Double Indexed Differential Linear Logic reconciling Resources and Differential operators

LPD & LSC seminar

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1. Background

The smooth semantics

Formulas:

- Each MALL formula is a finite dimentional vector space :
  $[\![1]\!] := \mathbb{R} \quad [\![A \otimes B]\!] := [\![A]\!] \otimes [\![B]\!] \quad [\![A \oplus B]\!] := [\![A]\!] \uplus [\![B]\!]$
  ...
- Exponentials are interpreted by infinite dimensional vector spaces:
  - $[\![?A]\!] := \mathcal{C}^\infty([\![A]\!]', \mathbb{R})$ (*functions*)
  - $[\![!A]\!] := \mathcal{C}^\infty([\![A]\!], \mathbb{R})'$ (*distributions*)
- Negation is duality: $[\![A^\perp]\!] := [\![A]\!]' = \mathcal{L}([\![A]\!], \mathbb{R})$

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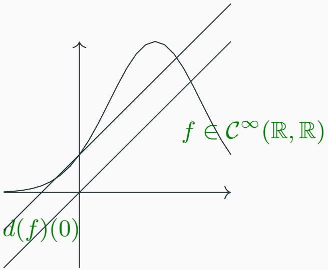
Proofs:

- Each proof is a **linear** map between the interpretation of the formulas.
- $A \Rightarrow B = !A \multimap B$ is $\mathcal{C}^\infty(A, B) \simeq \mathcal{L}(!A, B)$
- The **dereliction** states that $\mathcal{L}(A, B) \subseteq \mathcal{C}^\infty(A, B)$: it **forgets the linearity**.

Differential Linear Logic

Differential interaction nets. Ehrhard, Regnier (2004)



Differential linear logic is about linear extraction of a proof

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}\ \mathsf{d} \qquad\qquad \frac{f : !A \vdash B}{D_0(f) : A \vdash B}\ \bar{\mathsf{d}}$$

Differential Linear Logic

- Other rules has to be added (cut-elimination)

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ \mathsf{w} \quad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ \mathsf{c} \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ \mathsf{d} \quad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ \mathsf{p}$$

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$$\frac{}{\vdash !A}\ \bar{\mathsf{w}} \qquad \frac{\vdash \Gamma, !A \quad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\ \bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\ \bar{\mathsf{d}}$$

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$$\frac{}{\vdash \delta_0 : !A}\ \bar{\mathsf{w}} \qquad \frac{\vdash \Gamma, \psi : !A \quad \vdash \Delta, \phi : !A}{\vdash \Gamma, \Delta, \psi * \phi : !A}\ \bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, v : A}{\vdash \Gamma, D_0(\_)(v) : !A}\ \bar{\mathsf{d}}$$

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- They have nice **mathematical interpretation** (differential calculus)

$\bar{\mathsf{d}}/\mathsf{p}$ is the chain rule ...

Partial differential equations in the syntax

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Forgets linearity

$$\frac{f : !A \vdash B}{D_0(f) : A \vdash B}\ \bar{\mathsf{d}}$$

Applies D_0

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That is ℓ since $D_0(\ell) = \ell$

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Solves D

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~~*Apply D_0*~~

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Solution? (LPDO)

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~~*Apply* D_0~~

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Solution? (LPDO)

Type?

Graded linear logic

- *A core quantitative coeffect calculus.* Brunel et. al (2014)
- *Combining Effects and Coeffects via Grading.* Gaboardi et. al (2016)

Exponential rules of $\mathbf{B}_{\mathcal{S}}\mathbf{LL}$

$$\frac{\Gamma \vdash B}{\Gamma, !_0 A \vdash B}\ \mathsf{w} \qquad \frac{\Gamma, !_x A, !_y A \vdash B}{\Gamma, !_{x+y} A \vdash B}\ \mathsf{c} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !_1 A \vdash B}\ \mathsf{d}$$

$$\frac{!_{x_1} A_1, \ldots, !_{x_n} A_n \vdash B}{!_{x_1 \times y} A_1, \ldots, !_{x_n \times y} A_n \vdash !_y B}\ \mathsf{p} \qquad \frac{\Gamma, !_x A \vdash B \quad x \leq y}{\Gamma, !_y A \vdash B}\ \mathsf{d}_I$$

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Additive law

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Multiplicative law

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Order

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Order

$(\mathcal{S}, +, 0, \times, 1, \leq)$ is an ordered semiring

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- Type system for ressource consumption
- Coeffect analysis

