Double Indexed Differential Linear Logic reconciling Resources and Differential operators

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Abstract

Graded Linear Logic is a central development of Linear Logic, quantifying the use of resources in proofs and programs. Differential Linear Logic, on the other hand, dualises this focus on resources by adding co-structural rules defining proof differentiation. In this paper we use the Laplace transform as the final building block to reconcile co-structural rules with graded exponential. We define a graded differential linear logic where structural and co-structural rules act on four different exponential connectives. Contrarily to previous work, this proof calculus goes higher-order by including a promotion rule, and also provides a sound syntax for the recently introduced co-promotion rule. We provide two different models for this syntax: the first extends a previously known first-order model to higher order thanks to Köthe spaces, where the indices are either polynomials or linear partial differential operators. The second arises from the literature in functional analysis, the indices being Young functions and their convex conjugate.

Keywords: Linear Logic, Denotational Semantics, Graded Calculus, Differential Calculus, Functional analysis

Contents

1 Introduction and summary of contributions

In this paper, we reconcile two major developments of Linear Logic: Graded Linear Logic and Differential Linear Logic. This is motivated by the need to handle differential operators as supplementary data for formulas of Linear Logic, and is done thanks to the newly understood role of the Laplace transform in Differential Linear Logic. By designing this new logic, we solve several seemingly unrelated issues. First, we give a higher-order understanding of Differential Operators acting on Köthe spaces. We also define a syntax for the newly introduced codigging rule in differential categories, and prove cut-elimination. Finally, we allow for a higher-order extension of Fréchet and DF-spaces as a smooth model of Differential Linear Logic.

Context Linear logic (LL) has been introduced by Girard in the late 80s [Gir87] as a refinement of intuitionnistic logic. It decomposes the usual implication $\Rightarrow$ using two new connectors, the *linear implication* $\multimap$ and the *of course* connector !, which are related to the implication by the following equivalence:

$$A \Rightarrow B \equiv\, !A \multimap B.$$

This logic has a strong computational content, that can easily be understood in terms of resource consumption. A *linear proof*, which is either a proof of $\vdash A \multimap B$ or a proof of $A \vdash B$, corresponds to a program (or a proof) that uses exactly once the resource A (or the hypothesis A) to produce B (or to prove B). With this interpretation, the connector "of course" should be understood as an infinite bag of resources. In this setting, structural rules occurs only on the ! connector, defining the so-called structural rules of LL with the weakening w, the contraction c, the dereliction d and the promotion p. While the dereliction allows to forget the linearity of a proof, the promotion allows going higher-order to allow a cut between two non-linear proofs (or a composition between two non-linear functions in a categorical point of view). The structural rules of linear logic are the following.

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\,\mathsf{w} \quad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\,\mathsf{c} \quad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\,\mathsf{d} \quad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\,\mathsf{p}$$

They could also be presented in their one-sided version using the "why not" ? connector, dual to the ! connector.

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\,\mathsf{w} \quad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\,\mathsf{c} \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\,\mathsf{d} \quad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,\mathsf{p} \tag{1}$$

With the resource intuition in mind, the rise of indices on the ! connector is natural, as an index x on $!_x A$ would represent the number of time the hypothesis A is used. Bounded linear logic [GSS91] was one of the first development of LL, in which the exponentials are indexed by inequalities in order to bound the complexity of a proof. Revisiting Bounded Linear Logic with a more algebraic point of view, graded linear logic has exponentials indexed with elements of a semiring [GKO+16, GS14]. It been influential in other fields of research, such as differential privacy [GHH+13]. The structural rules in this case match naturally with the axioms of a semiring.

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_0 A}\,\mathsf{w} \qquad \frac{\vdash \Gamma, ?_x A, ?_y A}{\vdash \Gamma, ?_{x+y} A}\,\mathsf{c} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?_1 A}\,\mathsf{d} \tag{2}$$

$$\frac{\vdash \Gamma, ?_x A \qquad x \leq y}{\vdash \Gamma, ?_y A}\,\mathsf{d}_I \qquad \frac{\vdash ?_Y \Gamma, A}{\vdash ?_{x\times Y}\Gamma, !_x A}\,\mathsf{p}$$

In terms of resources, d means that a linear proof uses once a resource, and d_I that resources usage agree to the order of the semiring.

Differential linear logic (DiLL) on the other side is an orthogonal development of Linear Logic, defined by Ehrhard and Regnier in 2006 [ER06]. DiLL was the key to a new point of view of on programming language, via the Taylor expansion [ER08] or in the field of probabilistic programming [EPT18].Now proofs can not only be linear, but they can be *linearized*. This is done by adding to the previous structural rules three costructural rules: the coweakening $\bar{\mathsf{w}}$, the cocontraction $\bar{\mathsf{c}}$ and the codereliction $\bar{\mathsf{d}}$.

$$\frac{}{\vdash !A}\,\bar{\mathsf{w}} \qquad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\,\bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\,\bar{\mathsf{d}} \tag{3}$$

From a semantical point of view, the codereliction rule is the crucial one. Indeed, operating a cut rule between $\bar{\mathsf{d}}$ and a proof π of $!A \vdash B$ results in a proof of $A \vdash B$, that is a linear version of π. The cut-elimination procedure between structural and costructural rules then represents the basic rules of differential calculus (chain rule, addition and multiplication of functions).

Challenges Based on graded linear logic, Breuvart, Kerjean and Mirwasser have defined an Indexed Differential Linear Logic [BKM23], where the dereliction rule d has a new meaning. One can indeed understand the action of the dereliction and the codereliction as respectively solving a differential equation and applying a differential equation. This analogy works well in particular for equations involving Linear Partial Differential Operators with constant coefficients. The core idea behind IDiLL is to introduce two new exponential connectors, $!_D$ and $?_D$, which respectively represent the space of solutions and the space of parameters of the differential equation associated to D. The dereliction and the codereliction are now graded:

$$\frac{\vdash \Gamma, ?_{D_1} A}{\vdash \Gamma, ?_{D_1 \circ D_2} A}\ \mathsf{d}_I \qquad\qquad \frac{\vdash \Gamma, !_{D_1} A}{\vdash \Gamma, !_{D_1 \circ D_2} A}\ \bar{\mathsf{d}}_I$$

as well as indexed versions of the other structural and costructural rules. A very notable exception is that *no graded promotion* for IDiLL could be found, for several reasons: first, making differential operators act on higher-order functions, for example $f : \mathcal{C}^\infty(\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}), \mathbb{R})$ is a challenge. Second, the interaction between Linear partial differential operators ($\bar{\mathsf{d}}_I$) and the composition of non-linear functions (interpreting p) is intricate to describe and vastly different to the chain rule describe by DiLL.

This is not the only pitfall with IDiLL. Most notably, it exploits heavily the fact that any equation $D(\_) = g$, where D is a linear partial differential operator with constant coefficients, has a solution, and gives a most inelegant interpretation for the contraction rule.

Basically, structural and costructural rules in Graded DiLL do not seem compatible. In this paper, we provide a formal explanation for this fact exploiting a recent result by Kerjean and Lemay [KL24]: there is a reason why structural and costructural rules are symmetrical, and it is the good old Laplace transform.

The Laplace transform in play This functional operation is a well-known tool in applied mathematics, operating between real functions, as it turns differential equations into polynomial equations and back. In its higher-order version, it can be typed as a linear isomorphism between the two exponential connectors:

$$\mathscr{L} : !A \overset{\simeq}{\multimap} ?A$$

The fundamental interest to DiLL is that it changes in fact the interpretation of the costructural rules into the interpretation of the structural ones:

$$\mathscr{L}(\bar{\mathsf{w}}, \bar{\mathsf{c}}, \bar{\mathsf{d}}) = \mathsf{w}, \mathsf{c}, \mathsf{d}$$

This result extends to the codigging $\bar{\mu}$ [KL23] being transformed into the digging, which is the semantical counterpart of the promotion.

In this paper, we argue that $\mathscr{L}$ also acts on indices of exponential connectors. We justify the fact that structural and costructural rules should not act on the same set of indices, or said otherwise these rules should not act on the same pair of exponentials:

$$\begin{array}{cccccc} & ?^-_P A & \overset{\mathscr{L}}{\longleftarrow} & !^-_P A & \\ \mathsf{w}, \mathsf{c}, \mathsf{d}, \mu & (\_)^\perp \big\downarrow & & (\_)^\perp \big\downarrow & \bar{\mathsf{w}}, \bar{\mathsf{c}}, \bar{\mathsf{d}}, \bar{\mu} \\ & !^+_P A & \overset{\mathscr{L}}{\longrightarrow} & ?^+_P A & \end{array}$$

We use upper indices $+$, $-$ similarly to the notions of positive and negative connectors in Polarized Linear Logic [Lau02]. This analogy will be justified in section 5.3, exhibiting a polarized model of graded connectors.

A graded version of Differential Linear Logic should take this change of indices into account when defining the graded version of the costructural rules. Section 3 builds on these ideas the logic DIDiLL, a Double Indexed Differential Linear Logic. The key is that *now structural and costructural rules do not interact through cut-elimination.*

New developments This new setting allows all at once to solve several issues. First, it allows to give a syntactical presentation for the recently introduced codigging rule [KL23]. This transformation $\bar{\mu} : !!A \multimap !A$, defined in differential categories [BCLS20], is dual to the digging rule $\mu : !A \multimap !!A$ interpreting the promotion. However, a syntactical equivalent to the codigging was not possible, as no cut-elimination rule between the promotion and copromotion made sense. In a setting where structural and co-structural rules do not interact, the copromotion rule can be studied and defined.

Second, IDiLL and its model can be extended to higher-order. This is done by indexing structural rules by polynomials, and costructural rules by linear partial differential operators with constant coefficients. While IDiLL's proofs could only be interpreted by functions between finite dimensional vector spaces $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$, we now specify the setting to Köthe spaces [Ehr02]. This means that each vector space comes equipped with a countable basis X, and that differential operators now act with respect to this basis. We put a total order on these bases in order to have an interpretation for partial differentiation with respect to a vector x_j. In this graded version of Köthe spaces, we are able to consider functions with infinite-dimensional codomain. This is done in section 4.

Last but not least, DIDiLL is the right syntax for having finally a smooth and higher-order model of DiLL. By smooth, one means that the computations are not based in any way on a discrete set, contrarily to Köthe spaces. This quest for a higher-order, smooth model of classical DiLL is explained in section 5, in which we show that work by Ouerdiane *et al.* [GHOR00] [CEOO02] is a model of DIDiLL.

Contributions Let us sum up our contributions and the outline of the paper.

- Section 2 contains background material, on smooth models of DiLL (Section 2.1), on the codigging (Section 2.2) , and on previous work on indexed differential linear logic (Section 2.3).

- Section 3 introduces the calculus DIDiLL, a double indexed differential linear logic (Figure 1). It details its cut elimination procedure with the newly introduced copromotion rule (Figure 2) and proves a cut-elimination procedure (Theorem 3). It compares the calculus with polarized linear logic (Section 3.3).

