On a group of “associators”

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Outline

INTRODUCTION

Zeta functions with several complex indices

Let $\mathcal{H}_r = \{(s_1,\ldots,s_r) \in \mathbb{C}^r | \forall m = 1,\ldots,r, \Re(s_1)+\ldots+\Re(s_m) > m\}$, for $r \in \mathbb{N}_+$, the following zeta function converges for $(s_1,\ldots,s_r) \in \mathcal{H}_r$

$$\zeta(s_1,\ldots,s_r) = \sum_{n_1>\ldots>n_r>0} n_1^{-s_1}\ldots n_r^{-s_r}.$$

From a theorem by Abel, for $N \in \mathbb{N}, z \in \mathbb{C}, |z| < 1$, it can be obtained as

$$\zeta(s_1,\ldots,s_r) = \lim_{n\to+\infty} \mathrm{H}_{s_1,\ldots,s_r}(n) = \lim_{z\to 1} \mathrm{Li}_{s_1,\ldots,s_r}(z),$$

where the following functions are well defined for $(s_1,\ldots,s_r) \in \mathbb{C}^r$

$$\mathrm{Li}_{s_1,\ldots,s_r}(z) = \sum_{n_1>\ldots>n_r>0} \frac{z^{n_1}}{n_1^{s_1}\ldots n_r^{s_r}} \text{ and } \frac{\mathrm{Li}_{s_1,\ldots,s_r}(z)}{1-z} = \sum_{n\geq 0} \mathrm{H}_{s_1,\ldots,s_r}(n) z^n.$$

They do appear in the *regularization* of solutions of the following differential equation with noncommutative indeterminates in $X = \{x_0, x_1\}$

$$(DE) \quad dG = MG, \text{ with } M = \omega_0 x_0 + \omega_1 x_1, \omega_0(z) = \frac{dz}{z}, \omega_1(z) = \frac{dz}{1-z}.$$

Drinfel'd stated that[1] (DE) has a unique solution G_0 (resp. G_1), being group-like series, s.t. $G_0(z) \sim_0 e^{x_0 \log(z)}$ (resp. $G_1(z) \sim_1 e^{-x_1 \log(1-z)}$). There is then a unique group-like series $\Phi_{KZ} \in \mathbb{R}\langle\langle X \rangle\rangle$, so-called Drinfel'd associator, such that $G_0 = G_1 \Phi_{KZ}$.

[1] **V. Drinfel'd**, *On quasitriangular quasi-hopf algebra and a group closely connected with Gal($\bar{\mathbb{Q}}/\mathbb{Q}$)*, Leningrad Math. J., 4, 829-860, 1991.

Indexing by words (1/2)

Introducing $Y = \{y_k\}_{k\geq 1}$ and using the correspondence

$$(s_1,\ldots,s_r)\in\mathbb{N}_+^r \leftrightarrow y_{s_1}\ldots y_{s_r}\in Y^* \leftrightarrow x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1\in X^*x_1,$$

we will denote $\mathcal{Z} := \mathrm{span}_{\mathbb{Q}}\{\zeta(s_1,\ldots,s_r)\}_{(s_1,\ldots,s_r)\in\mathcal{H}_r\cap\mathbb{N}^r, r\in\mathbb{N}}$,

where $\zeta(y_{s_1}\ldots y_{s_r}) := \zeta(s_1,\ldots,s_r) =: \zeta(x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1)$.

We will also denote $\mathrm{H}_{y_{s_1}\ldots y_{s_r}} := \mathrm{H}_{s_1,\ldots,s_r}$ and $\mathrm{Li}_{x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1} := \mathrm{Li}_{s_1,\ldots,s_r}$.

The polylogarithms can be viewed as iterated integrals, w.r.t. ω_0,ω_1 and associated to words in X^* : $\mathrm{Li}_{s_1,\ldots,s_r}(z) = \alpha_0^z(x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1)$, where

$$\alpha_{z_0}^z(1_{X^*}) = 1_\Omega \quad\text{and}\quad \alpha_{z_0}^z(x_{i_1}\ldots x_{i_k}) = \int_{z_0}^{z}\omega_{i_1}(z_1)\ldots\int_{z_0}^{z_{k-1}}\omega_{i_k}(z_k).$$

Here, $1_\Omega : \Omega\to\mathbb{C}$, mapping z to 1, with $\Omega := \mathbb{C}\setminus(]-\infty,0]\cup[1,+\infty[)$ and $(z_0,z_1\ldots,z_k,z)$ is a subdivision of the path $z_0 \rightsquigarrow z$ in Ω.

They are single valued on Ω and alternatively they can be analytically continued and appear as multivalued functions over $B := \mathbb{C}-\{0,1\}$.

More rigorously, we have analytic functions on the universal cover $\widetilde{B}$, *i.e.* we choose a universal covering $(B,\widetilde{B},p)$ and a section $s:\Omega\to\widetilde{B}$ of p, lifted from the canonical embedding $j:\Omega\hookrightarrow B$

$$\begin{array}{ccc} & & \widetilde{B} \\ & s\nearrow & \downarrow p \\ \Omega & \underset{j}{\hookrightarrow} & B \end{array}$$

Indexing by words (2/2)

Next, using the correspondence $(s_1, \ldots, s_r) \in \mathbb{N}^r \leftrightarrow y_{s_1} \ldots y_{s_r} \in Y_0^*$, where $Y_0 = Y \cup \{y_0\}$, we will denote also $\mathrm{Li}^-_{y_{s_1} \ldots y_{s_r}} := \mathrm{Li}_{-s_1, \ldots, -s_r}$, $\mathrm{H}^-_{y_{s_1} \ldots y_{s_r}} := \mathrm{H}_{-s_1, \ldots, -s_r}$ and $\zeta^-(y_{s_1} \ldots y_{s_r}) := \zeta(-s_1, \ldots, -s_r)$.
Now, let $\theta_0 := z\partial_z, \theta_1 := (1-z)\partial_z$ (hence, $[\theta_0, \theta_1] = \theta_0 + \theta_1 = \partial_z$) and ι_0, ι_1 be sections of them, taking primitives for the corresponding differential operators w.r.t. the function (*i.e.* $\theta_0\iota_0 = \theta_1\iota_1 = \mathrm{Id}$). Then $\mathrm{Li}_{x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1} = (\iota_0^{s_1-1}\iota_1 \ldots \iota_0^{s_r-1}\iota_1)1_\Omega, \mathrm{Li}^-_{y_{t_1} \ldots y_{t_r}} = (\theta_0^{t_1+1}\iota_1 \ldots \theta_0^{t_r+1}\iota_1)1_\Omega$.

