Resources Control Graphs

Jean-Yves Moyen

LIPN – CNRS

Motivations

- Programs analysis deal about uniform properties:
  - Do **all** the executions terminate?
  - Are **all** the executions performed within a given time/space bound ?
- Importance of the use of resources.
  - Time and space are resources usually considered.
  - Termination is usually reduced to finding a decreasing well-founded ordering, *i.e.* total usage of a finite resource.
  - Only specific resources may be considered (non-overflow of a specific buffer or stack).
- Design a single tool for this kind of analysis.

The core idea

Control Flow Graphs

We're working here over counters machines.

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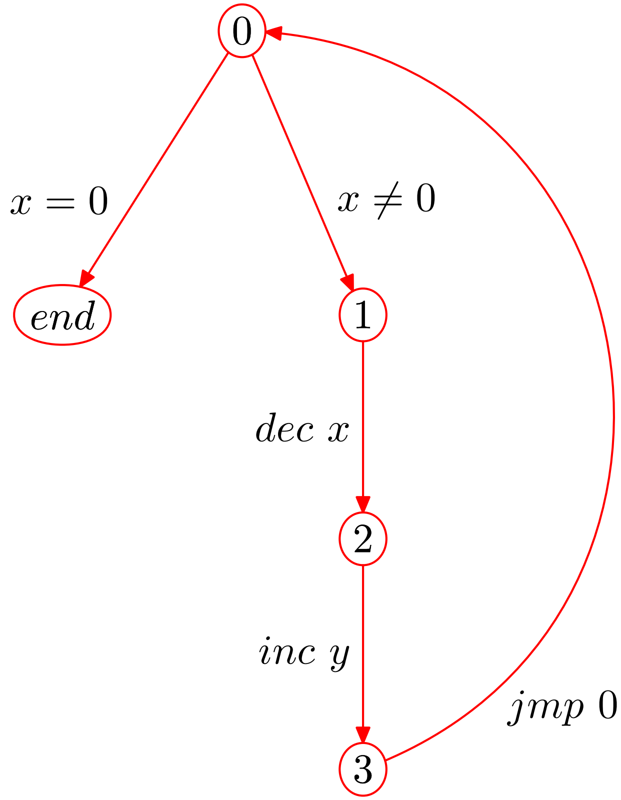

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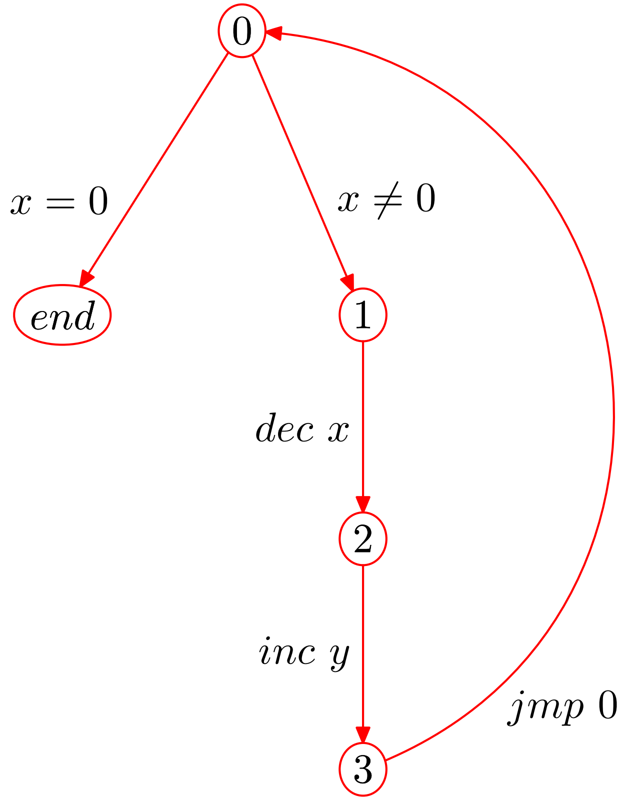

An execution of the program $\rightsquigarrow$ A path in the CFG.

