\frac{c_3 \cdot 2^i}{\epsilon}\right)^{2d}.$$

**Claim 6.** *Let  $\epsilon_i = \sqrt{(i+1)/2^i} \epsilon$ . With probability  $1 - \gamma$ ,  $A$  is simultaneously*

- (i) *a relative  $(\epsilon, \delta)$ -approximation for  $\mathcal{A}_k \cup \mathcal{B}_k$ , and*
- (ii) *a relative  $(\epsilon_i, \delta)$ -approximation for  $\mathcal{A}_i \cup \mathcal{B}_i$  for all  $i \in [0, k-1]$ , and*
- (iii) *a relative  $(\epsilon, \delta)$ -approximation for  $\mathcal{P}_0$ .*

*Proof.* (i) Each set in  $\mathcal{A}_k \cup \mathcal{B}_k$  has size less than  $\frac{\epsilon n}{2^k} \leq \epsilon n \delta \leq \epsilon n$ . Therefore, we apply Theorem A with  $\eta = \delta t \epsilon$  and take the union bound over  $|\mathcal{A}_k \cup \mathcal{B}_k| \leq 2|\mathcal{F}|$  sets which gives that for a large-enough value of  $c_2$ ,  $A$  fails to be an  $(\epsilon, \delta)$ -approximation for  $\mathcal{A}_k \cup \mathcal{B}_k$  with probability at most

$$2|\mathcal{F}| \cdot 2 \exp\left(-\frac{\delta^2 t^2 \epsilon^2 \cdot n}{2\epsilon n \delta \cdot t + \delta t \epsilon \cdot n}\right) = 2|\mathcal{F}| \cdot 2 \exp\left(-\frac{\delta \epsilon t}{3}\right) \leq \frac{\gamma}{3}.$$

(ii) For a fixed  $S \in \mathcal{A}_i \cup \mathcal{B}_i$ , we have  $|S| \leq \frac{\epsilon n}{2^i} \leq \epsilon_i n$ . Thus, applying Theorem A with  $\eta = \delta t \epsilon_i$  implies that the probability of failure for a fixed set  $S \in \mathcal{A}_i \cup \mathcal{B}_i$  is at most

$$2 \exp\left(-\frac{\delta^2 t^2 \epsilon_i^2 n}{2|S|t + \delta \epsilon_i t n}\right) \leq 2 \exp\left(-\frac{\delta^2 t \epsilon^2 (i+1)/2^i}{2\epsilon/2^i + \delta \epsilon \sqrt{(i+1)/2^i}}\right) \leq 2 \exp\left(-\frac{\epsilon \delta^2 t (i+1)}{4}\right).$$

Hence, by the union bound, the overall probability of failure is at most

$$\begin{aligned} \sum_{i=0}^{k-1} |\mathcal{A}_i \cup \mathcal{B}_i| \cdot 2 \exp\left(-\frac{\epsilon \delta^2 t (i+1)}{4}\right) &\leq \sum_{i=0}^{k-1} 2 \left(\frac{c_3 \cdot 2^i}{\epsilon}\right)^{2d} 2(\epsilon^d \gamma)^{c_2(i+1)/4} \leq \gamma \sum_{i=0}^{k-1} \frac{4(c_3 \cdot 2^{i-1})^{2d}}{2^{(d+1)c_2(i+1)/4}} \\ &\leq \gamma \sum_{i=1}^{\infty} \frac{1}{5^i} \leq \frac{\gamma}{3}, \end{aligned}$$

for  $c_2 = 8 \log_2 c_3 + 18 \geq 8 \left(\log_2 c_3 + \frac{\log_2(5)}{2d+2} + 1\right)$ .