1. For $w \in Y_0^*$, $\mathrm{H}^-_w \in \mathbb{Q}[n]$ (resp. $\mathrm{Li}^-_w, \in \mathbb{Z}[(1-z)^{-1}]$). It is of degree $d := \mid w \mid + (w)$ and of valuation 1. Hence, $\mathrm{H}^-_w(n) \sim_{+\infty} C^-_w n^d$ (resp. $\mathrm{Li}^-_w(z) \sim_1 B^-_w (1-z)^{-d}$), where $C^-_w \in \mathbb{Q}$ (resp. $B^-_w \in \mathbb{Z}$).
2. The families $\{\mathrm{Li}^-_{y_k}\}_{k \ge 0}$ and $\{\mathrm{H}^-_{y_k}\}_{k \ge 0}$ are $\mathbb{Q}$-linearly independent.
3. $\mathrm{span}_{\mathbb{Q}}\{\mathrm{Li}^-_{y_k}\}_{k \ge 0}$ is closed by Cauchy and Hadamard products.
4. Let $\mathcal{C} := (\mathbb{C}[z, z^{-1}, (1-z)^{-1}], \partial_z)$. Then the algebra $\mathcal{C}\{\mathrm{Li}_w\}_{w \in X^*}$ ($\cong \mathcal{C} \otimes_{\mathbb{C}} \mathbb{C}\{\mathrm{Li}_w\}_{w \in X^*}$) is closed by the operators $\{\theta_0, \theta_1, \iota_0, \iota_1\}$. Moreover, the bi-integro differential ring $(\mathcal{C}\{\mathrm{Li}_w\}_{w \in X^*}, \theta_0, \iota_0, \theta_1, \iota_1)$ is closed under the action of the group of transformations permuting the singularities in $\{0, 1, +\infty\}$,
   $\mathcal{G} := \{z \mapsto z, z \mapsto 1-z, z \mapsto z^{-1}, z \mapsto (1-z)^{-1}, z \mapsto 1-z^{-1}, z \mapsto z(1-z)^{-1}\}$:
   $$\forall h \in \mathcal{C}\{\mathrm{Li}_w\}_{w \in X^*}, \quad \forall g \in \mathcal{G}, \quad h(g(z)) \in \mathcal{C}\{\mathrm{Li}_w\}_{w \in X^*}.$$

Actions of $\mathcal{G}$ over $\mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}$

Since $\mathcal{G}$ is generated by the transformations $\{z\mapsto 1-z, z\mapsto z^{-1}\}$ then

- Let σ_{1-z} be the letter substitution defined by
$$\sigma_{1-z}(x_0)=-x_1 \text{ and } \sigma_{1-z}(x_1)=-x_0.$$
Then $\mathrm{L}(1-z)=\sigma_{1-z}(\mathrm{L}(z))Z_{ш}$.
Hence, $\mathrm{L}(1-z)\sim_0 e^{-x_1\log(z)}Z_{ш}$.

Example

$$\begin{aligned}
\mathrm{Li}_1(1-z) &= -\log(z)\\
\mathrm{Li}_2(1-z) &= -\mathrm{Li}_2(z)+\log(z)\,\mathrm{Li}_1(z)+\zeta(2),\\
\mathrm{Li}_3(1-z) &= -\mathrm{Li}_{2,1}(z)+\mathrm{Li}_1(z)\,\mathrm{Li}_2(z)-\tfrac{1}{2}\log(z)\,\mathrm{Li}_1(z)^2-\zeta(2)\,\mathrm{Li}_1(z)+\zeta(3),\\
\mathrm{Li}_{2,1}(1-z) &= -\mathrm{Li}_3(z)+\log(z)\,\mathrm{Li}_2(z)-\tfrac{1}{2}\log(z)^2\,\mathrm{Li}_1(z)+\zeta(3),\\
\mathrm{Li}_4(1-z) &= -\mathrm{Li}_{2,1,1}(z)+\mathrm{Li}_1(z)\,\mathrm{Li}_{2,1}(z)-\tfrac{1}{2}\mathrm{Li}_1(z)^2\,\mathrm{Li}_2(z)\\
&+ \tfrac{1}{6}\log(z)\,\mathrm{Li}_1(z)^3+\tfrac{1}{2}\zeta(2)\,\mathrm{Li}_1(z)^2-\zeta(3)\,\mathrm{Li}_1(z)+\tfrac{2}{5}\zeta(2)^2.
\end{aligned}$$

- Let $\sigma_{1/z}$ be the letter substitution defined by
$$\sigma_{1/z}(x_0)=-x_0+x_1 \text{ and } \sigma_{1/z}(x_1)=x_1.$$
Then $\mathrm{L}(1/z)=\sigma_{1/z}(\mathrm{L}(z)Z_{ш}^{-1})e^{i\pi x_1}Z_{ш}$.
Hence, for any $w\in X^*, \mathrm{Li}_w(1/z)\sim_{-\infty}(-1)^{|w|_{x_0}}\log^{|w|}(z)/\mid w\mid!$.

Example

$$\begin{aligned}
\mathrm{Li}_1(1/z) &= (i\pi)+\mathrm{Li}_1(z)+\log(z),\\
\mathrm{Li}_2(1/z) &= -\log(z)(i\pi)-\mathrm{Li}_2(z)+2\zeta(2)-\tfrac{1}{2}\log(z)^2,\\
\mathrm{Li}_3(1/z) &= \tfrac{1}{2}\log(z)^2(i\pi)+\mathrm{Li}_3(z)-2\log(z)\zeta(2)+\tfrac{1}{6}\log(z)^3,\\
\mathrm{Li}_{2,1}(1/z) &= -\tfrac{1}{2}\log(z)(i\pi)^2+(-\mathrm{Li}_2(z)+\zeta(2)-\tfrac{1}{2}\log(z)^2)(i\pi)\\
&- \mathrm{Li}_{2,1}(z)+\mathrm{Li}_3(z)-\log(z)\,\mathrm{Li}_2(z)+\zeta(3)-\tfrac{1}{6}\log(z)^3,\\
\mathrm{Li}_4(1/z) &= -\tfrac{1}{6}\log(z)^3(i\pi)-\mathrm{Li}_4(z)+\tfrac{4}{5}\zeta(2)^2+\log(z)^2\zeta(2)-\tfrac{1}{24}\log(z)^4.
\end{aligned}$$

Noncommutative, co-commutative bialgebras

$\mathbb{C}\langle X\rangle, \mathbb{C}\langle Y\rangle, \mathbb{C}\langle Y_0\rangle$: sets of polynomials over X, Y, Y_0, respectively.
$\mathbb{C}\langle\langle X\rangle\rangle, \mathbb{C}\langle\langle Y\rangle\rangle, \mathbb{C}\langle\langle Y_0\rangle\rangle$: duals of $\mathbb{C}\langle X\rangle, \mathbb{C}\langle Y\rangle, \mathbb{C}\langle Y_0\rangle$, respectively, (sets of formal power series).

- $(\mathbb{C}\langle X\rangle, ., \Delta_{\sqcup\!\sqcup}, 1_{X^*}, \mathrm{e})$: for $x, y \in X$ and $u, v \in X^*$, $u \sqcup\!\sqcup 1_{X^*} = u$ and $xu \sqcup\!\sqcup yv = x(u \sqcup\!\sqcup yv) + y(xu \sqcup\!\sqcup v)$;
$\Delta_{\sqcup\!\sqcup}(x) = x \otimes 1_{X^*} + 1_{X^*} \otimes x$.
$$\mathcal{D}_X := \sum_{w \in X^*} w \otimes w = \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{S_l \otimes P_l} \quad \text{(MRS-factorization)},$$
where $\mathcal{L}ynX$ is the set of Lyndon words over X (with $x_1 > x_0$), $\{P_l\}_{l \in \mathcal{L}ynX}$ is a (graded) basis of Lie algebra of primitive elements and $\{S_l\}_{l \in \mathcal{L}ynX}$ is a pure transcendence basis of $(\mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})$.
- $(\mathbb{C}\langle Y\rangle, ., \Delta_{⊎}, 1_{Y^*}, \mathrm{e})$: for $y_i, y_j \in Y$ and $u, v \in Y^*$, $u ⊎ 1_{Y^*} = u$ and $y_i u ⊎ y_j v = y_i(u ⊎ y_j v) + y_j(y_i u ⊎ v) + y_{i+j}(u ⊎ v)$;
$\Delta_{⊎}(y_i) = y_i \otimes 1_{Y^*} + 1_{Y^*} \otimes y_i + \sum_{k+l=i} y_k \otimes y_k$.
$$\mathcal{D}_Y := \sum_{w \in Y^*} w \otimes w = \prod_{l \in \mathcal{L}ynY}^{\searrow} e^{\Sigma_l \otimes \Pi_l} \quad \text{($⊎$ – extended MRS-factorization)},$$
where $\mathcal{L}ynY$ is the set of Lyndon words over Y (with $y_1 > \ldots$), $\{\Pi_l\}_{l \in \mathcal{L}ynY}$ is a (graded for the weight) basis of Lie algebra of primitive elements and $\{\Sigma_l\}_{l \in \mathcal{L}ynY}$ is a pure transcendence basis of $(\mathbb{C}\langle Y\rangle, ⊎, 1_{Y^*})$.