Petri nets and VASS

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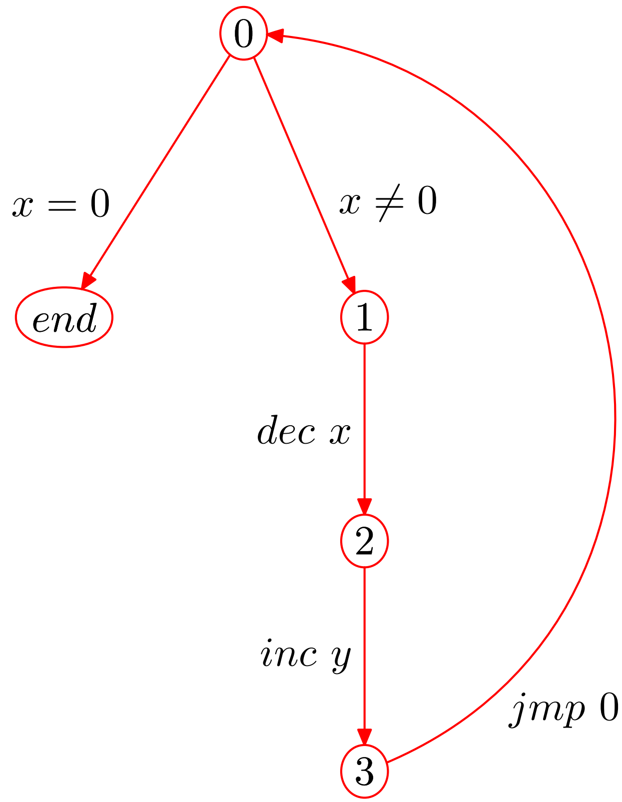

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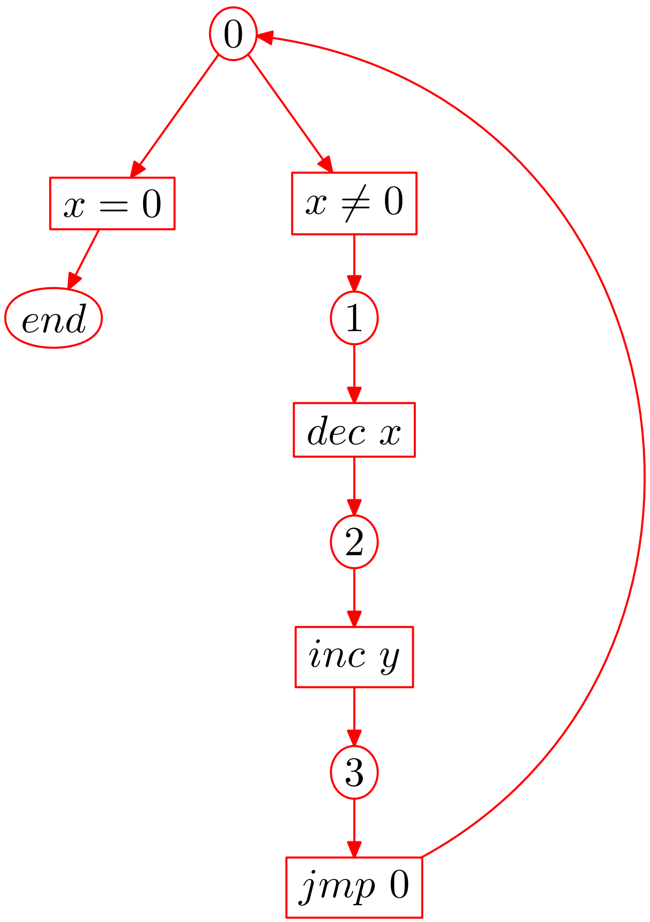