- Section 4 extends IDiLL to higher-order, by providing a model to DIDiLL indexed by polynomials and linear partial differential operators with constant coefficients. It recalls the fundamental of the interpretation of DiLL in Köthe spaces (Section 4.1). It defines a semiring on the set of polynomials, and defines the action of linear partial differential operators with constant coefficients on Köthe spaces in Section 4.2. It describes the action of the Laplace transform on Köthe spaces in propositions 11.

- Section 5 introduces a model of DIDiLL based on Fréchet or DF-Nuclear spaces, where indices quantify over the exponential growth of functions. After a quick recap on the historical context in Section 5.0.1, Section 5.1 recalls basic results on reflexive topological vector spaces. Section 5.2 present results in the literature by Ouerdiane and al [GHOR00], and Section 5.3 interprets DIDiLL thanks to them. Although this section is placed at the end of the paper, let us insist that it is central to our results, and that the syntax presented in section 3 was in fact designed following that model.

- Section 6 concludes on further possible developments to this work.

Detailed proofs are put in the appendix, in Section .1.

2 Differentiation, Grading and Polarization in Linear Logic

The syntax for formulas of Linear Logic is the following:

$$A, B := 0 \mid 1 \mid \top \mid \bot \mid A \otimes B \mid A \parr B \mid A \,\&\, B \mid A \oplus B.$$

The connectors $\parr$ and $\otimes$ are called respectively the multiplicative disjunction and multiplicative conjunction. The connectors $\oplus$ and & are called respectively the additive disjunction and multiplicative conjunction. They are introduced thanks to different variations of the introduction of disjunction and conjunctions, which are not recalled here (see [Gir87]). In the rest of the paper, we will favor a classical and one-sided presentation of Linear Logic, using the connector ? dual to !, as in Equation 1. Both ! and ? are called exponential connectors. The sequent calculus for LL is defined as the rules for additive and multiplicative connectors, as well as the dereliction, contraction, weakening and promotion rule presented in the introduction (Equation 1). The sequent calculus for DiLL adds the coweakening, cocontraction, codereliction rules of Equation 3 to the one for LL.

2.1 Smooth models of DiLL

We now give a few mathematical intuitions about the interpretation of the structural and costructural rules of DiLL. Several models of LL or DiLL are extensions of coherence spaces or of the relational model [Ehr05] [Ehr02] [DE11]. Here we focus on so-called smooth models [BET12, DK20, Ker18]: formulas A are interpreted as quite general topological $\mathbb{K}$-vector spaces $[\![A]\!]$ and non-linear proofs of $!A \vdash B$ are interpreted as smooth functions in $\mathcal{C}^\infty([\![A]\!], [\![B]\!])$. In these models, the exponential connectors ? and ! are interpreted respectively as spaces of scalar functions and as spaces of distributions with compact support

$$[\![?A]\!] = \mathcal{C}^\infty([\![A]\!]', \mathbb{K}) \qquad [\![!A]\!] = \mathcal{C}^\infty([\![A]\!], \mathbb{K})'.$$

The linear negation $(\_)^\perp$ is interpreted as the linear dual $(\_)' : \mathcal{L}(\_, \mathbb{K})$, being the space of scalar linear continuous functions. Categorically, models of DiLL are instances of linear differential categories [BCS06] [BCLS20]. The contraction rule is then interpreted as a natural transformation $\mathsf{c}_E : ?E \otimes ?E \to ?E$ representing the point wise scalar multiplication of functions, while $\mathsf{w}_E : \mathbb{K} \to ?E$ introduces the function cst_1 constant at $1 : \mathbb{K}$. Conversely, the cocontraction $\bar{\mathsf{c}}_E : !E \otimes !E \to !E$ represents the convolution $\phi * \psi$ of two distributions $\phi, \psi : !E$, while the coweakening $\bar{\mathsf{w}}_E : \mathbb{K} \to !E$ represents the introduction of δ_0, the dirac at 0. The dereliction $\mathsf{d}_E : E \to ?E$ maps a vector v to the evaluation of linear forms in E' along this vector v, which is a linear map but in particular a smooth map. The codereliction $\bar{\mathsf{d}}_E : E \to !E$ maps a vector v to the distribution $D_0(\_)(v)$ mapping functions to their differential at 0 along v. Finally, the promotion transforms a smooth map $f \in \mathcal{C}^\infty([\![A]\!], [\![B]\!])$ into the smooth map $\delta_f : a \mapsto (\delta_{f(a)} : g \mapsto g(f(a)))$. These intuitions are at the heart of the graded models developed in Sections 4 and 5.

Interpreting the exponential connectors as spaces of functions and distributions is what allows to *type a higher-order Laplace transform*. In everyday calculus, it is defined as:

$$\mathscr{L} : (f : \mathbb{R}^n \to \mathbb{R}) \overset{\simeq}{\mapsto} \left(x \in \mathbb{R}^n \mapsto \int f(t) e^{-xt} dt \right). \tag{4}$$

However, the use of integration does not generalize easily to general functions $f : E \to \mathbb{R}$, in particular to higher-order functions acting on infinite dimensional vector spaces. The key to the higher-order generalization of Equation 4 is to replace the use of integral by the use of generalized functions, otherwise called distribution.

Indeed, when $!A$ is seen as a space of distributions, one can generalize Equation 4 as:

$$\mathscr{L} : (\phi \in !E) \overset{\simeq}{\mapsto} \left(\ell \in E^\perp \mapsto \phi(t \in E \mapsto e^{\ell(t)}) \right) \in ?E. \tag{5}$$

Then $\mathscr{L}(\phi)$ when $\phi \in\, !A$ is indeed an element of $?A$, that is a non-linear function acting on A'.

2.2 The codigging

Let us also say a few words about the recently introduced codigging. In models of linear logic, the interpretation of ! has a comonadic structure, paired with the interpretation of d and a comultiplication $\mu : ! \to !!$, involved in the interpretation of p. DiLL symmetrizing all structural rules but p, it was a natural question to ask whether a copromotion rule $\bar{\mathsf{p}}$ existed, leading to a monadic structure on ! equipped with $\bar{\mathsf{d}}$ as a unit. Kerjean and Lemay answered positively to that question, from the categorical point of view [KL23]. Adding a codigging $\bar{\mu}$ to differential categories makes sense and several models of DiLL are in fact models of DiLL with codigging. This codigging is interpreted as an exponential operation of distributions, using the convolution $*$.

$$\bar{\mu}_A : \begin{cases} !!A & \to !A \\ \delta_\phi & \mapsto \sum_n \frac{1}{n!}\phi^{*^n} \end{cases}$$

The key point is that the monadic laws on ! ensure that in these model, *every map equals it Taylor Expansion at* 0.

There are two drawbacks to this work. First, several models of DiLL where every map equals it Taylor Expansion at 0 are *not* models of DiLL with codigging, as the existence of μ as a map poses strong conditions of convergence of these Taylor Expansions. This is the case for examples of the historical models of Finiteness or Köthe spaces [Ehr05] [Ehr02]. Second, *no syntactical copromotion rule* $\bar{\mathsf{p}}$, reflecting the codigging, was implemented, due to the impossibility to find a sound cut-elimination rule between p and $\bar{\mathsf{p}}$. The development of the syntax and semantics of DIDiLL solves both these issues.

2.3 Indexed differential linear logic

Kerjean [Ker18] has defined previously a logic which replaces the use of the differentiation at 0 in DiLL by the use of a single arbitrary linear partial differential operator with constant coefficient (LPDO). An LPDO is an operator of the form:

$$D = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \partial^\alpha \qquad \text{where} \quad \partial^\alpha : f \mapsto \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}(f)$$

when $\alpha = (\alpha_1, \dots, \alpha_n)$. Constant coefficients means $a_\alpha \in \mathbb{K}$. This hypothesis on coefficients simplify drastically the resolution of LPDOs thanks to the following theorem.

Theorem 1. *(Malgrange-Ehrenpreis) For each LPDO D, there is a distribution Φ_D such that $D(\Phi_D) = \delta_0$. This distribution is called the* fundamental solution *of D.*

This allows for an analogy of DiLL in terms of LPDO. If d is usually understood as *forgetting* the linearity of a map ℓ, it can then be seen as solving the differential equation associated to D_0 with parameter ℓ, the solution of the equation $D_0(\_) = \ell$ being ℓ itself. Hence, one can understand the action of the dereliction and the codereliction as respectively solving a differential equation and applying a differential operator.

This logic has been generalized by Breuvart, Kerjean and Mirwasser in 2023 into a graded setting [BKM23], to a logic IDiLL, featuring graded version of $\mathsf{w}, \mathsf{c}, \mathsf{d}, \bar{\mathsf{w}}, \bar{\mathsf{c}}, \bar{\mathsf{d}}$. Semantically, this generalization comes from the following well-know equation on LPDOs

$$\Phi_{D_1 \circ D_2} = \Phi_{D_1} * \Phi_{D_2}$$

which connects the fundamental solution of the composition of operators to the convolution of their fundamental solution. In this framework, it corresponds to a connection between the interpretation of the codereliction and the indexes on the exponential connectors. It hints for a logic, having an infinity of exponentials, indexed by a set of elements endowed with an algebraic operation (LPDO endowed with composition in our case). Dereliction becomes then

$$\frac{\vdash \Gamma, ?_{D_1} A}{\vdash \Gamma, ?_{D_1 \circ D_2} A}\ \mathsf{d}_I$$

which is naturally interpreted by $\mathsf{d}_I : f \mapsto \Phi_{D_2} * f$.

However, IDiLL still needs to be refined. A first reason to this is that it does not have the promotion rule. Several issues appear if one tries to add the promotion. From a syntactical point of view, the cut elimination procedure has to be strongly extended. Semantically, if one wants to consider differential equations with their parameters and solutions, the natural semantics is the following. The exponential connectors are interpreted by:

$$[\![?_D A]\!] = \{f \mid \exists g, D(f) = g\} \quad [\![!_D A]\!] = \{g \mid \exists f, D(f) = g\}.$$

Then, the rules are naturally interpreted, in the similar way of DiLL, but taking into account the LPDOs. However, the semantics of the contraction rule, which is

$$\mathsf{c} : \begin{cases} ?_{D_1} E \mathbin{\hat{\otimes}} ?_{D_2} E & \to ?_{D_1 \circ D_2} E \\ f \otimes g & \mapsto \Phi_{D_1 \circ D_2} * (D_1(f).D_2(g)) \end{cases} \tag{6}$$

does not follow the usual smooth interpretation of c. Grading does not appear naturally with the usual interpretation and must be forced through ad-hoc computations. Finally, introducing a promotion rule for IDiLL would mean interpreting the cut-elimination between p and d_I, meaning computing

$$D(g \circ f) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \partial^\alpha (g \circ f)$$

in terms of other primitive rules, which is no way obvious.

3 DIDiLL: a Double Indexed DiLL

In this section, we define and study the calculus DIDiLL. For our purposes, we will use a weaker definition of semiring. Beware that in graded linear logic, this restriction is not required. The additional axioms on the general case are used to interpret commutations case in proof nets (see for example [DG24, Part 1.3]) which are not necessary here. On the contrary, distributivity on both sides does not hold in the semirings interpreting DIDiLL in Section 4 and 5.