First structures of polylogarithms and harmonic sums

1. Completed with $\mathrm{Li}_{x_0^k}(z) := \log^k(z)/k!$, $\{\mathrm{Li}_w\}_{w\in X^*}$ is $\mathbb{C}$-linearly independent. Hence, the following morphism of algebras is injective
$$\mathrm{Li}_\bullet : (\mathbb{C}\langle X\rangle, ш, 1_{X^*}) \to (\mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}, ., 1), \quad u \mapsto \mathrm{Li}_u .$$
Thus, $\{\mathrm{Li}_l\}_{l\in\mathcal{L}ynX}$ (resp. $\{\mathrm{Li}_{S_l}\}_{l\in\mathcal{L}ynX}$) is algebraically independent.
2. The following morphism of algebras is injective
$$\mathrm{H}_\bullet : (\mathbb{C}\langle Y\rangle, \uplus, 1_{Y^*}) \to (\mathbb{C}\{\mathrm{H}_w\}_{w\in Y^*}, ., 1), \quad u \mapsto \mathrm{H}_u.$$
Hence, $\{\mathrm{H}_w\}_{w\in Y^*}$ is linearly independent. It follows that, $\{\mathrm{H}_l\}_{l\in\mathcal{L}ynY}$ (resp. $\{\mathrm{H}_{\Sigma_l}\}_{l\in\mathcal{L}ynY}$) is algebraically independent.
3. $\zeta : \dfrac{(\mathbb{Q}1_{X^*} \oplus x_0\mathbb{Q}\langle X\rangle x_1, ш, 1_{X^*})}{(\mathbb{Q}1_{Y^*} \oplus (Y - \{y_1\})\mathbb{Q}\langle Y\rangle, \uplus, 1_{Y^*})} \twoheadrightarrow (\mathcal{Z}, ., 1)$ such that, for any $l_1, l_2 \in \mathcal{L}ynX - X$, $\zeta(l_1 ш l_2) = \zeta((\pi_y l_1) \uplus (\pi_y l_2)) = \zeta(l_1)\zeta(l_2)$, where $\pi_Y : (\mathbb{C}\langle\langle X\rangle\rangle, .) \to (\mathbb{C}\langle\langle Y\rangle\rangle, .), x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \mapsto y_{s_1} \ldots y_{s_r}$.
4. There exists, at least, an associative law of algebra $\top$, in $\mathbb{Q}\langle Y_0\rangle$, not dualizable such that the following morphism is a onto
$$\mathrm{Li}_\bullet^- \quad : \quad (\mathbb{Q}\langle Y_0\rangle, \top) \quad \to \quad (\mathbb{Q}\{\mathrm{Li}_w^-\}_{w\in Y_0^*}, .), \quad w \mapsto \mathrm{Li}_w^-,$$
and $\ker \mathrm{Li}_\bullet^- = \mathbb{Q}\{w - w\top 1_{Y_0^*} | w \in Y_0^*\}$.

   Moreover, if $\top' : \mathbb{Q}\langle Y_0\rangle \times \mathbb{Q}\langle Y_0\rangle \to \mathbb{Q}\langle Y_0\rangle$ is a law such that $\mathrm{Li}_\bullet^-$ is a morphism for $\top'$ and $(1_{Y_0^*} \top' \mathbb{Q}\langle Y_0\rangle) \cap \ker(\mathrm{Li}_\bullet^-) = \{0\}$ then $\top' = g \circ \top$, where $g \in GL(\mathbb{Q}\langle Y_0\rangle)$ such that $\mathrm{Li}_\bullet^- \circ g = \mathrm{Li}_\bullet^-$.

GLOBAL RENORMALIZATIONS OF DIVERGENT ZETAS VALUES INDEXED BY INTEGRAL MULTI-INDICES

Abel like theorem for noncommutative generating series

$$S_{z_0\rightsquigarrow z} := (\alpha_{z_0}^z \otimes \mathrm{Id})\mathcal{D}_X = \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\alpha_{z_0}^z(S_l)P_l}, \qquad \mathrm{L} := (\mathrm{Li}_\bullet \otimes \mathrm{Id})\mathcal{D}_X = \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\mathrm{Li}_{S_l} P_l}, \qquad \mathrm{H} := (\mathrm{H}_\bullet \otimes \mathrm{Id})\mathcal{D}_Y = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\mathrm{H}_{\Sigma_l}\Pi_l}.$$

$$Z_{ш} := \prod_{l\in\mathcal{L}ynX-X}^{\searrow} e^{\zeta(S_l)P_l}, \qquad Z_{ш\!\!+} := \prod_{l\in\mathcal{L}ynY-\{y_1\}}^{\searrow} e^{\zeta(\Sigma_l)\Pi_l}.$$

L satisfies (DE) and $\mathrm{L}(z) \sim_0 e^{x_0 \log(z)}$ and $\mathrm{L}(z) \sim_1 e^{-x_1 \log(1-z)} Z_{ш}$.
$S_{z_0\rightsquigarrow z} = \mathrm{L}(z)\mathrm{L}^{-1}(z_0)$ is a unique solution of (DE) and $S_{z_0\rightsquigarrow z_0} = 1_{X^*}$.
L is a unique solution of (DE) and $\mathrm{L}(z) \sim_0 e^{x_0 \log(z)}$. Thus, $Z_{ш}$ is unique.

Since $\mathrm{Gal}_{\mathbb{C}}(DE) = \{e^C\}_{C\in\mathcal{L}ie_{\mathbb{C}}\langle\langle X\rangle\rangle} =: Haus_{ш}(\mathbb{C}\langle\langle X\rangle\rangle)$ then let $\mathbb{Q} \subset A \subset \mathbb{C}$ and $dm(A) := \{\overline{Z}_{ш} = Z_{ш} e^C \mid C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle, \langle e^C|x_0\rangle = \langle e^C|x_1\rangle = 0\}$.
Then $dm(A) = \mathrm{Gal}_{\mathbb{C}}^{\geq 2}(DE)$ is a strict normal subgroup of $\mathrm{Gal}_{\mathbb{C}}(DE)$.