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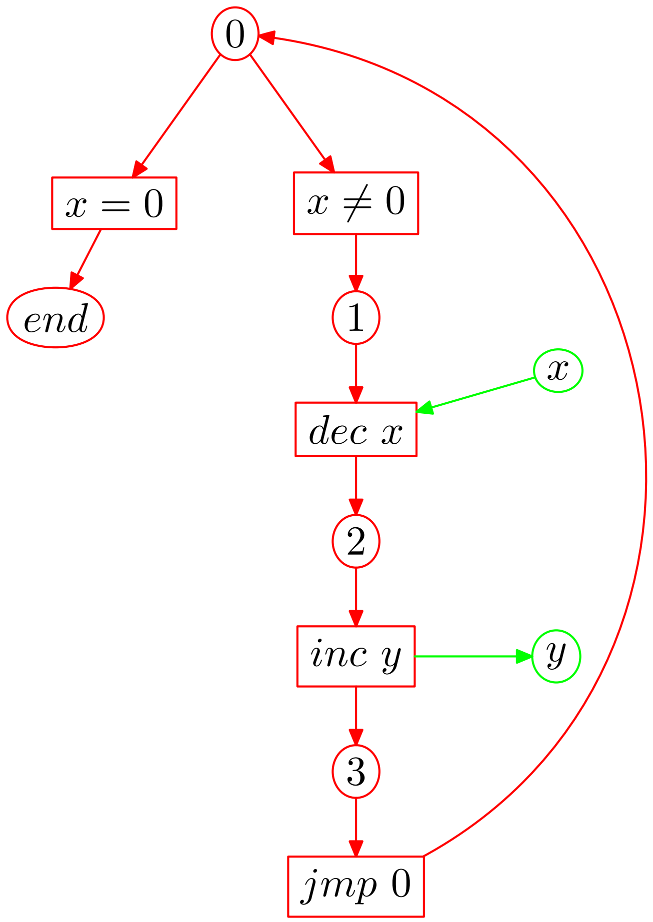

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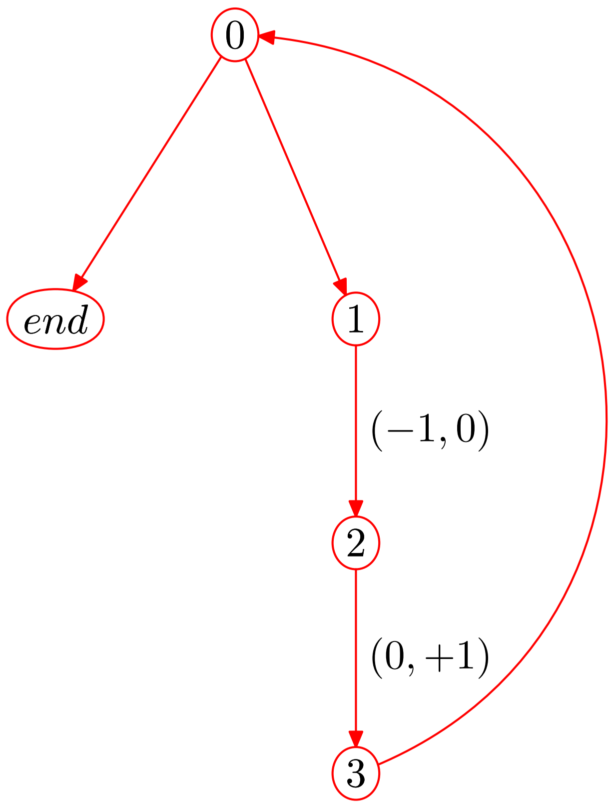