Definition 2. *Let $(\mathcal{S}, +, 0)$ be a commutative monoid, 1 an element of $\mathcal{S}$, and $\times$ an associative binary operation on $\mathcal{S}$ (the product). When the product is right-distributive over the sum, 0 is left-absorbing and 1 is a left-neutral for the product, i.e.*

$$(x+y)z = xz + yz \qquad 0 \times x = 0 \qquad 1 \times x = x$$

for each $x, y, z \in \mathcal{S}$, and $\leq$ is a partial order such that the sum is increasing monotone, $(\mathcal{S}, +, 0, \times, 1)$ is a weak semiring.

Let us now detail the sequent calculus of this logic. To the rules and connectors of polarized MALL, we now add 4 families of graded connectors: $?^+_x P$, $?^-_x N$, $!^+_x P$ and $!^-_x N$, where x ranges over a semiring (see Figure 1a). We denote by *polarity* the upper index $+$ or $-$, and simply denote by index the lower index $x \in \mathcal{S}$, where $\mathcal{S}$ is a semiring. We call $?^-$ and $!^-$ negative exponentials while $?^+$ and $!^+$ are positive.

As is usual, we define inductively the involutive negation such that $(?^+_x P)^\perp = !^-_x P^\perp$ and $!^+_x P^\perp = ?^-_x P^\perp$. We now define the rules of DIDiLL as the rules of MALL to which are added the exponential rules in figure 1c.

For the promotion rule, $?_X \mathcal{N}$ represents $?_{x_1} N_1, \ldots, ?_{x_n} N_n$, for $X = (x_1, \ldots, x_n)$ and $\mathcal{N} = N_1, \ldots, N_n$. Then, $y \times X = (yx_1, \ldots, yx_n)$. Notice that the structural and costructural rules act on negative exponentials only. Notice also that, to be as general as possible, we present this logic graded by a weak semiring with an order. However, in the concrete model that we present in Section 4, the order in defined through the sum: $x \leq z$ when there is y such that $x + y = z$.

3.1 The copromotion $\bar{\mathsf{p}}$

From a codigging $\bar{\mu}_E : !!E \to !E$ (see Section 2.2), several choices arise when designing a copromotion rule dual to the promotion rule. While the promotion was defined syntactically and then semantically interpreted by the digging, here we use the codigging, which is semantical, to define the copromotion.

The promotion rule can be decomposed into two rules: the functoriality of the bang ($!_\mathsf{f}$) and the digging.

$$\frac{\Gamma \vdash A}{!\Gamma \vdash !A}\,!_\mathsf{f} \qquad \frac{!!\Gamma \vdash B}{!\Gamma \vdash B}\,\mu \qquad \mathsf{p} = \frac{\dfrac{!\Gamma \vdash A}{!!\Gamma \vdash !A}\,!_\mathsf{f}}{!\Gamma \vdash !A}\,\mu$$

$$P, Q := 1 \mid 0 \mid P \otimes Q \mid P \oplus Q \mid ?^+_x P \mid !^+_x P$$
$$N, M := \top \mid \bot \mid N \parr M \mid N \,\&\, M \mid ?^-_x N \mid !^-_x N$$
$$A.B := N \mid P \mid A \otimes B \mid A \oplus B \mid A \parr B \mid A \,\&\, B$$

(a) Formulas of DIDiLL

$$(?^+_x P)^\perp = !^-_x P^\perp \qquad (!^+_x P)^\perp = ?^-_x P^\perp$$
$$(?^-_x N)^\perp = !^+_x N^\perp \qquad (!^-_x N)^\perp = ?^+_x N^\perp$$
$$\mathscr{L}(?^+_x P) = !^+_x P \qquad \mathscr{L}(!^+_x P) = ?^+_x P$$
$$\mathscr{L}(?^-_x N) = !^-_x N \qquad \mathscr{L}(!^-_x N) = ?^-_x N$$

(b) Syntactical transformations of DIDiLL

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?^-_0 N}\ \mathsf{w} \qquad \frac{}{\vdash !^-_0 N}\ \bar{\mathsf{w}}$$

$$\frac{\vdash \Gamma, ?^-_x N, ?^-_y N}{\vdash \Gamma, ?^-_{x+y} N}\ \mathsf{c} \qquad \frac{\vdash \Gamma, !^-_x N \qquad \vdash \Delta, !^-_y N}{\vdash \Gamma, \Delta, !^-_{x+y} N}\ \bar{\mathsf{c}}$$

$$\frac{\vdash \Gamma, N}{\vdash \Gamma, ?^-_1 N}\ \mathsf{d} \qquad \frac{\vdash \Gamma, N}{\vdash \Gamma, !^-_1 N}\ \bar{\mathsf{d}}$$

$$\frac{\vdash \Gamma, ?^-_x N \qquad x \leq y}{\vdash \Gamma, ?^-_y N}\ \mathsf{d}_I \qquad \frac{\vdash \Gamma, !^-_x N \qquad x \leq y}{\vdash \Gamma, !^-_y N}\ \bar{\mathsf{d}}_I$$

$$\frac{\vdash ?^-_X \mathcal{N}, P}{\vdash ?^-_{y \times X} \mathcal{N}, !^+_y P}\ \mathsf{p} \qquad \frac{\vdash !^-_x N, P}{\vdash !^-_{y \times x} N, ?^+_y P}\ \bar{\mathsf{p}}$$

(c) Exponential rules of DIDiLL

Figure 1: The logic DIDiLL

One can then define a copromotion rule, that will be dual to the promotion, but one major restriction has to be done. To be able to define a cut-elimination procedure, we had to restrict for only a one-formula context in $\bar{\mathsf{p}}$. This comes from the binary flavor of $\bar{\mathsf{c}}$, compared to the unary one of c. In the cut elimination case between p and c, the promoted part is duplicated, and then recombined through contractions. Adapting this idea in the costructural case is possible, but the recombination works only with exactly one formula in the context, because of the binarity. This gives the following rule

$$\bar{\mathsf{p}} = \dfrac{\dfrac{?A \vdash B}{??A \vdash ?B}\, ?_{\mathsf{f}}}{?A \vdash ?B}\, \bar{\mu} \qquad\qquad \dfrac{\vdash !A, B}{\vdash !A, ?B}\, \bar{\mathsf{p}}$$

which is presented in our graded polarized setting in figure 1c.

Notice that here, $\bar{\mathsf{p}}$ and p *do not interact through cut-elimination*, as they do not involve dual connectors. Grading, and involving the Laplace transform in structural rules, is thus a way to safely include $\bar{\mathsf{p}}$ as a syntactical rule in DiLL presented as a sequent calculus. We haven't developed a proof-net version of our calculus: DiLL fits well in this formalism but Graded Linear Logic does not. Note however that a tentative to symmetrize promotion was already explored in proof-nets by Gimenez [Gim09].

3.2 Cut-elimination

In LL and in DiLL, there are only two exponential connectors, the ! and the ?, which are dual to each other. Then, in LL, each structural rule $\mathsf{w}, \mathsf{d}, \mathsf{c}$ interacts with the promotion rule, in the cut elimination, but they do not interact together. When one considers DiLL, three costructural rules $\bar{\mathsf{w}}, \bar{\mathsf{d}}$ and $\bar{\mathsf{c}}$ are added. This adds many cases to the cut elimination: the principal formulas of costructural rules are of the form $!A$, and then each costructural rule interacts both with the promotion, and with every other structural rule.

Here, in the rules of DIDiLL, principal formulas of the costructural rules have the form $!^-_y N$, while the ones of structural rules have the form $?^-_y N^\perp$. But $!^-_y N$ *is not* the dual of $?^-_y N^\perp$, since duality changes the polarity. This implies that the number of possible interactions through the cut rule is very reduced compared to DiLL: $\bar{\mathsf{w}}, \bar{\mathsf{c}}$ and $\bar{\mathsf{d}}$ cannot interact with w, c and d. The only cut elimination cases to consider are the one with either a promotion rule, or a copromotion rule. Moreover, the promotion and the copromotion cannot interact together, for similar reasons. The cases have to be refined, compared to what is done in LL or DiLL since we use graded exponentials. The rewritings for the cuts between graded structural rules and graded promotion are presented in [BP15] for the non polarized case. They are the same here, adding the polarity. They are recalled in the appendix, in section .2. The other rewriting rules, that comes from the interactions between costructural rules and the copromotion are new, but similar to their structural counterpart, as they describe the same categorical structure. We present them in figure 2.

One can define a translation from DIDiLL to LL, using the mix rule of linear logic, to deduce the following theorem.

Theorem 3. *The logic DIDiLL enjoys a strongly normalizing cut-elimination procedure.*

$$\dfrac{\dfrac{\vdash !^-_y N, P}{\vdash !^-_{0\times y} N, ?^+_0 P}\,\bar{\mathsf{p}} \qquad \dfrac{}{\vdash !^-_0 P^\perp}\,\bar{\mathsf{w}}}{\vdash !^-_0 N}\,cut \quad\rightsquigarrow\quad \dfrac{}{\vdash !^-_0 N}\,\bar{\mathsf{w}}$$

$$\dfrac{\dfrac{\vdash !^-_y N, P}{\vdash !^-_{1\times y} N, ?^+_1 P}\,\bar{\mathsf{p}} \qquad \dfrac{\vdash \Gamma, P^\perp}{\vdash \Gamma, !^-_1 P^\perp}\,\bar{\mathsf{d}}}{\vdash \Gamma, !^-_y N}\,cut \quad\rightsquigarrow\quad \dfrac{\vdash !^-_y N, P \qquad \vdash \Gamma, P^\perp}{\vdash \Gamma, !^-_y N}\,cut$$

$$\dfrac{\dfrac{\vdash !^-_y N, P}{\vdash !^-_{(xz)y} N, ?^+_{xz} P}\,\bar{\mathsf{p}} \qquad \dfrac{\vdash !^-_z P^\perp, Q}{\vdash !^-_{xz} P^\perp, ?^+_x Q}\,\bar{\mathsf{p}}}{\vdash !^-_{(xz)y} N, ?^+_x Q}\,cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\vdash !^-_y N, P}{\vdash !^-_{zy} N, ?^+_z P}\,\bar{\mathsf{p}} \qquad \vdash !^-_z P^\perp, Q}{\vdash !^-_{zy} N, Q}\,cut}{\vdash !^-_{x(zy)} N, ?^+_x Q}\,\bar{\mathsf{p}}$$

$$\dfrac{\dfrac{\vdash !^-_z N, P}{\vdash !^-_{yz} N, ?^+_y P}\,\bar{\mathsf{p}} \qquad \dfrac{\vdash \Gamma, !^-_x P^\perp \qquad x \leq y}{\vdash \Gamma, !^-_y P^\perp}\,\bar{\mathsf{d}}_I}{\vdash \Gamma, !^-_{yz} N}\,cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\vdash !^-_z N, P}{\vdash !^-_{xz} N, ?^+_x P}\,\bar{\mathsf{p}} \qquad \vdash \Gamma, !^-_x P^\perp}{\vdash \Gamma, !^-_{xz} N}\,cut \qquad xz \leq yz}{\vdash \Gamma, !^-_{yz} N}\,\bar{\mathsf{d}}_I$$

$$\dfrac{\dfrac{\vdash !^-_z N, P}{\vdash !^-_{(x+y)z} N, ?^+_{x+y} P}\,\bar{\mathsf{p}} \qquad \dfrac{\vdash \Gamma, !^-_x P^\perp \qquad \vdash \Delta, !^-_y P^\perp}{\vdash \Gamma, \Delta, !^-_{x+y} P^\perp}\,\bar{\mathsf{c}}}{\vdash \Gamma, \Delta, !^-_{xz+yz} N}\,cut \quad\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\vdash !^-_z N, P}{\vdash !^-_{xz} N, ?^+_x P}\,\bar{\mathsf{p}} \qquad \vdash \Gamma, !^-_x P^\perp}{\vdash \Gamma, !^-_{xz} N}\,cut \qquad \dfrac{\dfrac{\vdash !^-_z N, P}{\vdash !^-_{yz} N, ?^+_y P}\,\bar{\mathsf{p}} \qquad \vdash \Delta, !^-_y P^\perp}{\vdash \Delta, !^-_{yz} N}\,cut}{\vdash \Gamma, \Delta, !^-_{xz+yz} N}\,\bar{\mathsf{c}}$$