The logic DB$_{\mathcal{S}}$LL

- A syntactical differentiation of B$_{\mathcal{S}}$LL

The exponential rules of DB$_{\mathcal{S}}$LL

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_0 A}\ \mathsf{w} \qquad \frac{\vdash \Gamma, ?_x A, ?_y A}{\vdash \Gamma, ?_{x+y} A}\ \mathsf{c} \qquad \frac{\vdash \Gamma, ?_x A \quad x \leq y}{\vdash \Gamma, ?_y A}\ \mathsf{d}_I$$

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Theorem

When $\mathcal{S}$ is additive splitting, and the order is define through the sum, DB$_{\mathcal{S}}$LL enjoys a cut elimination procedure.

Linear Partial Differential Operators

Definition

A **LPDOcc** is a linear operator defined as

$$D = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \qquad (a_\alpha \in \mathbb{R})$$

- A LPDO acts on smooth maps, or distributions.
- A **fundamental solution** is a distribution Φ_D s.t. $D(\Phi_D) = \delta_0$

- Monoidal structure: $(\mathcal{D}, \circ, id)$ is a monoid
- A crucial equality: $\Phi_{D_1 \circ D_2} = \Phi_{D_1} * \Phi_{D_2}$

The smooth semantics for IDiLL

Definition

$$\mathsf{w} : \begin{cases} \mathbb{R} & \to ?_{id}E \\ 1 & \mapsto cst_1 \end{cases} \qquad \bar{\mathsf{w}} : \begin{cases} \mathbb{R} & \to !_{id}E \\ 1 & \mapsto \delta_0 \end{cases}$$

$$\mathsf{c} : \begin{cases} ?_{D_1}E \mathbin{\hat{\otimes}} ?_{D_2}E & \to ?_{D_1 \circ D_2}E \\ f \otimes g & \mapsto \Phi_{D_1 \circ D_2} * (D_1(f).D_2(g)) \end{cases}$$

$$\bar{\mathsf{c}} : \begin{cases} !_{D_1}E \mathbin{\hat{\otimes}} !_{D_2}E & \to !_{D_1 \circ D_2}E \\ \psi \otimes \phi & \mapsto \psi * \phi \end{cases}$$

$$\mathsf{d}_I : \begin{cases} ?_{D_1}E & \to ?_{D_1 \circ D_2}E \\ f & \mapsto \Phi_{D_2} * f \end{cases} \qquad \bar{\mathsf{d}}_I : \begin{cases} !_{D_1}E & \to !_{D_1 \circ D_2}E \\ \psi & \mapsto \psi \circ D_2 \end{cases}$$

Theorem

The smooth semantics is **compatible** with the cut-elimination procedure.

Limitations

- $?(?A) = \mathcal{C}^\infty(\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}), \mathbb{R})$, how to describe it? Topological structure? Reflexivity?
- How would the promotion interact with $\bar{\mathsf{d}}_I$? Another chain rule?
- Which differential equations are represented?

We need linearly independant families to interpret partial derivatives:

- For finitary formulas, we are isomorphic to $\mathbb{R}^n$.
- For (E, V), we define $!_D(E, V)$ as $(!_D E, !_D V)$

2. From Laplace transform to polarization

How to combine differentiation, gradation and higher order?

- In linear logic

$$\frac{!A_1, \dots, !A_n \vdash B}{!A_1, \dots, !A_n \vdash !B}\ \mathsf{p} \qquad\qquad \frac{!A \xrightarrow{f} B}{!A \xrightarrow{\mathsf{p}_A} !!A \xrightarrow{!f} !B}$$

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For $f \in \mathcal{C}(A, B), g \in \mathcal{C}(B, C)$, ${\color{red}f; g}$ is:

$$!A \xrightarrow{p_A} !!A \xrightarrow{!f} !B \xrightarrow{g} C$$

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- In graded linear logic

$$\frac{!_{y_1} A_1, \dots, !_{y_n} A_n \vdash B}{!_{x \times y_1} A_1, \dots, !_{x \times y_n} A_n \vdash !_x B}\ \mathsf{p} \qquad \frac{!_y A \xrightarrow{f} B}{!_{x \times y} A \xrightarrow{\mathsf{p}_{A,x,y}} !_x !_y A \xrightarrow{!_x f} !_x B}$$

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- From monoid to semiring, requires discreteness

A polarized version?