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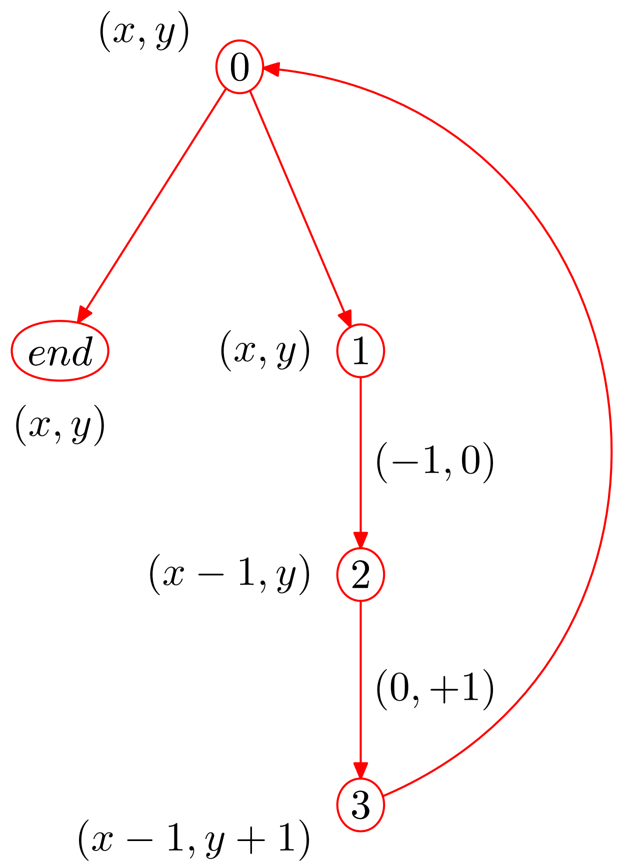

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- $V = \bigcup V_i$, $V^+ = \bigcup V_i^+$.
- $W_{i,j} = \mathcal{F}(V_i, V_j)$, $W = \bigcup W_{i,j}$ is the set of weights.
- $\omega : A \to W$ such that if $a = (s_i, s_j)$, then $\omega(a) \in W_{i,j}$.

Configuations and walks

Let $G = (S, A)$ be a graph.

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Let $R = (G, V, V^+, W, \omega)$ be a RSS.

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- A *walk* is a sequence of configurations $(s^1, x^1), \cdots, (s^n, x^n)$ such that:
  - $s^1, \cdots, s^n$ is a path.
  - If a^i is the edge between s^i and s^{i+1}, then $x^{i+1} = \omega(a^i)(x^i)$.

Turing Machines

A TM can be represented by a RSS in the following way:

- Underlying graph is the automaton of the TM.
- Valuations (and admissible valuations) are bi-infinite strings over $\{0, 1\}$.
- Weights perform the corresponding operations.

Each execution of the TM corresponds to an (admissible) walk in the RSS and each (admissible) walk in the RSS corresponds to an execution of the TM.

By Rice's theorem, extensionnal properties of RSS are not decidable.

Facts and notations

Let $R = (G, V, V^+, W, \omega)$ be a RSS. Let $(s^0, x^0), \cdots, (s^n, x^n)$ be a walk following edges $a^1, \cdots, a^n$.

- $x^n = \omega(a^n) \circ \cdots \circ \omega(a^1)(x^0)$.
- Functions composition is done in reverse order.
- Weight functions are usually somewhat uniform.
- We write $f \mathbin{⨟} g$ instead of $g \circ f$ and $x \circledast f$ instead of $f(x)$.
- $x^n = x^0 \circledast \omega(a^1) \mathbin{⨟} \cdots \mathbin{⨟} \omega(a^n)$.
- $⨟$ is associative.
- $\overline{W}$ is the closure of $\bigcup \omega(a_i)$ by $⨟$.

Ordering, termination, resource awarness

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- R *uniformely terminates* if there is no infinite admissible walk.
- R is f*-resource aware* if for each admissible walk $(s^0, x^0), \cdots, (s^i, x^i)$, $x^i \prec f(x^0)$ (f increasing).

A first result

R does not uniformely terminate $\Rightarrow \exists$ an admissible cycle $(s,x)\xrightarrow{*}(s,x')$ such that $x \preceq x'$.
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If the cycle exists, follow it infinitely many time. Monotonicity ensure that valuations reached increase and positivity that this keeps everything admissible.

Resource Control Graph

Let p be a program. It's Resource Control Graph (RCG) is a RSS where:

- The underlying graph is the Control Flow Graph.
- Admissible valuations approximate the state of memory.

Approximations allow to restrict weighting functions to some uniform familly of functions (*e.g.* $\lambda x.x + \alpha$).