Theorem (HNM, 2009)

Let $e^C \in \mathrm{Gal}_{\mathbb{C}}(DE)$. Putting $\overline{\mathrm{L}} := \mathrm{L}e^C$ and $\overline{Z}_{ш} := Z_{ш} e^C$, one has

$$\lim_{z\to 1} \exp\left[-y_1 \log \frac{1}{1-z}\right]\pi_Y \overline{\mathrm{L}}(z) = \lim_{n\to\infty} \exp\left[\sum_{k\geq 1} \mathrm{H}_{y_k}(n)\frac{(-y_1)^k}{k}\right]\overline{\mathrm{H}}(n) = \pi_Y \overline{Z}_{ш}.$$

Or equivalently,

$$\overline{\mathrm{L}}(z) \sim_1 \exp\left[x_1 \log \frac{1}{1-z}\right]\overline{Z}_{ш}, \overline{\mathrm{H}}(n) \sim_{+\infty} \exp\left[-\sum_{k\geq 1} \mathrm{H}_{y_k}(n)\frac{(-y_1)^k}{k}\right]\pi_Y \overline{Z}_{ш}.$$

Singular expansion, asymptotic expansion and finite parts

For $w \in X^*x_1$, by Abel like theorem, there exists $a_i, b_{i,j} \in \mathcal{Z}$ such that

$$\mathrm{Li}_w(z) \underset{z\to 1}{\asymp} \sum_{i=1}^{(w)} a_i \log^i(1-z) + \langle Z_{\sqcup\!\sqcup} | w\rangle + \sum_{i\in\mathbb{N}_+, j\in\mathbb{N}_-} b_{i,j} \frac{\log^i(1-z)}{(1-z)^j}$$

and $\alpha_i, \beta_{i,j}$ and $\gamma_{\pi_Y w} \in \mathcal{Z}[\gamma]$ such that

$$\mathrm{H}_{\pi_Y w}(n) \underset{n\to+\infty}{\asymp} \sum_{i=1}^{|w|} \alpha_i \log^i(n) + \gamma_{\pi_Y w} + \sum_{i,j\in\mathbb{N}_+} \beta_{i,j} \frac{\log^i(n)}{n^j}.$$

Example (Costermans' PhD dissertation, 2008)

$$\begin{aligned}
\mathrm{Li}_{2,1}(z) &= \zeta(3) + (1-z)\log(1-z) - (1-z)^{-1} - \tfrac{1}{2}(1-z)\log^2(1-z) \\
&+ \tfrac{1}{4}(1-z)^2(-\log^2(1-z) + \log(1-z)) + \ldots, \\
\mathrm{H}_{2,1}(n) &= \zeta(3) - \tfrac{1}{n}(\log(n) + 1 + \gamma) + \tfrac{1}{2n}\log(n) + \ldots, \\
\mathrm{Li}_{1,2}(z) &= 2 - 2\zeta(3) - \zeta(2)\log(1-z) - 2(1-z)\log(1-z) \\
&+ (1-z)\log^2(1-z) + \tfrac{1}{2}(1-z)^2((\log^2(1-z) - \log(1-z)) + \ldots, \\
\mathrm{H}_{1,2}(n) &= \zeta(2)\gamma - 2\zeta(3) + \zeta(2)\log(n) + \tfrac{1}{2n}(\zeta(2) + 2) + \ldots, \\
\zeta(2)\gamma &= .949481711114981524545564102231704933640005949473666\ldots.
\end{aligned}$$

The map $\gamma_{\bullet} : (\mathbb{C}\langle Y\rangle, ⧢, 1_{Y^*}) \to (\mathbb{C}, ., 1)$ is a character since its graph, Z_γ, is group-like. It then follows, from the same Abel like theorem, that

$$Z_\gamma = B(y_1)\pi_Y Z_{\sqcup\!\sqcup}, \quad \text{where} \quad B(y_1) = \exp\left(\gamma y_1 - \sum_{k\geq 2}\zeta(k)\frac{(-y_1)^k}{k}\right).$$

By the ⧢-extended MRS-factorization, the cancellation leads to

$$Z_{⧢} = \mathtt{Mono}(y_1)\pi_Y Z_{\sqcup\!\sqcup}, \quad \text{where} \quad \mathtt{Mono}(y_1) = \exp\left(-\sum_{k\geq 2}\zeta(k)\frac{(-y_1)^k}{k}\right).$$

Characters and the constants $\{\gamma_{s_1,\ldots,s_r}\}_{(s_1,\ldots,s_r)\in\mathbb{N}^r}$

Let us consider the following characters defined over an algebraic basis by

$$\begin{aligned}
\zeta_{ш} &: (\mathbb{Q}\langle X\rangle, ш, 1_{X^*}) \twoheadrightarrow (\mathcal{Z}, ., 1), \quad l \in \mathcal{L}ynX \mapsto \langle Z_{ш}|l\rangle,\\
\zeta_{ⅎ} &: (\mathbb{Q}\langle Y\rangle, ⅎ, 1_{Y^*}) \twoheadrightarrow (\mathcal{Z}, ., 1), \quad l \in \mathcal{L}ynY \mapsto \langle Z_{ⅎ}|l\rangle.
\end{aligned}$$

s.t. $\forall l \in \mathcal{L}ynX - X, \langle Z_{ш}|l\rangle = \langle Z_{ⅎ}|\pi_Y(l)\rangle = \zeta(l)$ and

$$\begin{aligned}
\langle Z_{ш}|x_0\rangle &= 0 = \log(1),\\
\langle Z_{ш}|x_1\rangle &= 0 = \mathrm{f.p.}_{z\to 1}\log(1-z), && \{(1-z)^a\log^b(1-z)\}_{a\in\mathbb{Z},b\in\mathbb{N}},\\
\langle Z_{ⅎ}|y_1\rangle &= 0 = \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_1(n), && \{n^a\mathrm{H}_1^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}.
\end{aligned}$$

Let $B_{i,j}$'s be Bell polynomials. For $k \in \mathbb{N}_+, w \in Y^+$, one has

$$\gamma_{y_1^k} = \langle Z_\gamma|y_1^k\rangle = \sum_{s_1,\ldots,s_k>0, s_1+\ldots+ks_k=k} \frac{(-1)^k}{s_1!\ldots s_k!}(-\gamma)^{s_1}\left(-\frac{\zeta(2)}{2}\right)^{s_2}\ldots\left(-\frac{\zeta(k)}{k}\right)^{s_k},$$

$$\gamma_{y_1^k w} = \langle Z_\gamma|y_1^k w\rangle = \sum_{i=0}^{k}\frac{\zeta(x_0[(-x_1)^{k-i} ш \pi_X w])}{i!}\left(\sum_{j=1}^{i} B_{i,j}(\gamma, -\zeta(2), 2\zeta(3), \ldots)\right).$$

Example (Costermans' PhD dissertation, 2008)

$$\begin{aligned}
\gamma_{1,1} &= [\gamma^2 - \zeta(2)]/2,\\
\gamma_{1,1,1} &= [\gamma^3 - 3\zeta(2)\gamma + 2\zeta(3)]/6,\\
\gamma_{1,1,1,1} &= [80\zeta(3)\gamma - 60\zeta(2)\gamma^2 + 6\zeta(2)^2 + 10\gamma^4]/240,\\
\gamma_{1,7} &= \zeta(7)\gamma + \zeta(3)\zeta(5) - 54\zeta(2)^4 175,\\
\gamma_{1,1,6} &= 4\zeta(2)^3\gamma^2/35 + [\zeta(2)\zeta(5) + 2\zeta(3)\zeta(2)^2 - 4\zeta(7)]\gamma/5 + \zeta(6,2) + 19\zeta(2)^4/35 + \zeta(2)\zeta(3)^2/2 - 4\zeta(3)\zeta(5),\\
\gamma_{1,1,1,5} &= 3\zeta(6,2)/4 - 14\zeta(3)\zeta(5)/3 + 3\zeta(2)\zeta(3)^2/4 + 809\zeta(2)^4/1400\\
&- (2\zeta(7) - 3\zeta(2)\zeta(5)/2 + \zeta(3)\zeta(2)^2/10)\gamma + (\zeta(3)^2/4 - \zeta(2)^3/5)\gamma^2 + \zeta(5)\gamma^3/6.
\end{aligned}$$