Polarization of the formulas

Discrete version of the semiring

A short expedition in hell

- In DiLL, cut elimination introduces sums of proofs
- Semantically, $D(f.g) = D(f)g + fD(g)$
- In the syntax: proofs get very technical

$$\dfrac{\dfrac{\vdash ?A, ?A}{\vdash ?A}\,\mathsf{c} \qquad \dfrac{\vdash A^\perp}{\vdash !A^\perp}\,\bar{\mathsf{d}}}{\vdash}\,cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash ?A, ?A \qquad \dfrac{\vdash A^\perp}{\vdash !A^\perp}\,\bar{\mathsf{d}}}{\vdash ?A}\,cut \qquad \dfrac{}{\vdash !A^\perp}\,\bar{\mathsf{w}}}{\vdash}\,cut \;+\; \dfrac{\dfrac{\vdash ?A, ?A \qquad \dfrac{}{\vdash !A^\perp}\,\bar{\mathsf{w}}}{\vdash ?A}\,cut \qquad \dfrac{\vdash A^\perp}{\vdash !A^\perp}\,\bar{\mathsf{d}}}{\vdash}\,cut$$

- No more sums with the grading

$$\dfrac{\dfrac{\vdash ?_x A, ?_y A}{\vdash ?_{x+y} A}\,\mathsf{c} \qquad \dfrac{\vdash A^\perp}{\vdash !_1 A^\perp}\,\mathsf{d}}{\vdash}\,cut \qquad x + y = 1 \Rightarrow x = 0, y = 1 \text{ or } x = 1, y = 0$$

- A very heavy syntax, a discrete semiring

The Laplace transform, at higher order

The Laplace transform, for engineers:

$$\mathscr{L}(f) = x \mapsto \int e^{-xt} f(t)\mathsf{d}t$$

The Laplace transform, at higher-order:

$$\mathscr{L}(\psi) = \ell \in ?A \mapsto \psi(x \in A \mapsto e^{\ell(x)})$$

At the core of DiLL:

Laplace transform turns costructural rules into structural ones.

$$\mathscr{L}(\delta_0) = cst_1 \qquad \mathscr{L}(\psi * \phi) = \mathscr{L}(\psi).\mathscr{L}(\phi) \qquad \mathscr{L}(D_0(\_)(v)) = eval_v$$

The Laplace transform, at higher order

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At the core of DiLL:

Laplace transform turns costructural rules into structural ones.

$$\mathscr{L}(\bar{\mathsf{w}}) = \mathsf{w} \qquad \mathscr{L}(\bar{\mathsf{c}}) = \mathsf{c} \qquad \mathscr{L}(\bar{\mathsf{d}}) = \mathsf{d}$$

Laplace and semiring

- In DiLL:

$$!A \quad \overset{\mathscr{L}}{\simeq} \quad ?A \quad \overset{(\_)^\perp}{\simeq} \quad !A$$

- In graded DiLL:

$$\begin{aligned} !_P A &= \{P(\partial)(\psi) \mid \psi \in !A\} && \text{(parameters)} \\ &\overset{\mathscr{L}}{\simeq} \{P.f \mid f \in ?A\} = ?_P A && \text{(paramaters)} \\ &\overset{(\_)^\perp}{\simeq} \{\phi \mid \exists \psi \in !A, \psi = P.\phi\} = {\color{red} !_P A} && \text{(solutions)} \end{aligned}$$

Going back to heaven

Two isomorphic semirings, and

$$
\begin{aligned}
P, Q &:= 1 \mid 0 \mid P \otimes Q \mid P \oplus Q \mid ?_{x^+} P \mid !_{x^+} P \\
N, M &:= \top \mid \bot \mid N \parr M \mid N \,\&\, M \mid ?_{x^-} N \mid !_{x^-} N \\
A, B &:= N \mid P \mid A \otimes B \mid A \oplus B \mid A \parr B \mid A \,\&\, B
\end{aligned}
$$

The Laplace transform, orthogonal to the negation:

$$
\begin{tikzcd}
\mathsf{w}, \mathsf{c}, \mathsf{d}, \mu & ?_{x^-} A \arrow[d, "(\_)^\perp"'] & !_{x^-} A \arrow[l, "\mathscr{L}"'] \arrow[d, "(\_)^\perp"'] & \bar{\mathsf{w}}, \bar{\mathsf{c}}, \bar{\mathsf{d}}, \bar{\mu} \\
& !_{x^+} A \arrow[r, "\mathscr{L}"] & ?_{x^+} A &
\end{tikzcd}
$$