Figure 2: Cut-elimination cases for the copromotion

3.3 Polarized Linear Logic

We now briefly compare DlDiLL with Polarized Linear Logic [Lau02], which refines Linear Logic by specializing the action of the connectors to so-called positive or negative formulas. Its grammar is as follows:

$$P, Q := 1 \mid 0 \mid P \otimes Q \mid P \oplus Q \mid !N$$
$$N, M := \top \mid \perp \mid N \parr M \mid N \,\&\, M \mid ?P$$

The syntactical use of polarization is in proof-search: negatives connectors are those whose introduction rule are reversible, while the introduction rule of positive connector are not reversible. But the polarity also has a semantical meaning, notably in game semantics, where two players face each other to define a strategy, interpreting a proof. In a smooth model of DiLL, which is detailed in Section 5, this notion comes at plays at the involutive duality interpreting the negation acts between two different kinds of vector spaces. Polarization finds also its way back in DlDiLL as we split exponentials, having positive and negative version of both ! and ?.

Some versions of polarized linear logic feature shifts connectors ↑ and ↓. These are connectors which change polarity in a covariant way, used for focusing, and that are traditionally included in the grammar of polarized proofs systems when the exponential connector do not change the polarity of the formulas [MT10].

$$P, Q := 1 \mid 0 \mid P \otimes Q \mid P \oplus Q \mid !P \mid \downarrow N$$
$$N, M := \top \mid \bot \mid N ⅋ M \mid N \,\&\, M \mid ?N \mid \uparrow P$$

Consider for example an exponential connector ! mapping negative formulas to negative formulas. Then to interpret minimal logic into the polarized grammar thanks to the usual call-by-name translation, one needs a shift, as the formula $!N \multimap M \equiv (!N)^{\perp} ⅋ M$ is not well constructed if ⅋ applies only to negative connector. The arrow is then translated as $N \Rightarrow M = \uparrow(!N)^{\perp} ⅋ M$.

In DIDiLL, exponential connectors do not change the polarity of formulas, even if we have four of them. The key is that the role usually played by shifts connectors ↑ and ↓ is no longer necessary, thanks to the class of unpolarized formulas denoted A, B in figure 1. For example, comparing with the previous translation, the call-by-name translation of the arrow is simply $N \Rightarrow M = (!^{-} N)^{\perp} ⅋ M$, which is well-defined (omitting indices).

Fast forwarding to content exposed in Section 5, notice that shifts do not have a natural interpretation in our standard model of Fréchet / DF-spaces, interpreting respectfully Negative and Positive connectors. There is no natural covariant way to transform a Fréchet space in a DF-space and conversely, without drastically changing their topology. However, the constraint on the interpretation of MALL connectors is also much softer in topological vector spaces than in polarized linear logic. Indeed, taking the example of the tensor product, while the projective tensor product of DF-spaces is indeed a DF-space, the projective tensor product of a Fréchet space and a DF-Space can also be constructed, making it a simple topological vector space. Hence, one can construct a category of formulas A, B on which every MALL operation is defined.

4 Grading **DIDiLL** with LPDOs

In this section, we interpret DIDiLL indices on structural rules by polynomials and indices on costructural ones by linear partial differential operators with constant coefficients. This generalizes IDiLL [BKM23] by providing an interpretation for the promotion rule: we are looking for a semiring structure on the set of multivariate polynomials, instead of a monoid as in IDiLL (see Section 2.3). Beware that non-indexed dereliction d and codereliction $\bar{\mathsf{d}}$, as well as the copromotion $\bar{\mathsf{p}}$ are sadly *not interpreted* in this model.

With the introduction of partial differentiation on higher-order functions, we need to refine the interpretation of formulas. In models of IDiLL, formulas were interpreted by finite dimensional real vector spaces $\mathbb{R}^n$ or by spaces of functions or distributions on them. That is, differential operators only acted on functions or distributions in $\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R})$ or $\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R})'$, where the variables of $\mathbb{R}^n$ are implicitly ordered. Now, partial differential operators must act on spaces $\mathcal{C}^\infty(E,\mathbb{R})$: what does $\frac{\partial\_}{\partial x_1}$ mean for $f \in \mathcal{C}^\infty(E,\mathbb{R})$?

Therefore, we pair topological vector spaces with a basis, which will be at most countable. More precisely, we interpret DIDiLL in Köthe spaces [Ehr02], which are a historical model of DiLL, where $\mathbb{K}$-vector spaces are paired with a basis, and their topology is induced by an orthogonality relation on sequences in $\mathbb{K}$. Section 4.1 recalls the notion of Köthe spaces while Section 4.2 interprets DIDiLL indexed by differential operators in to Köthe spaces. We begin by defining a semiring structure overs the space of polynomials.

Definition 4. *We define by induction:*

$$\mathcal{P}_0 = \mathbb{R} \qquad \mathcal{P}_{n+1} = \mathcal{P}_n[X_{n+1}] \qquad \mathcal{P}_\omega = \cup_{n\geq 0}\mathcal{P}_n$$

and we give the following operations for each $P, Q \in \mathcal{P}_\omega$ such that

$$P = \sum_{\alpha\in\mathbb{N}^\omega} a_\alpha X^\alpha \qquad Q = \sum_{\beta\in\mathbb{N}^\omega} b_\beta X^\beta$$

we define, $P \boxplus Q = P \times Q$, the usual product of polynomials, and $P \boxtimes Q$ as a composition:

$$P \boxplus Q = \sum_{\alpha,\beta} a_\alpha b_\beta X^{\alpha+\beta} \qquad P \boxtimes Q = \sum_\alpha a_\alpha Q^{|\alpha|}.$$

where $Q^{|\alpha|}$ denotes the multiplication of Q by itself $|\alpha| = \sum_i \alpha_i$ times. We denote

$$\widehat{P} = \sum_\alpha (-1)^{|\alpha|} a_\alpha X^\alpha.$$

Using the multinomial theorem, we prove the following.

Proposition 5. *$(\mathcal{P}_\omega, \boxplus, \boxtimes, 1, X_1)$ is a weak semiring.*

Definition 6. *Considering $P \in \mathcal{P}_\omega$, we define $P(\partial)$ as:*

$$P = \sum_\alpha a_\alpha X_1^{\alpha_1} \dots X_n^{\alpha_n} \qquad P(\partial) = \sum_\alpha a_\alpha \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots x_n^{\alpha_n}}.$$

In particular, for $P, Q \in \mathcal{P}_\omega$ one has $(P \times Q)(\partial) = P(\partial) \circ Q(\partial)$ and $1(\partial) = \mathsf{id}$.

4.1 Köthe spaces

Most of the material of this section is simply a recap of what can be found in the litterature [Ehr02]. Hence, the reader already familiar with Köthe spaces can safely skip this material, just remembering the fact that now the basis of each Köthe space should be totally ordered.

$$
\begin{array}{ccccc}
 & & \textsc{Parameters} & & \\
 & ?^{-}_{P}A = \{f \mid \exists g \in ?A,\ f = \hat{P}\cdot g\} & \overset{\mathscr{L}^{-}}{\longleftarrow} & !^{-}_{P}A = \{\psi \mid \exists \phi \in !A, \psi = P(\partial)(\phi)\} & \\
\mathsf{w},\mathsf{c},\mathsf{d},\mathsf{p} & (\_)^{\perp}\big\downarrow & & (\_)^{\perp}\big\downarrow & \bar{\mathsf{w}},\bar{\mathsf{c}},\bar{\mathsf{d}},\bar{\mathsf{p}} \\
 & !^{+}_{P}A = \{\psi \mid \exists \phi \in !A, \phi = \hat{P}\cdot\psi\} & \overset{\mathscr{L}^{+}}{\longrightarrow} & ?^{+}_{P}A = \{f \mid \exists g \in ?A, g = P(\partial)(f)\} & \\
 & & \textsc{Solutions} & &
\end{array}
$$

Figure 3: Laplace and Duality acting on exponential graded by polynomials

For $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, $\mathbb{K}^{\mathbb{N}}$ denotes the vector space of all sequences on $\mathbb{K}$. Following Ehrhard, we write for $E \subset \mathbb{K}^{\mathbb{N}}$:

$$E^{\perp} := \{(\alpha_n)_n \in \mathbb{K}^{\mathbb{N}} \mid \forall \lambda \in E, (\lambda_n \alpha_n)_n \in \ell_1\}.$$

This acts as an orthogonality, meaning that for any E one has $E \subseteq E^{\perp\perp}$ and $E^{\perp} = E^{\perp\perp\perp}$.

We now give the definition of Köthe spaces as used by Ehrhard, which coincide with the definition of *perfect sequence space* by Schaefer and Köthe [Sch71, Köt69].

Definition 7. *A perfect sequence space is the data (X, E_X) of a subset $X \subset \mathbb{N}$ and $E_X \subset \mathbb{K}^X$ such that $E_X^{\perp\perp} = E_X$. It is endowed with its* normal topology*, that is with the initial topology induced by the semi-norms: $q_\alpha : (\lambda_n)_n \mapsto ||(\lambda_n\alpha_n)_n||_1$ for all $\alpha \in {E_X}^{\perp}$. As X is clear from the context, we abusively note E_X to denote the perfect sequence space (X, E_X).*

Definition 8. *An important restriction to our setting is that we consider Köthe spaces $E = (X, E_X)$ where X is a totally ordered subset of $\mathbb{N} : X = (x_1, \dots, x_n, \dots)$. This order is redefined at each interpretation of linear logic connector.*

From now on, E_X and F_Y denote perfect sequence spaces. We now recall the constructions making Köthe spaces a model of Linear Logic.

The space $E \multimap F$ of linear continuous maps from E_X to F_Y correspond to the subset of $\mathbb{K}^{X\times Y}$ of all M such that the sum:

$$\sum_{i,j} M_{i,j} x_i y'_j$$

is absolutely converging for all $x \in E$ and $y' \in F^{\perp}$. $X \times Y$ is lexicographically ordered. The *tensor product* of two perfect sequence spaces E_X and F_Y is the perfect sequence space $(E \multimap F^{\perp})^{\perp}$. In particular, $E^{\perp} \simeq E \multimap \mathbb{K}$, where an element $\ell = (\ell_i)_i$ of $E^{\perp}$ acts on $x = (x_i)_i$ in E as $\ell(x) = \sum_i \ell_i x_i$.