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$$\boxed{\text{Unif. term. of the RCG} \Rightarrow \text{u.t. of the program.}}$$

Weighted graphs and Non Size Increasingness

Weighted graphs as RSS

- A weighted graph is a graph $G = (S, A)$ together with a weighting function $\omega : A \to \mathbb{Z}$.
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A weighted graph uniformely terminates if and only if it contains no cycle of weight ≥ 0.

It is $\lambda x.x + \alpha$-resource aware if it contains no cycle of weight > 0.

Both criterions can be decided in polynomial time.

Non Size Increasingness

Consider assembly-like programs working over lists of integers.

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This leads to the following RCG:

- $V^+ = \mathbb{N}$ (total space usage is ≥ 0).
- $\omega(\texttt{cons}) = \lambda x.x + 1$, $\omega(\texttt{tail}) = \lambda x.x - 1$.
- $\overline{W} = \bigcup \lambda x.x + \alpha$, $V = \mathbb{Z}$.

NSI (2)

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Admissible valuations play exactly the same role as M. Hofmann's diamonds: the valuation is equal to the number of diamonds needed in the current state.

Going further

- We can use more types if we have an appropriate size function.
- We can choose not to consider all the lists, *i.e.* only control some buffers.
- We can similarly control depth of data stacks (`push`, `pop`).
- We can similarly control depth of recursion stack (`call`, `return`).

Size Change Principle

Original SCP

(Lee, Jones, Ben Amram)

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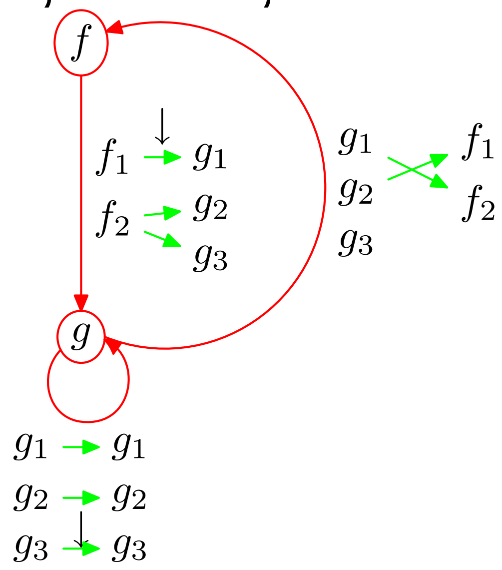

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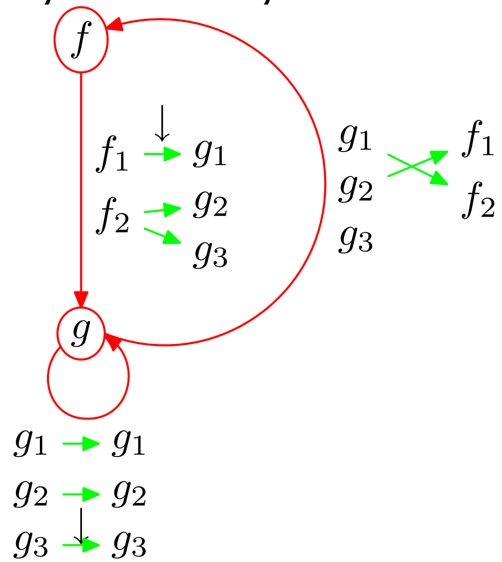

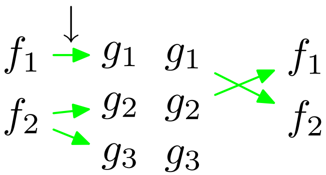

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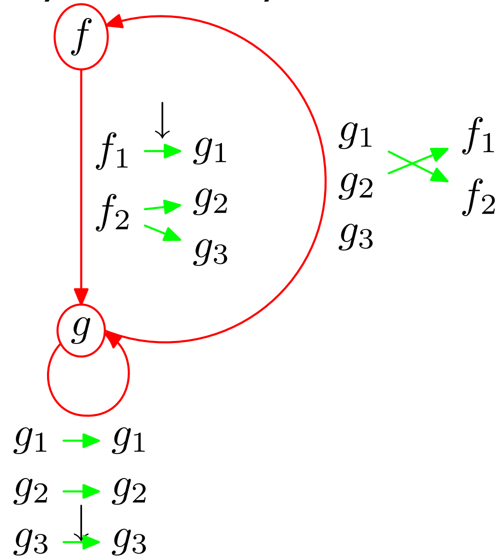