Gradation of $Z_{⧢}$ and second Abel like theorem

Proposition (HNM, 2003)

For any $k \geq 0$, *let* $\kappa_k(z, t_1, \cdots, t_k)$ *be the formal power series given by*

$$\kappa_k(z, t_1, \cdots, t_k) = e^{x_0[\log(z)-\log(t_1)]}x_1 \cdots e^{x_0[\log(t_{k-1})-\log(t_k)]}x_1 e^{x_0 \log(t_k)} = e^{x_0 \log(z)} \sum_{l_1,\cdots,l_k \geq 0} \prod_{i=1}^{k} \frac{\log^{l_i}(t_i)}{l_i!} \mathrm{ad}_{-x_0}^{l_i} x_1.$$

and[2] *let* $\circ$ *be defined by* $x_1 x_0^l \circ P = x_1(x_0^l ⧢ P)$, $l \in \mathbb{N}, P \in \mathbb{C}\langle X \rangle$. *Then*

$$\mathrm{L}(z) = \sum_{k \geq 0, w \in x_0^* ⧢ x_1^k} \mathrm{Li}_w(z) w = \sum_{k \geq 0} \int_0^z \omega_1(t_k) \cdots \int_0^{t_{k-1}} \omega_1(t_1) \kappa_k(z, t_1, \cdots, t_k),$$

$$Z_{⧢} = \sum_{k \geq 0, l_1, \cdots, l_k \geq 0} \zeta_{⧢}(x_1 x_0^{l_1} \circ \cdots \circ x_1 x_0^{l_k}) \prod_{i=0}^{k} \mathrm{ad}_{-x_0}^{l_i} x_1.$$

Theorem (Duchamp, HNM, Ngô, 2015)

$$\mathrm{L}^- := \sum_{w \in Y_0^*} \mathrm{Li}_w^- w, \quad \mathrm{H}^- := \sum_{w \in Y_0^*} \mathrm{H}_w^- w, \quad C^- := \sum_{w \in Y_0^*} \prod_{w=uv, v \neq 1_{Y_0^*}} \frac{1}{(v)+|v|} w.$$

$$\lim_{z \to 1} h^{\odot -1}((1-z)^{-1}) \odot \mathrm{Li}^-(z) = \lim_{n \to +\infty} g^{\odot -1}(n) \odot \mathrm{H}^-(n) = C^-,$$

where $h(t) = \sum_{w \in Y_0^*} ((w)+|w|)! t^{(w)+|w|} w$ *and* $g(t) = \sum_{w \in Y_0^*} t^{(w)+|w|} w.$

Moreover, H^- *and* C^- *are group-like, respectively, for* $\Delta_{⧺}$ *and* $\Delta_{⧢}$.

[2] $\{\mathrm{ad}_{-x_0}^{l_1} x_1 \cdots \mathrm{ad}_{-x_0}^{l_k} x_1\}_{k \geq 0}^{l_1, \cdots, l_k \geq 0}$ and $\{x_1 x_0^{l_1} \circ \cdots \circ x_1 x_0^{l_k}\}_{k \geq 0}^{l_1, \cdots, l_k \geq 0}$ are dual bases of $\mathcal{U}(\mathcal{J})$ and $\mathcal{U}(\mathcal{J})^\vee$ ($\mathcal{J}$ is the Lie algebra generated by $\{\mathrm{ad}_{x_0}^l x_1\}_{l \in \mathbb{N}}$).

Actions of Galois differential group

For any $e^{C} \in \mathrm{Gal}_{\mathbb{C}}(DE)$, let $\overline{Z}_{⧢} := Z_{⧢}\, e^{C}$. Let $\overline{Z}_{\gamma}$ be the noncommutative generating series of $\{\overline{\gamma}_w\}_{w \in Y^*}$, where

$$\forall w \in Y^*, \quad \overline{\gamma}_w = \mathrm{f.p.}_{n \to +\infty} \overline{\mathrm{H}}_w(n), \quad \{n^a \log^b(n)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}.$$

Then, for $\Delta_{⧢\!\!\!+}$, $\overline{Z}_{\gamma}$ is group-like series and $\overline{\gamma}_{\bullet}$ is a character. The first Abel like theorem yields $\overline{Z}_{\gamma} = B(y_1)\pi_Y \overline{Z}_{⧢}$.

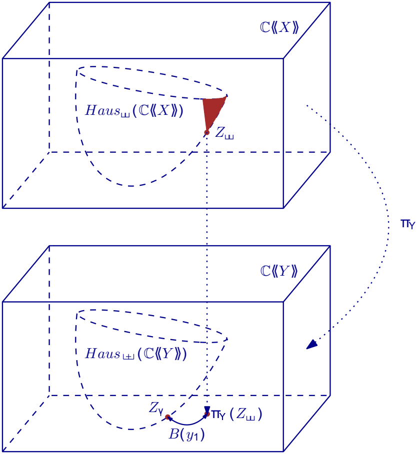

Figure: Illustration of $Z_{\gamma} = B(y_1)\pi_Y Z_{⧢}$.

Actions of sub-groups of Galois differential group

The monodromies around $0, 1$ of L are given, respectively, by

$$\mathcal{M}_0\mathrm{L} = \mathrm{L}e^{2\mathrm{i}\pi\mathfrak{m}_0} \quad \text{and} \quad \mathcal{M}_1\mathrm{L} = \mathrm{L}Z_{\shuffle}^{-1}\, e^{2\mathrm{i}\pi x_1} Z_{\shuffle} = \mathrm{L}e^{2\mathrm{i}\pi\mathfrak{m}_1},$$

$$\mathfrak{m}_0 := x_0 \qquad \text{and} \qquad \mathfrak{m}_1 := \prod_{l \in \mathcal{L}ynX - X}^{\searrow} e^{-\zeta(S_l)\,\mathrm{ad}_{P_l}(-x_1)}.$$

Their actions could not do neither simplification nor introducing the left factor $e^{\gamma y_1}$ in Z_γ and $Z_{\shuffle}$, respectively :

- For $C = 2\mathrm{i}\pi\mathfrak{m}_0$, one has $\overline{Z}_{\shuffle} = Z_{\shuffle}\, e^{2\mathrm{i}\pi x_0}$ and
$$\overline{Z}_\gamma = \exp\left(\gamma y_1 - \sum_{k \geq 2} \zeta(k) \frac{(-y_1)^k}{k}\right) \pi_Y Z_{\shuffle} = Z_\gamma.$$
- For $C = 2\mathrm{i}\pi\mathfrak{m}_1$, one has $\overline{Z}_{\shuffle} = e^{-2\mathrm{i}\pi x_1} Z_{\shuffle}$ and
$$\overline{Z}_\gamma = \exp\left(\underbrace{(\gamma - 2\mathrm{i}\pi)}_{=:T} y_1 - \sum_{k \geq 2} \zeta(k) \frac{(-y_1)^k}{k}\right) \pi_Y Z_{\shuffle} = e^{-2\mathrm{i}\pi y_1} Z_\gamma.$$

Now, let $\overline{Z}_{\shuffle} \in dm(A)$, then $\overline{Z}_\gamma = e^{\gamma y_1} \overline{Z}_{\stuffle}$ and it follows that

Corollary (action of $dm(A)$, HNM, 2009)

If $\overline{Z}_{\shuffle} \in dm(A)$ then $(\overline{Z}_\gamma = B(y_1)\pi_Y \overline{Z}_{\shuffle} \Leftrightarrow \overline{Z}_{\stuffle} = \mathtt{Mono}(y_1)\pi_Y \overline{Z}_{\shuffle})$.