The product and coproduct constructions, interpreting & and $\oplus$, are defined as the product and coproduct of topological vector spaces and preserve perfect sequence spaces. Their basis are ordered with the lexicographical order.

The interpretation of exponentials formulas in Köthe spaces embodies the intuition that non-linear proofs should be represented as analytic functions. Consider a set X and $\mathcal{M}(X)$ the set of all finite multisets of X. If $\mu \in \mathcal{M}(X)$ and $x \in E$, we write:

$$x^\mu \;=\; \prod_n x_n^{\mu(n)}$$

and $x^!$ the element of $\mathbb{K}^{\mathcal{M}(X)}$ such that $(x^!)_\mu = x^\mu$. Then one defines:

$$!E = (\mathcal{M}(X), \{x^! | x \in E\}^{\perp\perp}).$$

Equivalently $!E$ could be defined as the dual of the set of scalar entire maps $E \Rightarrow \mathbb{K}$, for a notion of entire maps that coincides with the usual definition for absolutely converging power series with infinite radius of convergence when X is a singleton and $\mathbb{K}^X \;=\; \mathbb{K}$.

The interpretation for the codereliction is the linear continuous morphism $\bar{\mathsf{d}}_E : E \multimap !E$ such that $\bar{\mathsf{d}}_E(x) : M \in E \Rightarrow \mathbb{K} \mapsto \sum_{a \in X} M_{\{a\}} x_a$. Note that from a more analytic point of view, $M_{\{a\}}$ corresponds to $\frac{\partial f}{\partial x_a}(0)$ and M_μ to $\frac{\partial^{|\mu|} f}{\partial x_1^{\mu(1)} \dots x_n^{\mu(n)} \dots}(0)$.

The intuitions detailed in Section 2.1 about DiLL also apply to Köthe spaces: contraction corresponds to the scalar multiplication of functions while cocontraction corresponds to the convolution product of distributions [Ehr02].

4.2 DIDiLL indexed by polynomials

Consider DIDiLL indexed by the semiring $(\mathcal{P}_\omega, \boxplus, \boxtimes, 1, X_1)$. We interpret MALL formulas and non-indexed exponentials as done by Ehrhard [Ehr02] and recalled in Section 4.1. This interpretation is indifferent to the polarity of formulas.

Definition 9. *Consider (X, E_X) a Köthe space. We consider X ordered: $X = (x_1, \dots x_n \dots)$. Consider $P = \sum_{\alpha \in \mathbb{N}^\omega} a_\alpha X^\alpha \in \mathcal{P}_\omega$, $f : E \Rightarrow \mathbb{K}$ a function and $\phi : !E \equiv (E \Rightarrow \mathbb{K})^\perp$ a distribution.*

- *For $x \in E$, denoting λ_i the coordinate of x along x_i such that $x = \sum_i \lambda_i x_i$, we define*

$$P(x) := \sum_{\alpha \in \mathbb{N}^\omega} a_\alpha \prod_i \lambda_i^{\alpha_i}$$

- *$P \cdot f$ denote the analytic function $x \mapsto P(x) \cdot f(x)$, where $\_ \cdot \_$ is the scalar multiplication.*

- *$P \cdot \phi$ denote the distribution $g \mapsto \phi(P \cdot g)$.*

- *$P(\partial) f$ denotes the partial differential operator $P(\partial)$ applied to f, according to the basis X:*

$$P(\partial) f : z \mapsto \sum_\alpha a_\alpha \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots x_n^{\alpha_n}}(z).$$

- *$P(\partial)\phi$ denotes the distribution $\phi \circ \widehat{P}(\partial)$. This is the standard definition of differential operators applied to distributions.*

Note that, if X is finite with n elements, and x_m appears in P with $n \leq m$, it would generate an issue in this interpretation. In this case, we identity X to $(x_1, \dots, x_n, 0, \dots)$ and each λ_i for $i > n$ will be 0.

Consider (X, E_X) a Köthe space. Let us detail how differential operators act on entire functions $f : E \Rightarrow \mathbb{K}$. Consider $P \in \mathcal{P}_\omega$ and write f as $f(x) = \sum_{\mu \in \mathcal{X}} f_\mu x^\mu$. For $\mu \in \mathcal{X}$ let's write μ_i the (possibly null) coefficient of x_i in μ. Then for $a \in E$ $\frac{\partial f}{\partial x_i}(a) = \sum_\mu f_\mu \mu_i \frac{a^\mu}{a_i}$ and as such, with the notations of Definition 6 we have:

$$(P(\delta)(f))\,(x) = \sum_\alpha a_\alpha f_{(\mu+\alpha)} \prod_i (\mu_i + \alpha_i) x^\mu.$$

Definition 10. *We in interpret as follows graded exponentials:*

$$[\![?^-_P A]\!] = \hat{P} \cdot ([\![?A]\!]) \qquad [\![!^-_P A]\!] = P(\partial)([\![!A]\!])$$

$$[\![!^+_P A]\!] = (\hat{P})^{-1} \cdot ([\![!A]\!]) \qquad [\![?^+_P A]\!] = P(\partial)^{-1}(?A)$$

These definitions and their properties are summarized in Figure 3.

This also defines four functors acting on the category of Köthe spaces and linear maps:

$$?^-_P E = \hat{P} \cdot (?E) \qquad !^-_P E = P(\partial)(!E)$$

$$!^+_P E = \hat{P}^{-1} \cdot (!E) \qquad ?^+_P E = P(\partial)^{-1}(?E)$$

We now turn to the heart of the interpretation, meaning the cut-elimination invariant interpretation. To that end, we give the interpretation of each exponential rules of DIDiLL. The costructural rules $\bar{\mathsf{d}}_I$, $\bar{\mathsf{c}}$ and $\bar{\mathsf{w}}$ are interpreted as in IDiLL [Ker18], representing the action of differential operators. More elegantly than in IDiLL, the structural rules d_I, c and w now represent operations on polynomials. A new and important addition to IDiLL is the interpretation of the promotion rule, representing the composition of higher-order analytic functions.

The usual smooth interpretation of DiLL rules (see Section 2.1) adapts well to the bi-graded case:

- The weakening rule is interpreted by the introduction of the polynomial constant at 1, wich is indeed a function of $[\![?^-_0 N]\!] = \{1 \cdot f | f : N^\perp \Rightarrow \mathbb{K}\}$. The coweakening is interpreted as the introduction of the Dirac at 0, which is indeed an element of $[\![!^-_0 N]\!] = \{\phi \circ \mathsf{id} \mid \phi : (N \Rightarrow \mathbb{K})^\perp\}$.
- The contraction is interpreted as usual, and the scalar multiplication of $f \in [\![!^-_P N]\!]$ and $g \in [\![!^-_Q N]\!]$ leads indeed to a function in $[\![!^-_{P \times Q} N]\!]$. The cocontraction is interpreted as usual by the convolution product, and the convolution $\phi \circ P(\partial) * \psi \circ Q(\partial)$ equals indeed $(\phi * \psi) \circ (P(\partial) \circ Q(\partial)) =$

$\phi * \psi \circ (P \times Q)(\partial)$. Note that, contrary to IDiLL's model (see Equation 6), here the interpretation for the contraction follows the usual one (see Section 2.1), and grading appears naturally.

Graded dereliction and codereliction work likewise. Remember that the order on $\mathcal{P}_\omega$ is interpreted with the respect to the additive law: $P \leq Q$ if there is H such that $Q = H \times P$. Indexed dereliction d_I is interpreted by the composition by a polynomial: to a function $f = P \cdot f'$ it maps $H \cdot f$. Indexed codereliction $\bar{\mathsf{d}}_I$ is then interpreted as in IDiLL by the precomposition of a distribution $\phi \circ P(\partial)$ with $H(\partial)$.

Non indexed dereliction and codereliction are however *not* interpreted in this model. Indeed, the interpretation of d would map an element $x = \sum \lambda_i x_i$ to a map $x_1 \cdot f$ with $f \in ?E$. Following the usual interpretation in Köthe spaces, the interpretation would be the function $y = \sum \lambda'_i x_i \in E^\perp \mapsto \sum \lambda_i \lambda'_i$. If $\lambda_1 = 0$, this interpretation cannot be factorized as $x_1 \cdot f$. Likewise, interpreting $\bar{\mathsf{d}}$ would mean writing $x \mapsto (f \mapsto D_0(\_)(x))$ as a distribution $\phi \circ D$ with D a LPDOcc. As $f \mapsto D_0(f)$ is clearly not a linear partial differential operator, this tentative fails.

Graded promotion requires the definition of a new form of composition. Consider informally two functions $f = P \cdot f' : A \Rightarrow B$ and $g = Q \cdot g' : B \Rightarrow C$. The idea is that the composition $g \circ f$ should be defined pairwise on polynomials and functions: $g \tilde{\circ} f = (Q \boxtimes P) \cdot (g' \circ f')$. Note that when $Q = P = 1$, this is the usual composition. When $A = B = C = \mathbb{K}$ and $f' = g' = cst_1$, then this is also the usual composition of univariate polynomials. This composition is associative and has $1 \cdot \mathsf{id}$) as left and right neutral.

We now define the graded digging:

$$\mu_{Q,P} : \begin{cases} !^+_{Q \boxtimes P} E & \to \quad !^+_Q !^+_P E \\ \quad \phi & \mapsto \quad (Q \cdot f' \in ?^-_Q ?^-_P E^\perp \\ & \qquad \mapsto f'(P \cdot g' \in ?^-_P \\ & \qquad \mapsto \phi(x \in E^\perp \\ & \qquad \mapsto (Q \boxtimes P)(x) \cdot f'(g'(x))))) \end{cases}$$

Consider a linear map $\ell \in [\![!^+_X \mathcal{N}^\perp]\!] \multimap [\![P]\!]$ interpreting a proof of $\vdash ?^-_X \mathcal{N}, P$. Then the proof resulting from a graded promotion rule is as usual the composition $!_Q \ell \circ \mu_{Q,P}$.

All these rules are invariant by cut-elimination, as they follow the classical constructions of smooth models of DiLL.

We now study a graded version of the Laplace transform interpreted in Köthe spaces:

$$\mathscr{L}^- : \begin{cases} !^-_P E & \to \quad ?^-_P E \\ \quad \phi & \mapsto \quad (\ell \in E^\perp \mapsto \phi(x \in E \mapsto e^{\ell(x)})) \end{cases} \tag{7}$$

$$\mathscr{L}^+ : \begin{cases} !^+_P E & \to \quad ?^+_P E \\ \quad \phi & \mapsto \quad (\ell \in E^\perp \mapsto \phi(x \in E \mapsto e^{\ell(x)})) \end{cases} \tag{8}$$

This is a straightforward adaptation of Equation 5.

Through a careful computation of the Laplace transform on a Köthe space, and using strongly the fact that E and $E^\perp$ have the same basis, one proves the following proposition.