Polarized exponential rules

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_{0^-} N}\ \mathsf{w} \qquad \frac{\vdash \Gamma, ?_{x^-} N, ?_{y^-} N}{\vdash \Gamma, ?_{x^- + y^-} N}\ \mathsf{c} \qquad \frac{\vdash \Gamma, N}{\vdash \Gamma, ?_{1^-} N}\ \mathsf{d}$$

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$$\frac{\vdash \Gamma, ?_{x^-} N \qquad x^- \leq y^-}{\vdash \Gamma, ?_{y^-} N}\ \mathsf{d}_I \qquad \frac{\vdash \Gamma, !_{x^-} N \qquad x^- \leq y^-}{\vdash \Gamma, !_{y^-} N}\ \bar{\mathsf{d}}_I$$

$$\frac{\vdash ?_{X^-} \mathcal{N}, Q}{\vdash ?_{y^- \times X^-} \mathcal{N}, !_{y^+} Q}\ \mathsf{p} \qquad \frac{\vdash !_{x^-} N, P}{\vdash !_{y^- \times x^-} N, ?_{y^+} P}\ \bar{\mathsf{p}}$$

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$$\frac{}{\vdash !_{0^-} N}\,\bar{\mathsf{w}} \qquad \frac{\vdash \Gamma, !_{x^-} N \qquad \vdash \Delta, !_{y^-} N}{\vdash \Gamma, \Delta, !_{x^- + y^-} N}\,\bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, N}{\vdash \Gamma, !_{1^-} N}\,\bar{\mathsf{d}}$$

$$\frac{\vdash \Gamma, ?_{x^-} N \qquad x^- \leq y^-}{\vdash \Gamma, ?_{y^-} N}\,\mathsf{d}_I \qquad \frac{\vdash \Gamma, !_{x^-} N \qquad x^- \leq y^-}{\vdash \Gamma, !_{y^-} N}\,\bar{\mathsf{d}}_I$$

$$\frac{\vdash ?_{X^-} \mathcal{N}, Q}{\vdash ?_{y^- \times X^-} \mathcal{N}, !_{y^+} Q}\,\mathsf{p} \qquad \boxed{\frac{\vdash !_{x^-} N, P}{\vdash !_{y^- \times x^-} N, ?_{y^+} P}\,\bar{\mathsf{p}}}$$

From codigging to copromotion

- Analogy with the promotion:

$$\frac{\Gamma \vdash A}{!\Gamma \vdash !A}\,!_{\mathsf{f}} \qquad \frac{!!\Gamma \vdash B}{!\Gamma \vdash B}\,\mu \qquad \mathsf{p} = \dfrac{\dfrac{!\Gamma \vdash A}{!!\Gamma \vdash !A}\,!_{\mathsf{f}}}{!\Gamma \vdash !A}\,\mu$$

- The codigging rule:

$$\bar{\mu} : \begin{cases} !!A & \to !A \\ \delta_\phi & \mapsto \sum_n \frac{1}{n}{\phi^*}^n \end{cases}$$

- It gives:

$$\bar{\mathsf{p}} = \dfrac{\dfrac{?A \vdash B}{??A \vdash ?B}\,?_{\mathsf{f}}}{?A \vdash ?B}\,\bar{\mu} \qquad \frac{\vdash !A, B}{\vdash !A, ?B}\,\bar{\mathsf{p}}$$

- Restricted to one formula, because of the binarity of $\bar{\mathsf{c}}$

A simple cut elimination

- Negative connectives cannot interact together
- Only two rules with positive connectives: p and $\bar{\text{p}}$
- $!_{x^+}A$ in p $\rightarrow$ interacts with structural rules
- $?_{x^+}A$ in $\bar{\text{p}}$ $\rightarrow$ interacts with costructural rules
- $\bar{\text{p}}$ and costructural rules is very similar to p and structural rules
- No interaction between p and $\bar{\text{p}}$

We simply use the cut elimination of LL, and an easy translation:

Theorem

The logic DIDiLL enjoys a cut elimination procedure.

Algebraic conditions on the semiring

Definition

A weak semiring is a tuple $(\mathcal{S}, +, \times, 0, 1)$ which satisfies:

- $+$ is commutative (c and $\bar{\text{c}}$), 0 neutral element;
- $\times$ is associative (p and $\bar{\text{p}}$);
- right distributivity: $(x + y)z = xz + yz$ (p/c and $\bar{\text{p}}/\bar{\text{c}}$);
- 0 left absorbing: $0 \times x = 0$ (p/w and $\bar{\text{p}}/\bar{\text{w}}$);
- 1 left neutral: $1 \times x = x$ (p/d and $\bar{\text{p}}/\bar{\text{d}}$);
- $\times$ is right monotonous over $\leq$: $x \leq y \Rightarrow xz \leq yz$ (p/d_I and $\bar{\text{p}}/\bar{\text{d}}_I$).