Finally, if $\gamma \notin A$ then γ is transcendent over the A-algebra generated by convergent polyzetas.

Homogenous polynomials relations among local coordinates

$$Z_\gamma = B(y_1)\pi_Y Z_{\sqcup\!\sqcup}$$

	Relations among $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Relations among $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2}) = \zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4}) = \frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1}) = \frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1}) = \frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2}) = \frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2}) = \frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3}) = \frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2}) = 3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2}) = -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1}) = -\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1}) = -\frac{3}{2}\zeta(S_{x_0^4x_1}) + \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - \frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3}) = -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2}) = \frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2}) = \frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3}) = \frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4}) = \zeta(S_{x_0^4x_1})$
6	$\zeta(\Sigma_{y_6}) = \frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5x_1}) = \frac{8}{35}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_2}) = \zeta(\Sigma_{y_3})^2 - \frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4x_1^2}) = \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_5y_1}) = \frac{2}{7}\zeta(\Sigma_{y_2})^3 - \frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1x_0x_1}) = \frac{4}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_3y_1y_2}) = -\frac{17}{30}\zeta(\Sigma_{y_2})^3 + \frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1^3}) = \frac{23}{70}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_2y_1}) = 3\zeta(\Sigma_{y_3})^2 - \frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2x_1x_0x_1^2}) = \frac{2}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_1^2}) = \frac{3}{10}\zeta(\Sigma_{y_2})^3 - \frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^2x_0x_1}) = -\frac{89}{210}\zeta(S_{x_0x_1})^3 + \frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2^2y_1^2}) = \frac{11}{63}\zeta(\Sigma_{y_2})^3 - \frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^4}) = \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_1^3}) = \frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0x_1x_0x_1^3}) = \frac{8}{21}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2y_1^4}) = \frac{17}{50}\zeta(\Sigma_{y_2})^3 + \frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0x_1^5}) = \frac{8}{35}\zeta(S_{x_0x_1})^3$

(Bùi's PhD dissertation, 2016)

Noetherian rewriting system & irreducible coordinates

$$Z_\gamma = B(y_1)\pi_Y Z_{ш}$$

	Rewriting among $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Rewriting among $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1}) \to \frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2}) \to \zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4}) \to \frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1}) \to \frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1}) \to \frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2}) \to \frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2}) \to \frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3}) \to \frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2}) \to 3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2}) \to -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1}) \to -\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1}) \to -\frac{3}{2}\zeta(S_{x_0^4x_1}) + \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1}) \to \frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - \frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3}) \to -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2}) \to \frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2}) \to \frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3}) \to \frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4}) \to \zeta(S_{x_0^4x_1})$
6	$\zeta(\Sigma_{y_6}) \to \frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5x_1}) \to \frac{8}{35}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_2}) \to \zeta(\Sigma_{y_3})^2 - \frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4x_1^2}) \to \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_5y_1}) \to \frac{2}{7}\zeta(\Sigma_{y_2})^3 - \frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1x_0x_1}) \to \frac{4}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_3y_1y_2}) \to -\frac{17}{30}\zeta(\Sigma_{y_2})^3 + \frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1^3}) \to \frac{23}{70}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_2y_1}) \to 3\zeta(\Sigma_{y_3})^2 - \frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2x_1x_0x_1^2}) \to \frac{2}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_1^2}) \to \frac{3}{10}\zeta(\Sigma_{y_2})^3 - \frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^2x_0x_1}) \to -\frac{89}{210}\zeta(S_{x_0x_1})^3 + \frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2^2y_1^2}) \to \frac{11}{63}\zeta(\Sigma_{y_2})^3 - \frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^4}) \to \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_1^3}) \to \frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0x_1x_0x_1^3}) \to \frac{8}{21}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2y_1^4}) \to \frac{17}{50}\zeta(\Sigma_{y_2})^3 + \frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0x_1^5}) \to \frac{8}{35}\zeta(S_{x_0x_1})^3$

(Bùi's PhD dissertation, 2016)

POLYLOGARITHMS AND HARMONIC SUMS INDEXED BY NONCOMMUTATIVE RATIONAL SERIES

Rational series, $\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$

$\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$: Sweedler dual of $\mathbb{C}\langle X\rangle$, for $\Delta_{\mathtt{conc}}$
(set of noncommutative rational series[3]).

Theorem (Schützenberger, 1961)

$S \in \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ iff there is a linear representation, (ν,μ,η) of dimension $n>0$, i.e. $\nu \in M_{1,n}(\mathbb{C})$, $\eta \in M_{n,1}(\mathbb{C})$ and $\mu : X^* \to M_{n,n}(\mathbb{C})$ such that

$$S = \nu\Big(\sum_{w\in X^*} \mu(w)\ w\Big)\eta = \nu((\mu\otimes \mathrm{Id})\mathcal{D}_X)\eta.$$

Theorem (HNM, 1995)

Let (ν,μ,η) be a linear representation of $S \in \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$.
Then the series $\sum_{w\in X^*}\langle S|w\rangle\alpha_{z_0}^z(w)$ is convergent.
Noting this extension by $\alpha_{z_0}^z(S)$, one has

$$\alpha_{z_0}^z(S) = \nu\Big(\prod_{l\in\mathcal{L}ynX} e^{\alpha_{z_0}^z(S_l)\ \mu(P_l)}\Big)\eta.$$

And, for any $T = R$ or $S \in \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$, one has the convergent series $\alpha_{z_0}^z(T) = \sum_{w\in X^*}\langle T|w\rangle\alpha_{z_0}^z(w)$ and identity $\alpha_{z_0}^z(R \sqcup\!\sqcup S) = \alpha_{z_0}^z(R)\alpha_{z_0}^z(S)$.

[3] A series S is called *rational* iff it belongs to the closure, by $\{+, \mathrm{conc}, {}^*\}$, of $\mathbb{C}\langle X\rangle$ in $\mathbb{C}\langle\langle X\rangle\rangle$.

Exchangeable series, $\mathbb{C}_{\mathrm{exc}}\langle\langle X\rangle\rangle$

The power series S belongs to $\mathbb{C}_{\mathrm{exc}}\langle\langle X\rangle\rangle$, iff

$$(\forall u,v\in X^*)((\forall x\in X)(|u|_x=|v|_x)\quad\Rightarrow\quad\langle S|u\rangle=\langle S|v\rangle).$$

If $\quad S=\sum_{i_0,i_1\geq 0} s_{i_0,i_1}x_0^{i_0}\,ш\,x_1^{i_1}\quad$ then $\quad\alpha_{z_0}^z(S)=\sum_{i_0,i_1\geq 0} s_{i_0,i_1}\frac{(\alpha_{z_0}^z(x_0))^{i_0}}{i_0!}\frac{(\alpha_{z_0}^z(x_1))^{i_1}}{i_1!}.$

Lemma (Duchamp, HNM, Ngô, 2016)