Proposition 11. *For $\phi \in !^-_P E$ we have that $\mathscr{L}^-(\phi)(\ell)$ is well-defined for every $\ell \in E^\perp$, and that $\mathscr{L}^-(\phi) \in ?^-_P E$.*

One shows likewise that $\mathscr{L}^+$ is well-defined. This shows that the diagram in Figure 3 is well-defined. It commutes thanks to the involutivity of $(\_)^\perp$ in Köthe spaces.

Sadly, looking in particular at the proof of proposition 11, we see no reason why $\mathscr{L}^-$ or $\mathscr{L}^+$ should be isomorphism. As such, we cannot obtain the graded codigging as the reverse image by the Laplace transform, as done in Section 5. It seems that, as in usual Köthe spaces [KL19, Section 5.1], graded Köthe spaces offer no interpretation for the copromotion.

5 A Smooth Higher Order Model of DlDiLL

In this section we extend to higher-order the smooth and polarized model of DiLL made of Fréchet and DF-spaces, which was previously restricted to first-order, by grading its exponentials Young Functions.

5.0.1 Historical context

A first smooth model was built by Blute, Ehrhard and Tasson [BET12] based on work by Frölicher, Kriegl and Michor [KM97]. This was however an intuitionistic model, as it does not reflect the involutive linear negation of LL. DiLL does not hold in an intuitionistic version: its symmetric nature makes it deeply classical.

Finding a smooth model of DiLL encompassing this classical nature turns out to be quite a challenge: vector spaces that are invariant under double linear dual, the so-called reflexive spaces, hold bad topological stability properties. An intricate solution based on the so-called Arens dual can be constructed [DK20]. A more natural solution appears to find polarized models: two classes of spaces, dual one to another, with a contravariant linear duality between both of them. This is the case of Nuclear Fréchet spaces. Section 5.1 briefly introduces duality in topological vector spaces and a *first-order* polarized model of DiLL made of Nuclear Fréchet and duals of Fréchet (DF) topological vector spaces previously exposed by Kerjean [Ker18]. In section 5.2 we give a higher-order graded refinement of this model, based on work by Ouerdiane and collaborators [GHOR00] and already exposed briefly in [KL23]. Section 5.3 shows that this model is also a model of DlDiLL, using the graded nature of these spaces.

5.1 Reflexive topological spaces

This section can be skipped until Definition 14 if one is only interested by the logical structure, admitting that Nf spaces and Ndf spaces are classes of topological vector spaces which are dual to each other and invariant under double linear negation (i.e. double linear dual).

Definition 12. *A* ***Hausdorff and locally convex topological vector space*** *(lcs) is a $\mathbb{K}$-vector space endowed with a Hausdorff topology making scalar multiplication and addition continuous, and such that every point has a basis of convex open neighborhoods.*

The space of all linear continuous functions between two lcs E and F is denoted $\mathcal{L}(E, F)$. The dual of a lcs E is denoted $E' := \mathcal{L}(E, \mathbb{K})$. Given a lcs E, the vector space E' can be made a lcs by endowing it with a topology. We will only consider in this paper the *strong* topology: the space E' is endowed with the topology of uniform convergence on bounded subsets of E.

Definition 13. *A space is said to be reflexive when $E \simeq E''$, that is when E and E'' are linearly homeomorphic.*

Reflexive spaces have poor stability properties: they are not stable by (topological) tensor product and does not generalize well to higher-order. Some subclasses of reflexive spaces are well better behaved however:

Fréchet spaces are metrisable complete lcs. However, the dual of a metrisable space is not metrisable, and hence Fréchet spaces are not spaces by duality. We call *DF-spaces* the duals of Fréchet spaces. The precise definition of DF-spaces can be found in the litterature [Jar81] and will not make further use of the details this definition. What is interesting to us is that when added the condition of nuclearity, these spaces are reflexive.

Nuclear spaces are those lcs E for which, for every Banach space B the projective tensor product with B and the injective tensor product with B are linearly homeomorphic. Projective and injective tensor products are two different topologies on tensor product related to different continuity conditions on bilinear maps, we will not detail them here. Again, we refer to the literature for a more thorough presentation [Jar81, 12.4, 21.1].

Definition 14. *We denote* NDF *the category of nuclear DF-spaces,* NF *the category of Nuclear Fréchet spaces.*

Kerjean [Ker18] defined a first-order model of DiLL where the *positive formulas are interpreted by Nuclear DF-spaces*, *negative formulas by Nuclear Fréchet spaces*, the linear negation by the strong dual $(\_)'$, the additives disjunction and conjunction by the coproduct and the product, the multiplicative conjunction by the projective tensor product and the multiplicative disjunction by its dual.

Exponential connectors then only apply on formulas interpreted by finite dimensional vector spaces: if A is interpreted by $\mathbb{R}^n$, then $?A$ is interpreted by $\mathcal{C}^\infty((\mathbb{R}^n)', \mathbb{R})$ and $!A$ by the space of distribution $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$. This model was used in [BKM23] to define a first-order model for D-DiLL, a differential and graded LL without promotion. Work by Kerjean and Lemay extends this model to higher-order [KL19], without being able to interpret the promotion rule however.

$$\begin{array}{ccccc} & [\![?^-_\theta N]\!] = \mathscr{F}_\theta([\![N]\!]') & \xleftarrow[\text{Fréchet Nuclear spaces}]{\mathscr{L}^-} & [\![!^-_{\theta^*} N]\!] = \mathscr{G}'_\theta([\![N]\!]) & \\ \mathsf{w},\mathsf{c},\mathsf{d},\mu & (\_)^\perp \updownarrow & & (\_)^\perp \updownarrow & \bar{\mathsf{w}},\bar{\mathsf{c}},\bar{\mathsf{d}},\bar{\mu} \\ & [\![!^+_\theta P]\!] = \mathscr{F}'_\theta([\![P]\!]) & \xrightarrow[\text{DF Nuclear spaces}]{\mathscr{L}^+} & [\![?^+_\theta P]\!] = \mathscr{G}_{\theta^*}([\![P]\!]') & \end{array}$$

Figure 4: Laplace and Duality acting on exponential graded by exponential growth

5.2 The spaces $\mathscr{F}_\theta$ and $\mathscr{G}_\theta$

From now on we consider lcs on $\mathbb{C}$. The key to construct a higher-model of DiLL based on NF and NDF is in fact to consider a good decomposition of these objects. It will allow to construct Fréchet spaces of smooth maps over DF spaces, and as such a higher-order smooth model of DiLL, but graded and polarized.

Proposition 15. *[Jar81, Chapter 21] The topology on any* NF *space N can be defined through a coutable family of Hilbertian norms $|\_|_p$, $p \in \mathbb{N}$, and if one denote N_p the Hilbert space resulting of the completion of N with respect to $|\_|_p$, we have that N is the limit of all N_p, while N' is the colimit of all $(N_p)'$:*

$$\bigcap_p N_p = N \qquad \bigcup_p (N_p)' = N'.$$

Definition 16. *[GHOR00] For a Young function θ and for a Banach space B, let $Exp(B, \theta, m)$ denote the Banach space of holomorphic functions from B to $\mathbb{C}$ such that:*

$$|f(z)| \leq K e^{\theta(m||z||)}. \tag{9}$$

The space $Exp(\theta, m, p)$ is Banach when endowed with the norm : $||\_||_{\theta,m} : f \mapsto sup\{|f(z)|e^{-\theta(m||z||)} | z \in B\}$.

This define two types of functions with exponential growth, depending if they take their arguments on an NF lcs N or on a NDF lcs N'.

$$\mathscr{F}_\theta(N') = \bigcap_{m,p} \mathrm{Exp}((N_p)', \theta, m) \quad \mathscr{G}_\theta(N) = \bigcup_{m,p} \mathrm{Exp}(N_p, \theta, m).$$

Through an isomorphism with spaces of formal power series, one can show that $\mathscr{F}_\theta(N)$ is a NF space [GHOR00, Prop 2]. As such, its dual $\mathscr{F}'_\theta(N)$, i.e. the space of distributions acting on $\mathscr{F}_\theta(N)$, is a NDF space. As linear morphisms are bounded, $\mathscr{G}_\theta : \mathrm{NF} \to \mathrm{NDF}$ and $\mathscr{F}_\theta : \mathrm{NDF} \to \mathrm{NF}$ are indeed functors.

Theorem 17. *[GHOR00, Thm 1] Suppose that $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ is a convex and growing function, such that $\lim_\infty \frac{\theta(x)}{x} = \infty$. For the conjugate Young function $\theta^* := sup_{t \geq 0}(tx - \theta(t))$, we have that the Laplace transform results in an* *isomorphism:*

$$\mathscr{L}^+ : \begin{cases} \mathscr{F}'_\theta(N') & \simeq & \mathscr{G}_{\theta^*}(N) \\ \phi & \mapsto & \big(\ell \in N' \mapsto \phi(x \in N \mapsto e^{\ell(x)} \in \mathbb{C})\big) \end{cases} \tag{10}$$

The conjugate of θ, also called the convex conjugate, is related to inverses: if the function θ can be defined as $\theta = \int \mu(t)dt$, then $\theta^* = \int \mu^{-1}(t)dt$.

Proposition 18. *The scalar multiplication of two functions $f_1 \in \mathscr{F}_{\theta_1}(F')$ and $f_2 \in \mathscr{F}_{\theta_2}(F')$ belongs to $\mathscr{F}_{\theta_1+\theta_2}(F')$.*

Taking the dual equation 10, we get:

$$\mathscr{L}^- : \mathscr{G}'_{\theta^*}(N) \simeq \mathscr{F}_\theta(N')$$

Applying it to the Proposition 18, we get:

Proposition 19. *The convolution of two distributions $\phi_1 \in \mathscr{G}'_{\theta_1}(F)$ and $f_2 \in \mathscr{G}'_{\theta_2}(F)$ belongs to $\mathscr{G}'_{(\theta_1^*+\theta_2^*)^*}(F')$.*

In their paper [CEOO02, Lemma 1], the authors also describe convolution operator of type $\mathscr{F}'_\theta(F') \otimes \mathscr{F}'_\theta(F') \to \mathscr{F}'_\theta(F')$. That is, on $\mathscr{F}'$, the convolution does not change the indices. Applying the Laplace isomorphism 10, we get a scalar multiplication of type $\mathscr{G}_\theta(F) \otimes \mathscr{G}_\theta(F) \to \mathscr{G}_\theta(F)$. From this, we get a slogan: *grading acts on negative exponential connectors, leaving positive connector unchanged*. In hindsight, this justifies the choice of upper indices $+\ -$ chosen for the exponentials in DIDiLL.