(iii) Since  $|\mathcal{P}_0| \leq \left(\frac{c_3}{\epsilon}\right)^{2d}$ , Theorem 1 implies that this failure probability is at most  $\frac{\gamma}{3}$  if  $t \geq \frac{3}{\epsilon \delta^2} \ln \frac{2(c_3/\epsilon)^{2d}}{\gamma/3}$ .  $\square$

Observe that for any set  $S \in \mathcal{F}$ , there exists a set  $S_k \in \mathcal{P}_k$ , with  $A_k = S \setminus S_k \in \mathcal{A}_k$  and  $B_k = S_k \setminus S \in \mathcal{B}_k$ , such that  $S = (S_k \setminus B_k) \cup A_k$ . Similarly, one can express  $S_k$  in terms of  $S_{k-1} \in \mathcal{P}_{k-1}$ ,  $A_{k-1} \in \mathcal{A}_{k-1}$ ,  $B_{k-1} \in \mathcal{B}_{k-1}$  and so on until we reach  $S_0 \in \mathcal{P}_0$ . Thus using Claim 6, with probability at least  $1 - \gamma$ ,

$$\begin{aligned} \left| \frac{|S|}{n} - \frac{|A \cap S|}{t} \right| &= \left| \frac{|S_k|}{n} - \frac{|B_k|}{n} + \frac{|A_k|}{n} - \left( \frac{|A \cap S_k|}{t} - \frac{|A \cap B_k|}{t} + \frac{|A \cap A_k|}{t} \right) \right| \\ &\stackrel{(i)}{\leq} \left| \frac{|S_k|}{n} - \frac{|A \cap S_k|}{t} \right| + \delta \max\left\{ \epsilon, \frac{|A_k|}{n} \right\} + \delta \max\left\{ \epsilon, \frac{|B_k|}{n} \right\} = \left| \frac{|S_k|}{n} - \frac{|A \cap S_k|}{t} \right| + 2\delta \epsilon \leq \dots \\ &\stackrel{(ii)}{\leq} \left| \frac{|S_0|}{n} - \frac{|A \cap S_0|}{t} \right| + 2\delta \sum_{j=0}^{k-1} \epsilon_j + 2\delta \epsilon \\ &\stackrel{(iii)}{\leq} \delta \max\left\{ \epsilon, \frac{|S_0|}{n} \right\} + 14\delta \epsilon \leq \delta \frac{|S|}{n} + 16\delta \epsilon \leq 2\delta \max\left\{ \frac{|S|}{n}, 16\epsilon \right\}, \end{aligned}$$

where the second-last step uses the fact that  $|S_0| \leq |S| + \sum_{j=0}^k |B_j| \leq |S| + \sum_{j=0}^{\infty} \frac{\epsilon n}{2^j} \leq |S| + 2\epsilon n$ .

Therefore,  $A$  is a relative  $(16\epsilon, 2\delta)$ -approximation of  $\mathcal{F}$  with probability at least  $1 - \gamma$ . Repeating the same arguments with  $\delta' = \delta/2$  and  $\epsilon' = \epsilon/16$ , we get a relative  $(\epsilon, \delta)$ -approximation of  $\mathcal{F}$ , as required.  $\square$

### 3 Proof of Theorem 3

The proof uses an argument similar to the discrepancy-based argument used for  $\epsilon$ -approximations [MWW93], though it is somewhat simpler as it does not need discrepancy, and it applies to the more general notion of a relative  $(\epsilon, \delta)$ -approximation.

To see the intuition, observe that since  $|\mathcal{F}| \leq (\epsilon|X|/d)^d$ , the bound of Theorem 1 depends only on  $|X|$ —in particular that a random sample  $A_1 \subseteq X$  of size  $O\left(\frac{1}{\epsilon \delta^2} \ln |X|^d\right) = O\left(\frac{d}{\epsilon \delta^2} \ln |X|\right)$  is a relative  $(\epsilon, \delta)$ -approximation. The size of  $A_1$  is much smaller than that of  $X$  and so applying Theorem 1 again to  $\mathcal{F}|_{A_1}$  gives a relative  $(\epsilon, \delta)$ -approximation  $A_2 \subseteq A_1$  for  $\mathcal{F}|_{A_1}$ , with