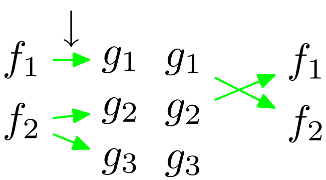

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$FLOW^{\omega}$, $DESC^{\omega}$

$FLOW^{\omega} = DESC^{\omega} \Rightarrow$ termination
(PSPACE-complet).

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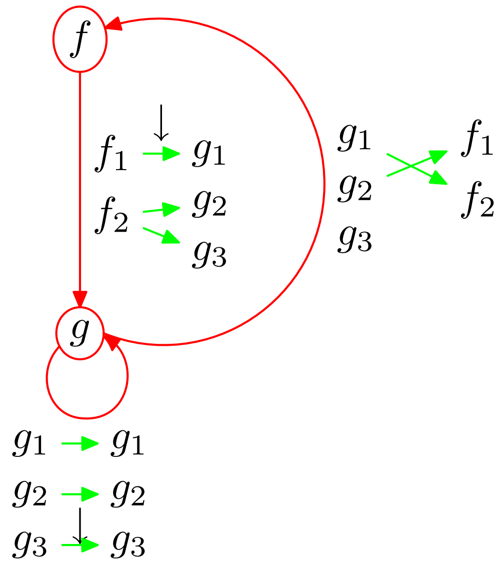

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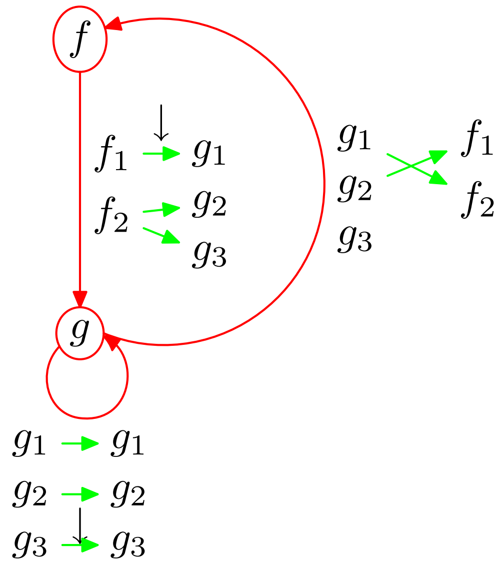

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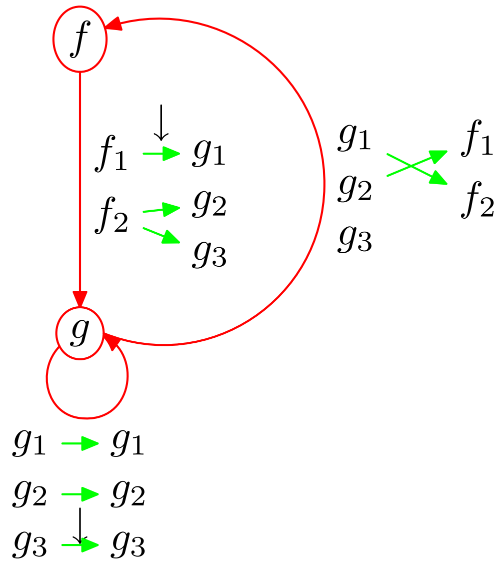