We also recall a proposition proved when studying the codigging [KL23, Prop V.8], while considering the composition of linear functions $f : \mathscr{F}'_{\theta_1}(N_1) \to N_2$ and $g : \mathscr{F}'_{\theta_2}(N_2) \to N_3$

Proposition 20. *For any Young functions θ_1, θ_2 we have a natural transformation in the category* NDF*:*

$$\mu_P : \mathscr{F}'_{\left(\theta_2 e^{\theta_1}\right)}(P) \to \left(\mathscr{F}'_{\theta_2}(\mathscr{F}'_{\theta_1}(P))\right).$$

The literature [CEOO02] [KL23] uses a codigging acting on distributions of type $\mathscr{F}'_\theta$. To interpret DIDiLL, we need a codigging acting on $\mathscr{G}'_\theta$. The exact same proof than in the litterature [CEOO02, Theorem 1], using the isomorphism $\mathscr{L}^-$, one gets a convolutional exponential interpreting the codigging of type:

$$\bar{\mu}_P : \Big(\mathscr{G}'_{\theta_2^*}(\mathscr{G}'_{\theta_1^*}(P))\Big) \to \mathscr{G}'_{\left(\theta_2 e^{\theta_1}\right)^*}(P'). \tag{11}$$

5.3 A model for DIDiLL

In this section, we build a denotational model for DIDiLL. Formulas N (resp. P) of DIDiLL are interpreted by Nuclear Fréchet (resp. Nuclear DF) lcs $[\![N]\!]$, (resp. $[\![P]\!]$) and proofs are interpreted by linear continuous maps.

Definition 21. *Let us define*

$$\Theta := \{\theta : \mathbb{R}_+ \to \mathbb{R}_+ \mid \theta \text{ continuous, convex, growing } \}$$

Addition $\boxplus$ *on* Θ *is defined as the pointwise addition of function, the neutral element being* cst_0 *the function constant at* 0*. The multiplication on* Θ *is defined as* $\theta_1 \boxtimes \theta_2 := x \mapsto \theta_1 \circ e^{\theta_2(x)}$*; with a left-identity being the function* id*. We now quotient this set with the equivalence relation at* ∞*:* $\Theta_\sim := \Theta/\sim_\infty$*. Addition and multiplication as defined above preserve equivalence, and thus apply to* $\Theta_\sim$*.*

Proposition 22. *The set* $(\Theta_\sim, cst_0, \boxplus, id, \boxtimes)$ *is a weak semiring.*

Definition 23. *We interpret 4 exponential connectors as follows. The interpretation is summarized in Figure 4, where the set of indices is* $\Theta_\sim$*.*

- $?^+_\theta P$*, where* P *is a positive formula, is a positive formula interpreted by* $\mathscr{G}_{\theta^*}(\llbracket P \rrbracket')$*.*
- $!^-_\theta N$*, where* N *is a negative formula, is the dual of* $?^-_\theta(N^\perp)$ *and a negative formula interpreted by* $\mathscr{G}'_\theta(\llbracket N \rrbracket)$*.*
- $?^-_\theta(N)$*, where* N *is a negative formula, is a negative formula interpreted by* $\mathscr{F}_\theta(\llbracket N \rrbracket')$*.*
- $!^+_\theta P$*, where* P *is a positive formula, is the dual of* $?^-_\theta(P^\perp)$ *and a positive formula interpreted by* $\mathscr{F}'_\theta(\llbracket P \rrbracket')$*.*

Colluding syntax and semantics, we can see how the Laplace transform and the duality act on these connectors:

$$\mathscr{L}(\llbracket !^+_\theta P \rrbracket) \simeq \llbracket ?^+_\theta P \rrbracket \qquad \llbracket (!^+_\theta P)^\perp \rrbracket \simeq \llbracket ?^-_\theta P^\perp \rrbracket$$

Let us show that the interpretation of rules in smooth models of DiLL (see Section 2.1) adapts well to this graded setting.

- The weakening rule w on a function corresponds to the introduction of the constant function on $\llbracket P \rrbracket'$, which is indeed an element of $\mathscr{G}_0(\llbracket P \rrbracket)$.
- The coweakening $\bar{\mathsf{w}}$ is interpreted as the introduction of δ_0, which acts linearly and continuously on every $\mathscr{F}_0(P)$.
- Contraction c and cocontraction $\bar{\mathsf{c}}$ are interpreted by scalar multiplication and convolution, as detailed in proposition 19 and proposition 18.
- The indexed dereliction d_I has an immediate interpretation and no computational content, as for $\theta' \leq \theta$ in the semi-ring, that is $\theta \leq \theta'$ pointwise, we have $(\theta')^* \leq \theta^*$ pointwise, and thus $\llbracket ?^+_{\theta'} P \rrbracket = \mathscr{G}_{(\theta')^*}(\llbracket P \rrbracket) \subseteq \llbracket ?^+_\theta P \rrbracket = \mathscr{G}_{\theta^*}(\llbracket P \rrbracket)$. The indexed codereliction $\bar{\mathsf{d}}_E$ works likewise by inclusion.

The interpretation of the promotion and copromotion rule stems from Section 5.2.

- Consider the interpretation $\ell \in \mathcal{L}(\mathscr{F}'_{\theta_1}[\![N]\!]', [\![Q]\!])$ of the sequent $\vdash ?^-_x N, Q$. Then by the functorality of $\mathscr{F}'_{\theta_2}$, and precomposing by $\mu_{[\![N]\!]'} : \mathscr{F}'_{\left(\theta_2 e^{\theta_1}\right)}([\![N]\!]') \to \left(\mathscr{F}'_{\theta_2}(\mathscr{F}'_{\theta_1}([\![N]\!]'))\right)$ described in Proposition 20, one get a linear map in $\mathcal{L}(\mathscr{F}'_{\left(\theta_2 e^{\theta_1}\right)}([\![N]\!]'), \mathscr{F}'_{\theta_2}([\![N]\!]'))$ interpreting the sequent $\vdash ?^-_{y\times x} N, !^+_y Q$.

- The interpretation of the copromotion rules work likewise, thanks to the natural transformation $\bar{\mu}$ described in equation 11.

If we drop the assumption that $\lim \frac{\Theta(x)}{x} = +\infty$, then we have an immediate interpretation for the non-indexed dereliction and codereliction are immediate, with $1 = id$. Notice however that id does not satisfy the hypothesis of Theorem 17 and as such $\mathscr{L}$ does not apply on $!^-_1$.

- Consider a Nuclear Fréchet space $N = \lim N_p$, then $\mathsf{d} : N \to ?^-_1 N$ maps $\ell \in (N')'$ to the same map in $\mathscr{G}_1(N')$, as ℓ is indeed bounded by any function $e^{\|\text{-}\|_p}$ on every N_p.

- Likewise, the map $\bar{\mathsf{d}} : N \to !^-_1 N$ mapping a vector v to $f \mapsto D_0(f)(v)$ is well typed, as $D_0(\_)(v)$ acts continuously on $\mathscr{F}'_{id}(N)$.

Cut-elimination rules of DIDiLL are preserved, as they follow the usual interpretation of calculus in smooth models of DiLL.

6 Conclusion

In this paper, we define DIDiLL a double indexed differential linear logic, and study two concrete models of this logic. This double indexation comes from two different ideas. The first one is the *graduation* of exponential connectors by a semiring. The semiring is interpreted by differential operators, or by the growth of some smooth functions.

The second index is a polarity on the exponential connectors of differential linear logic, coming from semantical intuitions. The Laplace transform is central in the models that we study, and polarity and graduation allows to syntactically represent its action on the exponentials. DIDiLL features the usual rules of differential linear logic, but graded on exponential connectives which are not dual to each other but linked with a Laplace transformation. In addition, we provide a *copromotion* rule, which is the syntactical counterpart of the codigging introduced by Kerjean and Lemay [KL23], and is the reverse Laplace transform of the promotion. This logic enjoys a strong normalizing cut-elimination property.

We provide a first model, based on Köthe spaces, which is graded by differential operators and multivariates polynomials. The second model that we study consists in smooth spaces of functions and distributions, indexed by Young functions.

Future works The next step in this work would be to develop the categorical framework corresponding to DIDiLL. To define this, work by Lemay and Vienney [LV23] should be the right starting point. They categorically make the join between differential categories and graded monads. This paper however does not feature an indexed dereliction d_I and as such is still quite far from being a model of DIDiLL. It should also incorporate correctly the fundamental $*$-autonomous nature of our models. Work by Kerjean and Lemay [KL24] on Laplace transform in differential categories should also be incorporated in the categorical definition of models of DIDiLL.

Another line of work would be to extend our model on Köthe spaces to more general differential operators. While the constant coefficient hypothesis is crucial in IDiLL to define the semantics (as in Equation 6), it would not be needed in DIDiLL if one removes duality, meaning if one focuses on intuitionnistic models of it. Working on more general operators, and methods for solving differential equations, should of course have connections with work on fixpoints and differentiation by Galal and Lemay [GPL24], as they may be the right notion to consider resolution of differential equations.

Finally, our paper confirms the Laplace transform as a central tool to study Differential Linear Logic and its extensions. While no codigging could be defined on our version of graded Köthe space, we would have much more chance have one if grading introduced some notion of boundedness for functions. In that case, the Laplace transform would have a chance to be an isomorphism, and the codigging could be defined as the reverse image of the digging. This strongly motivate the search for more refined models of DIDiLL, with restrictions on functions allowing both for the codigging and for the resolution of more intricate differential equations.

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.1 Proofs

We start by recalling theorem 3, and give its full proof.

Theorem 3. *The logic* DIDiLL *enjoys a strongly normalizing cut-elimination procedure.*

Proof. We define a map $\mathcal{T}$ that transforms formulas and proofs of DIDiLL into formulas and proofs of LL. For atoms and MALL connectors, $\mathcal{T}$ is the identity, and for exponential formulas we define:

$$\mathcal{T}(!_x^+ A) := !(\mathcal{T}(A)) \qquad \mathcal{T}(!_x^- A) := ?(\mathcal{T}(A))$$

$$\mathcal{T}(?_x^+ A) := ?(\mathcal{T}(A)) \qquad \mathcal{T}(?_x^- A) := ?(\mathcal{T}(A)).$$

Then, for the proofs, $\mathcal{T}$ is the identity for each rule, except for the costructural ones. $\mathcal{T}$ transforms $\bar{\mathsf{d}}, \bar{\mathsf{d}}_I, \bar{\mathsf{p}}$ respectively into d, d_I and p. For $\bar{\mathsf{w}}$ and $\bar{\mathsf{c}}$ we define:

$$\dfrac{\dfrac{}{\vdash}\,mix_0}{\vdash ?N}\,\mathsf{w} \qquad \dfrac{\dfrac{\vdash \Gamma, ?N \qquad \vdash \Delta, ?N}{\vdash \Gamma, \Delta, ?N, ?N}\,mix_2}{\vdash \Gamma, \Delta, ?N}\,\mathsf{c}$$

which concludes the definition of $\mathcal{T}$. This proper This map is such that if $\pi \rightsquigarrow_{cut} \pi'$, then $\mathcal{T}(\pi) \rightsquigarrow_{cut} \mathcal{T}(\pi')$. This property is easy to prove, except for the case $\bar{\mathsf{p}}/\bar{\mathsf{c}}$. Here, we have to use a commutation between mix_2 and cut. For such a cut, in a proof π, $\mathcal{T}(\pi)$ is

$$\dfrac{\dfrac{\vdash ?N, P}{\vdash ?N, !P}\,\mathsf{p} \qquad \dfrac{\dfrac{\vdash \Gamma, ?P^\perp \qquad \vdash \Delta, ?P^\perp}{\vdash \Gamma, \Delta ?P^\perp, ?P^\perp}\,mix_2}{\vdash \Gamma, \Delta, ?P^\perp}\,\mathsf{c}}{\vdash \Gamma, \Delta, ?N}\,cut$$

which reduces in LL to the proof π_1

$$\dfrac{\dfrac{\dfrac{\vdash ?N, P}{\vdash ?N, !P}\,\mathsf{p} \qquad \dfrac{\dfrac{\vdash ?N, P}{\vdash ?N, !P}\,\mathsf{p} \qquad \dfrac{\vdash \Gamma, ?P^\perp \qquad \vdash \Delta, ?P^\perp}{\vdash \Gamma, \Delta, ?P^\perp, ?P^\perp}\,mix_2}{\vdash \Gamma, \Delta, ?N, ?P^\perp}\,cut}{\Gamma, \Delta, ?N, ?N}\,cut}{\Gamma, \Delta, ?N}\,\mathsf{c}$$

while $\mathcal{T}(\pi')$, where $\pi \rightsquigarrow_{cut} \pi'$, is

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash ?N, P}{\vdash ?N, !P}\,\mathsf{p} \qquad \vdash \Gamma, ?P^\perp}{\vdash \Gamma, ?N}\,cut \qquad \dfrac{\dfrac{\vdash ?N, P}{\vdash ?N, !P}\,\mathsf{p} \qquad \vdash \Delta, ?P^\perp}{\vdash \Delta, ?N}\,cut}{\vdash \Gamma, \Delta, ?N, ?N}\,mix_2}{\vdash \Gamma, \Delta, ?N}\,\mathsf{c}$$

which is equal to π_1 modulo the following commutation rule between mix_2 and *cut* (see [Pro23, p. 50]) by applying it twice.