1. $\mathbb{C}_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle X\rangle\rangle:=\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle\bigcap\mathbb{C}_{\mathrm{exc}}\langle\langle X\rangle\rangle=\mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle\,ш\,\mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle.$
2. *For any* $x\in X$, *one has* $\mathbb{C}^{\mathrm{rat}}\langle\langle x\rangle\rangle=\mathrm{span}_{\mathbb{C}}\{(ax)^*\,ш\,\mathbb{C}\langle x\rangle|a\in\mathbb{C}\}$.
3. *The family* $\{x_0^*,x_1^*\}$ *is algebraically independent over* $(\mathbb{C}\langle X\rangle,\,ш\,,1_{X^*})$ *within* $(\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle,\,ш\,,1_{X^*})$.
4. *The module* $(\mathbb{C}\langle X\rangle,\,ш\,,1_{X^*})[x_0^*,x_1^*,(-x_0)^*]$ *is* $\mathbb{C}\langle X\rangle$*-free and* $\{(x_0^*)^{ш\,k}\,ш\,(x_1^*)^{ш\,l}\}^{(k,l)\in\mathbb{Z}\times\mathbb{N}}$ *forms a* $\mathbb{C}\langle X\rangle$*-basis of it.*

   Hence, $\{w\,ш\,(x_0^*)^{ш\,k}\,ш\,(x_1^*)^{ш\,l}\}_{w\in X^*}^{(k,l)\in\mathbb{Z}\times\mathbb{N}}$ *is a* $\mathbb{C}$*-basis of it.*

Theorem (extension of $\mathrm{Li}_\bullet$, Duchamp, HNM, Ngô, 2016)

$$\begin{array}{rcl}\mathrm{Li}_\bullet:(\mathbb{C}[x_0^*,x_1^*,(-x_0)^*]\,ш\,\mathbb{C}\langle X\rangle,\,ш\,,1_{X^*}) & \longrightarrow & (\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*},.,1_\Omega),\\ R & \longmapsto & \mathrm{Li}_R\,.\end{array}$$

$\mathrm{Li}_\bullet$ *is surjective and* $\ker\mathrm{Li}_\bullet$ *is the shuffle ideal generated by* $x_0^*\,ш\,x_1^*-x_1^*+1$.

Examples of polylogarithms indexed by rational series

For any $a, b \in \mathbb{C}$, $\alpha_1^z((ax_0)^*) = z^a$ and $\alpha_0^z((bx_1)^*) = (1-z)^{-b}$. Then

1. One has
$$\mathrm{Li}_{x_0^*}(z) = z, \quad \mathrm{Li}_{x_1^*}(z) = (1-z)^{-1}, \quad \mathrm{Li}_{(ax_0+bx_1)^*}(z) = z^a(1-z)^{-b}.$$
2. Since $(nx)^* = (x^*)^{\sqcup\!\sqcup\, n}$ then one has
$$\mathrm{Li}_{(nx_0^*)}(z) = z^n, \mathrm{Li}_{(-kx_1^*)}(z) = (1-z)^{-k}, \mathrm{Li}_{(nx_0^*) \sqcup\!\sqcup (kx_1^*)}(z) = z^n(1-z)^{-k}.$$
3. Since $(ax)^{*n} = (ax)^* \sqcup\!\sqcup (1-ax)^{n-1}$ then
$$\mathrm{Li}_{(ax_0)^{*n}}(z) = z^a \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(a\log z)^k}{k!},$$
$$\mathrm{Li}_{(ax_1)^{*n}}(z) = \frac{1}{(1-z)^a} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{1}{k!} \left(a \log \frac{1}{1-z}\right)^k.$$
4. For any $(s_1, \ldots, s_r) \in \mathbb{N}_+^r$ and $(t_1, \ldots, t_r) \in (\mathbb{C} - \mathbb{N}_+)^r$,
$$\mathrm{Li}_{(t_1x_0)^{*s_1} x_0^{s_1-1} x_1 \ldots (t_rx_0)^{*s_r} x_0^{s_r-1} x_1}(z) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{(n_1-t_1)^{s_1} \ldots (n_r-t_r)^{s_r}}.$$
In particular, for $s_1 = \ldots = s_r = 1$,
$$\mathrm{Li}_{(t_1x_0)^* x_1 \ldots (t_rx_0)^* x_1}(z) = \sum_{n_1, \ldots, n_r > 0} \mathrm{Li}_{x_0^{n_1-1} x_1 \ldots x_0^{n_r-1} x_1}(z)\; t_0^{n_1-1} \ldots t_r^{n_r-1}.$$

Polylogarithms and harmonic sums by rational series

Corollary (elements of polylogarithmic transseries)

$$\begin{aligned}
\mathbb{C}\{\mathrm{Li}_S\}_{S\in\mathbb{C}\langle X\rangle \,\text{ш}\, \mathbb{C}[x_0^*] \,\text{ш}\, \mathbb{C}[(-x_0^*)] \,\text{ш}\, \mathbb{C}[x_1^*]} &= \mathrm{span}_{\mathbb{C}}\left\{\frac{z^a}{(1-z)^b}\,\mathrm{Li}_w(z)\right\}_{w\in X^*}^{a\in\mathbb{Z},b\in\mathbb{N}} \\
&\subset \mathrm{span}_{\mathbb{C}}\{\mathrm{Li}_{s_1,\ldots,s_r}\}_{s_1,\ldots,s_r\in\mathbb{Z}^r} \\
&\quad \oplus \mathrm{span}_{\mathbb{C}}\{z^a | a\in\mathbb{Z}\}, \\
\mathbb{C}\{\mathrm{Li}_S\}_{S\in\mathbb{C}\langle X\rangle \,\text{ш}\, \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \,\text{ш}\, \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle} &= \mathrm{span}_{\mathbb{C}}\left\{\frac{z^a}{(1-z)^b}\,\mathrm{Li}_w(z)\right\}_{w\in X^*}^{a,b\in\mathbb{C}} \\
&\subset \mathrm{span}_{\mathbb{C}}\{\mathrm{Li}_{s_1,\ldots,s_r}\}_{s_1,\ldots,s_r\in\mathbb{C}^r} \\
&\quad \oplus \mathrm{span}_{\mathbb{C}}\{z^a | a\in\mathbb{C}\}.
\end{aligned}$$

More generally, one has $\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle \,\text{ш}\, \mathbb{C}\langle X\rangle \subset \mathrm{Dom}(\mathrm{Li}_\bullet) \cap \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$, and, on it, letting (ν,μ,η) be a linear representation of $S\in\mathrm{Dom}(\mathrm{Li}_\bullet)\cap\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$, one has

$$\begin{aligned}
\mathrm{Li}_S(z) &= \sum_{n\geq 0}\frac{\langle S|x_0^n\rangle}{n!}\log^n(z) + \sum_{k\geq 1}\sum_{w\in x_0^* \,\text{ш}\, x_1^k}\langle S|w\rangle\,\mathrm{Li}_w(z) \\
&= \nu\left(\prod_{l\in\mathcal{L}ynX-\{x_0\}} e^{\mathrm{Li}_{S_l}(z)\mu(P_l)}\right)e^{\log(z)\mu(x_0)}\eta.
\end{aligned}$$

Indexing polylogarithms $\{\mathrm{Li}^-_w\}_{w\in Y_0^*}$ by rational series

Using θ_0 and by denoting $\lambda : z \mapsto z(1-z)^{-1}$ and $S_2(k_i,j)$ the Stirling numbers of second kind, one has

$$\mathrm{Li}^-_{y_{s_1}\ldots y_{s_r}} = \sum_{k_1=0}^{s_1}\sum_{k_2=0}^{s_1+s_2-k_1}\ldots\sum_{k_r=0}^{\substack{(s_1+\ldots+s_r)-\\(k_1+\ldots+k_{r-1})}}\binom{s_1}{k_1}\binom{s_1+s_2-k_1}{k_2}\ldots\binom{s_1+\ldots+s_r-k_1-\ldots-k_{r-1}}{k_r}(\theta_0^{k_1}\lambda)\ldots(\theta_0^{k_r}\lambda),$$

$$\theta_0^{k_i}(\lambda(z)) = \frac{1}{1-z}\sum_{j=1}^{k_i} S_2(k_i,j)j!(\lambda(z))^j, \quad \text{for } k_i > 0.$$