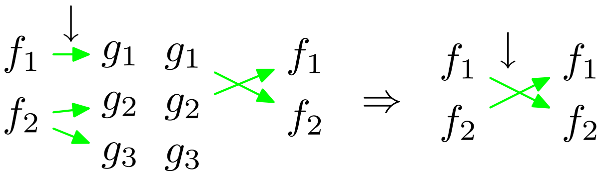

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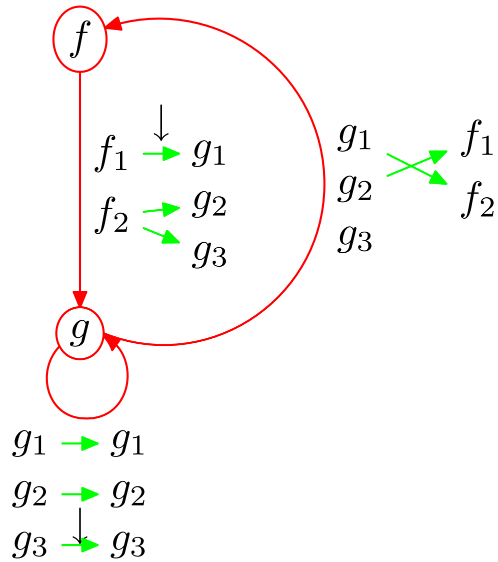

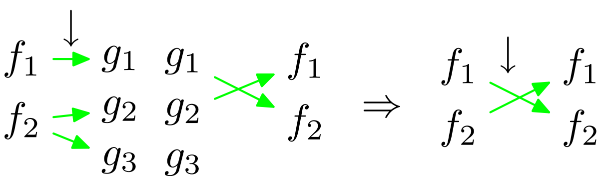

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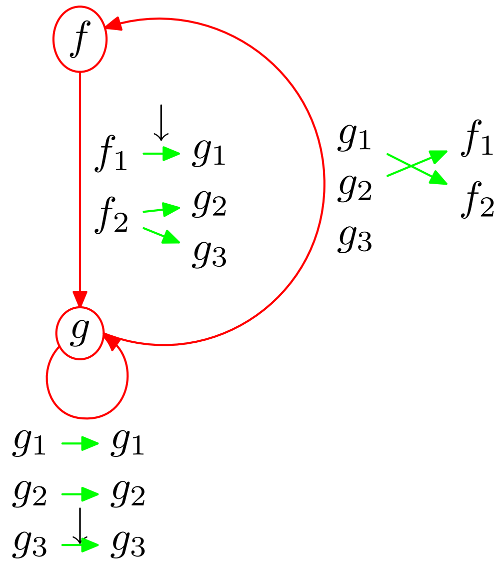

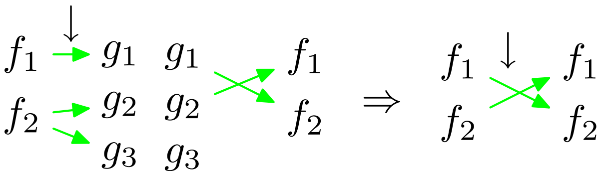

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SCP $\equiv$ uniform termination for all n.
Not u.t. $\Leftrightarrow \exists$ cycle of dec. idempotent weight.

Using matrices

(Abel and Altenkirch)

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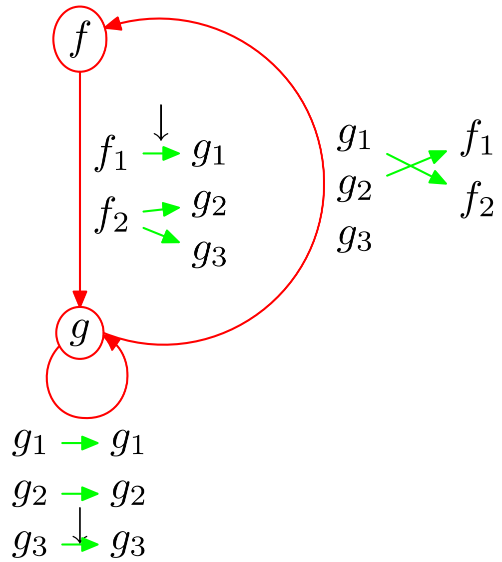