3. A model with differential operators

A semiring of LPDO

Set of operators:

$$\mathcal{D} = \left\{ \sum_{\alpha \in \mathbb{N}^n} a_\alpha \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \right\}$$

Two operations:

$$D_1 \boxplus D_2 = D_1 \circ D_2 \qquad\qquad \sum_\alpha a_\alpha \partial^\alpha \boxtimes D = \sum_\alpha a_\alpha D^\alpha$$

Its Laplace transform:

$$\mathscr{L}(\sum_\alpha a_\alpha \partial^\alpha) = \sum_\alpha a_\alpha X_1^{\alpha_1} \dots X_n^{\alpha_n} = \text{polynomials}$$

Kothe spaces (simplified)

- Spaces of sequences: $E \subseteq \mathbb{R}^{\mathbb{N}}$
- Based on an orthogonality:

$$E^{\perp} = \{(\alpha_n) \in \mathbb{R}^{\mathbb{N}} \mid \forall(\lambda_n) \in E, (\lambda_n \alpha_n) \in \ell_1\}$$

- A Kothe space: (X, E_X) with $X \subseteq \mathbb{N}$, $E_X \subseteq \mathbb{R}^X$ s.t. $E_X^{\perp\perp} = E_X$
- In our setting: X is totally ordered

$$X = (x_1, \ldots, x_n, \ldots) \qquad \text{basis of the vector space}$$

Kothe spaces and linear logic

- $E \multimap F$: the subset of $\mathbb{R}^{X \times Y}$ such that

$$\sum_{i,j} M_{i,j} x_i y'_j \text{ converges absolutely}, x \in E_X, y' \in F_Y^\perp$$

  $X \times Y$ is lexicographically ordered
- Similar constructions for all LL connectives

For the exponential:

- The basis is the set of finite multisets of X
- For $x \in E$, $x^!_\mu = \Pi_n x_n^{\mu(n)}$ and $!E = \{x^! \mid x \in E\}^{\perp\perp}$
- Order: sizes of the multisets, and weighted lexicographical order

It corresponds to the dual of the set of scalar entire maps

Mixing Kothe spaces and LPDO

For $x \in E$, $x = \sum_i \lambda_i x_i$. We define

$$P(x) := \sum_{\alpha \in \mathbb{N}^\omega} a_\alpha \prod_i \lambda_i^{\alpha_i}$$

$$\begin{array}{ccccccc} & ?^-_P E = P \cdot (?E) & \xleftarrow{\mathscr{L}} & !^-_P E = P(\partial)(!E) & \\ \mathsf{w}, \mathsf{c}, \mathsf{d}, \mu & (\_)^\perp \big\downarrow & & (\_)^\perp \big\downarrow & \bar{\mathsf{w}}, \bar{\mathsf{c}}, \bar{\mathsf{d}} \\ & !^+_P E = P^{-1} \cdot (!E) & \xrightarrow{\mathscr{L}} & ?^+_P E = P(\partial)^{-1}(?E) & \end{array}$$

The exponential rules are interpreted as in DiLL

The promotion corresponds to the composition

$$(P \cdot f) \tilde{\circ} (Q \cdot g) = (P \boxtimes Q) \cdot (f \circ g)$$

An second model, with a codigging

At the origin of the codigging: $\mathcal{F}_\theta$ and $\mathcal{G}_\theta$

- Spaces of functions with a growth controlled by θ (Young function)
- Laplace transforms θ in its convex conjugate
- Natural operations on the set of θ, which is a weak semiring
- It gives a model of our logic, with copromotion

Conclusion

We have a syntax which is:

- Graded, polarized, with promotion, copromotion and a cut elimination
- A model with differential operators based on Kothe spaces
- A model with a copromotion based on $\mathcal{F}_\theta$

Conclusion

We have a syntax which is:

- Graded, polarized, with promotion, copromotion and a cut elimination
- A model with differential operators based on Kothe spaces
- A model with a copromotion based on $\mathcal{F}_\theta$

What we want to do next:

- Study what happens in hell
- A model with differential operators and copromotion
- The categorical model

Thank you!