$$|A_2| = O\left(\frac{1}{\epsilon \delta^2} \ln |A_1|^d\right) = O\left(\frac{d}{\epsilon \delta^2} \ln \left(\frac{d}{\epsilon \delta^2} \ln |X|\right)\right) = O\left(\frac{d}{\epsilon \delta^2} \ln \frac{d}{\epsilon \delta} + \frac{d}{\epsilon \delta^2} \ln \ln |X|\right).$$

The size of  $A_2$  is again much smaller than that of  $A_1$ . Furthermore, it follows immediately from the definition of relative  $(\epsilon, \delta)$ -approximations that  $A_2$  is a relative  $(\epsilon, 3\delta)$ -approximation for  $\mathcal{F}$ . With each successive application of Theorem 1, the size of the set decreases rapidly, while the error of approximation increases only linearly, giving the required bound that is independent of  $|\mathcal{F}|$ .

Now we turn to the formal proof of Theorem 3. Let  $T(\epsilon, \delta, \gamma)$  be the smallest integer such that a uniform random sample of size at least  $T(\epsilon, \delta, \gamma)$  from  $X$  is a relative  $(\epsilon, \delta)$ -approximation for  $\mathcal{F}$  with probability at least  $1 - \gamma$ . Further define  $\delta_0 = 0$  and  $\delta_i = \frac{3^{i-1}}{\sqrt{|X|}}$  for  $i = 1, \dots, \left\lceil \frac{1}{2} \log_3(\sqrt{|X|}) \right\rceil + 1$ . We prove that for all  $i$ , for all  $\epsilon, \gamma \in (0, 1/2)$  and for all  $\delta \in (\delta_{i-1}, \delta_i]$ , it holds that  $T(\epsilon, \delta, \gamma) \leq \frac{c_1}{\epsilon\delta^2} \cdot \left( d \ln \frac{1}{\epsilon\delta} + \ln \frac{1}{\gamma} \right)$ , which is equivalent to the desired statement. The proof is by induction on  $i$ .

**Base case ( $i = 1$ ):** When  $\delta \in (0, \delta_0]$ , we have  $|X| \leq \frac{1}{\delta^2}$  and thus  $T(\epsilon, \delta, \gamma)$  is upper-bounded as required for any  $\epsilon, \gamma \in (0, 1/2)$ .

**Inductive hypothesis ( $j \leq i$ ):** Assume that the statement holds for all  $j \leq i$ , that is, for any  $\delta \in (0, \delta_i]$  and  $\epsilon, \gamma \in (0, 1/2)$ , we have  $T(\epsilon, \delta, \gamma) \leq \frac{c_1}{\epsilon\delta^2} \cdot \left( d \ln \frac{1}{\epsilon\delta} + \ln \frac{1}{\gamma} \right)$ .

**Inductive step ( $i \rightarrow i + 1$ ):** Let  $\delta \in (\delta_i, \delta_{i+1}]$ . Since  $\frac{\delta}{3} \in (0, \delta_i]$ , the inductive hypothesis gives that a random sample  $A' \subseteq X$  of size  $T(\epsilon, \frac{\delta}{3}, \frac{\gamma}{2}) \leq \frac{9c_1}{\epsilon\delta^2} \cdot \left( d \ln \frac{3}{\epsilon\delta} + \ln \frac{2}{\gamma} \right)$ , is a relative  $(\epsilon, \frac{\delta}{3})$ -approximation for  $\mathcal{F}$  with probability at least  $1 - \frac{\gamma}{2}$ . By Theorem 1, a uniform random sample  $A$  of  $A'$  of size