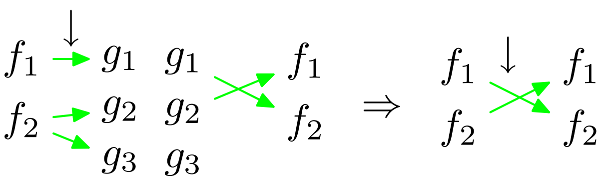

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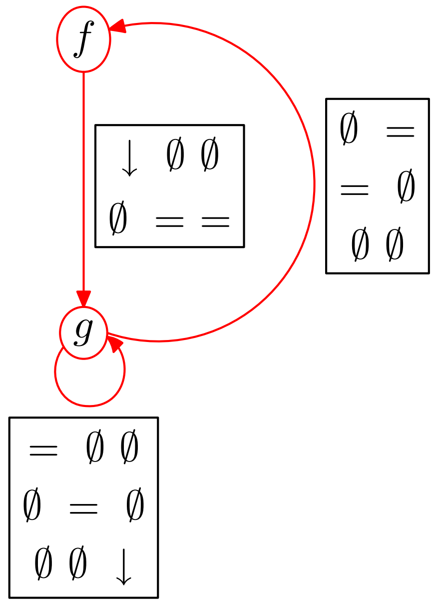

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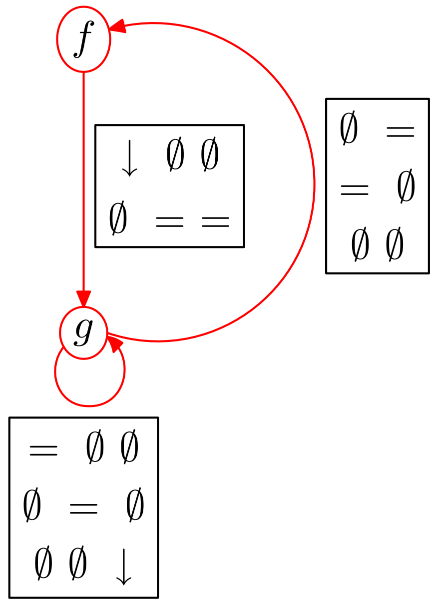

$$\boxed{\begin{matrix} \downarrow & \emptyset & \emptyset \\ \emptyset & = & = \end{matrix}} \otimes \boxed{\begin{matrix} \emptyset & = \\ = & \emptyset \\ \emptyset & \emptyset \end{matrix}} \Rightarrow \boxed{\begin{matrix} \emptyset & \downarrow \\ = & \emptyset \end{matrix}}$$

W: matrices over the three valued set.

$\fatsemi$: matrices multiplication.

Matrices Multiplication Systems with States

MMSS

A MMSS is a RSS where:

- $V_i = bZ^{k_i}$, $V_i^+ = \mathbb{N}^{k_i}$
- $W_{i,j} = \mathcal{M}_{k_i,k_j}$
- $\fatsemi = \circledast = \times$

If we use column-vectors instead of row-vectors, we need to transpose matrices and perform multiplication in reverse order.

Uniform termination

$\boxed{\text{Uniform termination of MMSS is not decidable}}$

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Reduce to uniform termination of counter machines.

- $x++, x--$: keep 1 as first component of vector.
- $x \neq 0$: $x--; x++$.
- $x = 0$: multiplication by -1.

Difficulty of instructions

- $x++, x--, x \neq 0$ can be modelised with VASS (*i.e.* with vectors).
- $x = 0$ needs MMSS (*i.e.* a matrix) to be modelised. (it is a well-known fact that Petri nets cannot test if a place is empty)
- First order program can be represented by several matrices, *i.e.* a tensor (given an enumeration of the edges).
- Tensor Multiplication Systems should be able to modelise high order programs.
- The higher the order, the more dimensions are required.

Polynomial bound

(Niggl and Wunderlich)

- Use a matrice over $\{0, 1, \infty\}$ as a certificate for polynomiality of instructions.
- Certificate for $P_1; P_2$: $M_1 \times M_2$.
- Certificate for tests: $\max$.
- Certificate for loops: closure.

Shape of the certificate of a program can lead to polytime bound.

Conclusion

- New tool, generalisation of existing one (Petri nets/VASS).
- Several analysis can be rewritten using this tool.
- Is it possible to combine these analysis ?
- Can we also rewrite other analysis ?
- Can we discover new analysis ?