This property on $\mathcal{T}$ implies that there is no infinite chain of reduction in DIDiLL, otherwise it would induce one in LL through $\mathcal{T}$, which is impossible by strong normality of the cut elimination of LL [Gir87]. This proves the strong normalization of the cut elimination for DIDiLL. □

Now we prove one of the main statement of Section 4.

Proposition 11. *For $\phi \in !_P^- E$ we have that $\mathscr{L}^-(\phi)(\ell)$ is well-defined for every $\ell \in E^\perp$, and that $\mathscr{L}^-(\phi) \in ?_P^- E$.*

Proof. Consider $P \in \mathcal{P}_\omega$, E a Köthe space with basis $X = (x_i)_{i\in\mathbb{N}}$.

Consider $\phi \in !_P^- E$, that is there is $\phi \in !E$ such that $\phi = \psi \circ P(\delta)$. Then we have in particular $\phi \in !E$. Let us show that $\mathscr{L}^-(\phi)(\ell)$ is well defined, for which we need to show that ϕ can be applied to $\mathscr{L}^-(\phi)(\ell) : x \mapsto e^{\ell(x)}$, meaning that $x \mapsto e^{\ell(x)}$ is in $\{x^! | x \in E\}^\perp$.

The linear function in $\{x^! | x \in E\}^\perp$ corresponding to $\mathscr{L}^-(\phi)(\ell)$ is exactly $(\widetilde{\mathscr{L}^-(\phi)}(\ell)) : x^! \mapsto e^{\ell(x)}$. It is linear in $x^!$, as writing $x = (\lambda_i)_i$ and $\ell = (\ell_i)_i$ one has

$$e^{\ell(x)} = \sum_n \frac{\ell(x)^n}{n!} = \sum_n \frac{1}{n!}\left(\sum_i \ell_i x_i\right)^n = \sum_{\mu\in\mathcal{M}(X)} \frac{1}{|\mu|!}\ell^\mu x^\mu$$

Let us show that $\mathscr{L}^-(\phi)$ is indeed in $?_P^- E$.

$$\begin{aligned}
\mathscr{L}^-(P(\partial)(\phi)) &= \ell \mapsto \phi \circ \hat{P}(\partial)(x \mapsto e^{\ell(x)}) \\
&= (\ell \in E^\perp) \mapsto \phi(x \in E \\
&\mapsto \left(\sum_\alpha (-1)^{|\alpha|} \prod \ell(x_i)^{\alpha_i} \cdot e^{\ell(x)}\right)) \\
&= \ell \mapsto (\sum_\alpha (-1)^{|\alpha|} \prod l(x_i)^{\alpha_i}) \cdot (\mathscr{L}^-(\phi))(\ell)
\end{aligned}$$

On the Köthe space $E^\perp$, the basis is also X, and now $(\ell(x_i)_i$ represents the coefficients of ℓ along this basis. Hence, $\sum_\alpha (-1)^{|\alpha|} \prod \ell(x_i)^{\alpha_i} = \hat{P}(\ell)$ and

$$\mathscr{L}^-(P(\partial)(\phi)) = \hat{P} \cdot (\mathscr{L}^-(\phi))$$

Hence $\mathscr{L}^-$ maps an object in $!_P^- E$ to an object in $?_P^- E$. □

Proposition 18. *The scalar multiplication of two functions $f_1 \in \mathscr{F}_{\theta_1}(F')$ and $f_2 \in \mathscr{F}_{\theta_2}(F')$ belongs to $\mathscr{F}_{\theta_1+\theta_2}(F')$.*

Proof. Consider $p, m \in \mathbb{N}$. Let us show that there is $K \in \mathbb{R}$ such that for all $x \in F'$ $|f_1(x)f_2(x)| \le K e^{\theta_1(m||z||_p)} e^{\theta_2(m||z||_p)}$. This is immediate as there is K_i such that $|f_i(x)| \le K_i e^{\theta_i(m||z||_p)}$ for both indices. □

Proposition 22. *The set* $(\Theta_\sim, cst_0, \boxplus, id, \boxtimes)$ *is a weak semiring.*

Proof. Addition is straightforwardly commutative, associative, and has 0 as left and right adjoint. The multiplication is associative:

$$\theta_1 \boxtimes (\theta_2 \boxtimes \theta_3) = \theta_1 \circ e^{\theta_2 \circ e^{\theta_3}(x)} = (\theta_1 \boxtimes \theta_2) \boxtimes \theta_3$$

and has *id* as a left unit. Addition is left-distributive over the multiplication, and any element is left anihilidated by cst_0. □

.2 Cut-elimination rule for the promotion

The cut elimination rules for the promotion are given in Figure 5. These rules were presented in the graded but not polarized case by Breuvart and Pagani [BP15]. Here, we present the same rules with our double indexed connectives.

$$\dfrac{\dfrac{\vdash ?^-_Y\mathcal{N}, P}{\vdash ?^-_{0\times Y}\mathcal{N}, !^+_0 P}\,\mathsf{p} \qquad \dfrac{\vdash \Gamma}{\vdash \Gamma, ?^-_0 P^\perp}\,\mathsf{w}}{\vdash \Gamma, ?^-_0\mathcal{N}}\,cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\vdash \Gamma}{\vdash \Gamma, ?^-_0 N_1}\,\mathsf{w}}{\vdots}\,\mathsf{w}}{\vdash \Gamma, ?^-_0\mathcal{N}}\,\mathsf{w}$$

$$\dfrac{\dfrac{\vdash ?^-_Z\mathcal{N}, P}{\vdash ?^-_{1\times Z}\mathcal{N}, !^+_1 P}\,\mathsf{p} \qquad \dfrac{\vdash \Gamma, P^\perp}{\vdash \Gamma, ?^-_1 P^\perp}\,\mathsf{d}}{\vdash \Gamma, ?^-_{1\times Z}\mathcal{N}}\,cut \quad\rightsquigarrow\quad \dfrac{\vdash ?^-_Z\mathcal{N}, P \qquad \vdash \Gamma, P^\perp}{\vdash \Gamma, ?^-_Z\mathcal{N}}\,cut$$

$$\dfrac{\dfrac{\vdash ?^-_Y\mathcal{N}, P}{\vdash ?^-_{xz_1\times Y}\mathcal{N}, !^+_{xz_1} P}\,\mathsf{p} \qquad \dfrac{\vdash ?^-_Z\mathcal{M}, ?^-_{z_1}P^\perp, Q}{\vdash ?^-_{x\times Z}\mathcal{M}, ?^-_{xz_1}P^\perp, !^+_x Q}\,\mathsf{p}}{\vdash ?^-_{xz_1\times Y}\mathcal{N}, ?^-_{x\times Z}\mathcal{M}, !^+_x Q}\,cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\vdash ?^-_Y\mathcal{N}, P}{\vdash ?^-_{z_1\times Y}\mathcal{N}, !^+_{z_1} P}\,\mathsf{p} \qquad \vdash ?^-_Z\mathcal{M}, ?^-_{z_1}P^\perp, Q}{\vdash ?^-_{z_1\times Y}\mathcal{N}, ?^-_Z\mathcal{M}, Q}\,cut}{\vdash ?^-_{xz_1\times Y}\mathcal{N}, ?^-_{x\times Z}\mathcal{M}, !^+_x Q}\,\mathsf{p}$$

$$\dfrac{\dfrac{\vdash ?^-_Z\mathcal{N}, P}{\vdash ?^-_{y\times Z}\mathcal{N}, !^+_y P}\,\mathsf{p} \qquad \dfrac{\vdash \Gamma, ?^-_x P^\perp \qquad x \leq y}{\vdash \Gamma, ?^-_y P^\perp}\,\mathsf{d}_I}{\vdash \Gamma, ?^-_{y\times Z}\mathcal{N}}\,cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\dfrac{\vdash ?^-_Z\mathcal{N}, P}{\vdash ?^-_{x\times Z}\mathcal{N}, !^+_x P}\,\mathsf{p} \qquad \vdash \Gamma, ?^-_x P^\perp}{\vdash \Gamma, ?^-_{x\times Z}\mathcal{N}}\,cut}{\vdots}\,\mathsf{d}_I}{\vdash \Gamma, ?^-_{y\times Z}\mathcal{N}}\,\mathsf{d}_I$$

$$\dfrac{\dfrac{\vdash ?^-_Z\mathcal{N}, P}{\vdash ?^-_{(x+y)\times Z}\mathcal{N}, !^+_{x+y} P}\,\mathsf{p} \qquad \dfrac{\vdash \Gamma, ?^-_x P^\perp, ?^-_y P^\perp}{\vdash \Gamma, ?^-_{x+y} P^\perp}\,\mathsf{c}}{\vdash \Gamma, ?^-_{(x+y)\times Z}\mathcal{N}}\,cut \quad\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash ?^-_Z\mathcal{N}, P}{\vdash ?^-_{x\times Z}\mathcal{N}, !^+_x P}\,\mathsf{p} \qquad \dfrac{\dfrac{\vdash ?^-_Z\mathcal{N}, P}{\vdash ?^-_{y\times Z}\mathcal{N}, !^+_y P}\,\mathsf{p} \qquad \vdash \Gamma, ?^-_x P^\perp, ?^-_y P^\perp}{\vdash \Gamma, ?^-_{y\times Z}\mathcal{N}, ?^-_x P^\perp}\,cut}{\vdash \Gamma, ?^-_{x\times Z}\mathcal{N}, ?^-_{y\times Z}\mathcal{N}}\,cut}{\vdots}\,\mathsf{c}}{\vdash \Gamma, ?^-_{x\times Z+y\times Z}\mathcal{N}}\,\mathsf{c}$$

Figure 5: Cases with the promotion