$$\frac{3}{\epsilon(\delta/3)^2} \ln \frac{2|\mathcal{F}|_{A'}}{(\gamma/2)}$$

is a relative  $(\epsilon, \frac{\delta}{3})$ -approximation for  $\mathcal{F}|_{A'}$  with probability  $1 - \frac{\gamma}{2}$ . Thus  $A$  is a uniform random sample of  $X$  that is a relative  $(\epsilon, \delta)$ -approximation for  $\mathcal{F}$  with probability at least  $1 - \gamma$ , implying the recurrence

$$T(\epsilon, \delta, \gamma) \leq |A| = \frac{3}{\epsilon(\delta/3)^2} \ln \frac{2|\mathcal{F}|_{A'}}{(\gamma/2)} \leq \frac{27}{\epsilon\delta^2} \ln \left( \frac{4}{\gamma} \left( \frac{e T(\epsilon, \frac{\delta}{3}, \frac{\gamma}{2})}{d} \right)^d \right).$$

The required bound on  $T(\epsilon, \delta, \gamma)$  now follows by the inductive hypothesis. As  $\left(1 + \frac{1}{d} \ln \frac{2}{\gamma}\right)^d \leq \frac{2}{\gamma}$ ,

$$\frac{27}{\epsilon\delta^2} \ln \left( \frac{4}{\gamma} \left( \frac{e \frac{9c_1}{\epsilon\delta^2} \left( d \ln \frac{3}{\epsilon\delta} + \ln \frac{2}{\gamma} \right)}{d} \right)^d \right) \leq \frac{27}{\epsilon\delta^2} \ln \left( \frac{4}{\gamma} \left( \frac{e 27 c_1}{\epsilon^2 \delta^3} \right)^d \left( 1 + \frac{1}{d} \ln \frac{2}{\gamma} \right)^d \right) \leq \frac{c_1}{\epsilon\delta^2} \ln \frac{1}{(\epsilon\delta)^d \gamma},$$

for any constant  $c_1 \geq 318$ , which concludes the proof of Theorem 3.  $\square$

**Acknowledgement.** We would like to thank the anonymous reviewers for their valuable comments which improved this paper.

## References

- [AB09] M. Anthony and P. L. Bartlett. *Neural network learning: Theoretical foundations*. Cambridge University Press, 2009.
- [AS12] N. Alon and J. Spencer. *The Probabilistic Method*. John Wiley, 2012.
- [BCM93] H. Brönnimann, B. Chazelle, and J. Matoušek. Product range spaces, sensitive sampling, and derandomization. *Proc. Symposium on Foundations of Computer Science*, pages 400–409, 1993.
- [Cha00] B. Chazelle. *The Discrepancy Method: Randomness and Complexity*. Cambridge University Press, New York, NY, USA, 2000.
- [DGL96] L. Devroye, L. Györfi, and G. Lugosi. *A Probabilistic Theory of Pattern Recognition*. Springer, Berlin, 1996.

- [FK16] A. Frieze and M. Karoński. *Introduction to random graphs*. Cambridge University Press, 2016.
- [Hau95] D. Haussler. Sphere Packing Numbers for Subsets of the Boolean n-Cube with Bounded Vapnik-Chervonenkis Dimension. *J. Comb. Theory, Ser. A*, 69(2):217–232, 1995.
- [HP11] S. Har-Peled. *Geometric Approximation Algorithms*. American Mathematical Society, Boston, MA, USA, 2011.
- [HS11] S. Har-Peled and M. Sharir. Relative  $(p, \epsilon)$ -Approximations in Geometry. *Discrete & Computational Geometry*, 45(3):462–496, 2011.
- [HW87] D. Haussler and E. Welzl.  $\epsilon$ -nets and simplex range queries. *Discrete & Computational Geometry*, 2:127–151, 1987.
- [KPW92] J. Komlós, J. Pach, and G. Woeginger. Almost tight bounds for  $\epsilon$ -nets. *Discrete & Computational Geometry*, 7:163–173, 1992.
- [KT59] A. N. Kolmogorov and V. M. Tikhomirov.  $\epsilon$ -entropy and  $\epsilon$ -capacity. *Uspekhi Mat. Nauk*, 14:3–86, 1959.
- [KV94] M. J. Kearns and U. V. Vazirani. *An Introduction to Computational Learning Theory*. MIT Press, 1994.
- [LLS01] Y. Li, P. M. Long, and A. Srinivasan. Improved Bounds on the Sample Complexity of Learning. *J. Comput. Syst. Sci.*, 62(3):516–527, 2001.
- [Mat99] J. Matoušek. *Geometric Discrepancy: An Illustrated Guide*. Springer, 1999.
- [Mat02] J. Matoušek. *Lectures in Discrete Geometry*. Springer-Verlag, New York, NY, 2002.
- [Mus16] N. H. Mustafa. A simple proof of the shallow packing lemma. *Discrete & Computational Geometry*, 55(3):739–743, 2016.
- [Mus22] N. H. Mustafa. *Sampling in Combinatorial and Geometric Set Systems*. AMS Press, 2022.
- [MWW93] J. Matoušek, E. Welzl, and L. Wernisch. Discrepancy and approximations for bounded VC-dimension. *Combinatorica*, 13(4):455–466, 1993.
- [Tal94] M. Talagrand. Sharper bounds for Gaussian and empirical processes. *Annals of Probability*, 22:28–76, 1994.
- [Tal16] M. Talagrand. *Upper and Lower Bounds for Stochastic Processes: Modern Methods and Classical Problems*. Springer Berlin Heidelberg, 2016.
- [VC71] V. Vapnik and A. Chervonenkis. On the uniform convergence of relative frequencies of events to their probabilities. *Theory of Probability and its Applications*, 16(2):264–280, 1971.

## Appendix

**Proof of Theorem A.**  $|A \cap S|$  follows hypergeometric distribution with expectation  $\frac{|S|t}{n}$ . Thus, we can apply the standard Chernoff’s tail estimate [FK16, Theorem 21.6 and Section 21.5] to get

$$\begin{aligned}
 \mathbb{P} \left[ |A \cap S| \notin \left( \frac{|S|t}{n} - \eta, \frac{|S|t}{n} + \eta \right) \right] &= \mathbb{P} \left[ |A \cap S| \leq \frac{|S|t}{n} - \eta \right] + \mathbb{P} \left[ |A \cap S| \geq \frac{|S|t}{n} + \eta \right] \\
 &\leq \exp \left( -\frac{\eta^2}{2(|S|t/n - \eta/3)} \right) + \exp \left( -\frac{\eta^2}{2(|S|t/n + \eta/3)} \right) \\
 &\leq 2 \exp \left( -\frac{n\eta^2}{2|S|t + n\eta} \right).
 \end{aligned}$$

□

**Proof of Theorem 1.** By Theorem A, a uniform random sample  $A$  of size  $t$  fails to be a relative  $(\epsilon, \delta)$ -approximation for a fixed  $S \in \mathcal{F}$  with probability at most  $2 \exp\left(-\frac{\epsilon \delta^2 t}{3}\right)$ . By the union bound,

$$\mathbb{P}\left[\exists S \in \mathcal{F} \text{ s.t. } |A \cap S| \notin \left(\frac{|S|t}{n} - \delta t \max\left\{\frac{|S|}{n}, \epsilon\right\}, \frac{|S|t}{n} + \delta t \max\left\{\frac{|S|}{n}, \epsilon\right\}\right)\right] \leq |\mathcal{F}| \cdot 2 \exp\left(-\frac{\epsilon \delta^2 t}{3}\right) \leq \gamma.$$

Therefore, with probability at least  $1 - \gamma$ ,  $A$  is a relative  $(\epsilon, \delta)$ -approximation for any set  $S \in \mathcal{F}$ .  $\square$