Proposition (Encoding polylogarithms by rational series)

$\mathrm{Li}^-_{y_{s_1}\ldots y_{s_r}} = \mathrm{Li}_{R_{y_{s_1}\ldots y_{s_r}}}$, *where* $R_{y_{s_1}\ldots y_{s_r}} \in (\mathbb{Z}[x_1^*], ⧢, 1_{X^*})$ *given by*

$$R_{y_{s_1}\ldots y_{s_r}} = \sum_{k_1=0}^{s_1}\sum_{k_2=0}^{s_1+s_2-k_1}\ldots\sum_{k_r=0}^{\substack{(s_1+\ldots+s_r)-\\(k_1+\ldots+k_{r-1})}}\binom{s_1}{k_1}\binom{s_1+s_2-k_1}{k_2}\ldots\binom{\sum_{i=1}^r s_i - \sum_{i=1}^{r-1} k_i}{k_r} \rho_{k_1} ⧢ \ldots ⧢ \rho_{k_r},$$

$$\rho_{k_i} = \begin{cases} x_1^* - 1_{X^*}, & \text{if } k_i = 0,\\ x_1^* ⧢ \sum_{j=1}^{k_i} S_2(k_i,j)j!(x_1^*-1_{X^*})^{⧢ j}, & \text{if } k_i > 0.\end{cases}$$

Extensions of $R_\bullet$ over $\mathbb{Z}\langle Y_0\rangle$

By linearity, $R_\bullet$ is extended over $\mathbb{Z}\langle Y_0\rangle$. Hence, for any $k, l \geq 0$, one has

$$\mathrm{Li}_{R_{y_k} ⧢ R_{y_l}} = \mathrm{Li}_{R_{y_k}} \mathrm{Li}_{R_{y_l}} = \mathrm{Li}^-_{y_k} \mathrm{Li}^-_{y_l} = \mathrm{Li}^-_{y_k \top y_l} = \mathrm{Li}_{R_{y_k \top y_l}} .$$

Theorem

1. $\{\mathrm{Li}_{R_{y_k}}\}_{k\geq 0}$ *is* $\mathbb{Q}$*-linearly independent and the restriction* $\mathrm{Li}_\bullet : (\mathbb{Z}[x_1^*], ⧢, 1_{X^*}) \to (\mathbb{Z}[(1-z)^{-1}], ., 1_\Omega)$ *is bijective.*
2. *For any* $k, l \geq 0$*, one has* $R_{y_k} ⧢ R_{y_l} = R_{y_k \top y_l}$.
3. *One has* $(\mathbb{Z}[x_1^*], ⧢, 1_{X^*}) \cong (\mathbb{Z}Y_0, \top, 1_{Y_0^*})$ *and then the morphism* $R_\bullet : (\mathbb{Z}\langle Y_0\rangle, ⧺, 1_{Y_0^*}) \to (\mathbb{Z}[x_1^*], ⧢, 1_{X^*})$ *is bijective.*

Corollary

For any $l \in \mathcal{L}ynY$*, there exists a unique polynomial* $p \in \mathbb{Z}[t]$ *of degree* $(l) + |l|$ *and of valuation* 1 *such that*

$$\begin{array}{rcll} R_l &=& \check{p}(x_1^*) & \in (\mathbb{Z}[x_1^*], ⧢, 1_{X^*}), \\ \mathrm{Li}_{R_l}(z) &=& p(e^{-\log(1-z)}) & \in (\mathbb{Z}[e^{-\log(1-z)}], ., 1), \\ \mathrm{H}_{\pi_Y R_l}(n) &=& \tilde{p}((n)_\bullet) & \in (\mathbb{Q}[(n)_\bullet], ., 1), \end{array}$$

where $(n)_\bullet : \mathbb{N} \to \mathbb{N}, i \mapsto n(n-1)\ldots(n-i+1)$, $\tilde{p}$ *is the exponential transformed of* p *and* p *is obtained as the exponential transformed of* $\check{p}$.

Example

Since $\mathrm{Li}_{R_{y_1}}(z) = z(1-z)^{-2}$ then $R_{y_1} = (2x_1)^* - x_1^*$ and $p(t) = t^2 - t$. Since $\pi_Y(tx_1)^* = (ty_1)^*$ then, via the Newton-Girard identity, $\mathrm{H}_{(ty_1)^*} = \sum_{k\geq 0} \mathrm{H}_{y_1^k} t^k = \exp(-\sum_{k\geq 1} \mathrm{H}_{y_k}(-t)^k/k)$, one has $\mathrm{H}_{\pi_Y R_{y_1}}(n) = n(n+1)/2$.

The constants $\{\gamma_{-s_1,\ldots,-s_r}\}_{(s_1,\ldots,s_r)\in\mathbb{N}^r, r\in\mathbb{N}}$

Theorem (extended double regularization)

$$\zeta_{\sqcup\!\sqcup}((tx_1)^*) = 1 \text{ and } \gamma_{\pi_Y(tx_1)^*} = \exp\left(\gamma t - \sum_{n\geq 2}\zeta(n)\frac{(-t)^n}{n}\right) = \frac{1}{\Gamma(1+t)}.$$

Corollary

For any $l \in \mathcal{L}ynY$, there exists a unique polynomial $p \in \mathbb{Z}[t]$ of degree $(l)+|l|$ and of valuation 1 such that $R_l = \check{p}(x_1^*) \in (\mathbb{Z}[x_1^*], \sqcup\!\sqcup, 1_{X^*})$ and

$$\zeta_{\sqcup\!\sqcup}(R_l) = p(1) \in \mathbb{Z} \quad \text{and} \quad \gamma_{\pi_Y R_l} = \tilde{p}(1) \in \mathbb{Q},$$

where $\tilde{p}$ is the exponential transformed of p and p is obtained as the exponential transformed of $\check{p}$.

Example (Ngô's PhD dissertation, 2016)

$$\mathrm{Li}_{-1,-1} = -\mathrm{Li}_{x_1^*} + 5\,\mathrm{Li}_{(2x_1)^*} - 7\,\mathrm{Li}_{(3x_1)^*} + 3\,\mathrm{Li}_{(4x_1)^*},$$
$$\mathrm{Li}_{-2,-1} = \mathrm{Li}_{x_1^*} - 11\,\mathrm{Li}_{(2x_1)^*} + 31\,\mathrm{Li}_{(3x_1)^*} - 33\,\mathrm{Li}_{(4x_1)^*} + 12\,\mathrm{Li}_{(5x_1)^*},$$
$$\mathrm{Li}_{-1,-2} = \mathrm{Li}_{x_1^*} - 9\,\mathrm{Li}_{(2x_1)^*} + 23\,\mathrm{Li}_{(3x_1)^*} - 23\,\mathrm{Li}_{(4x_1)^*} + 8\,\mathrm{Li}_{(5x_1)^*},$$
$$\mathrm{H}_{-1,-1} = -\mathrm{H}_{\pi_Y(x_1^*)} + 5\mathrm{H}_{\pi_Y((2x_1)^*)} - 7\mathrm{H}_{\pi_Y((3x_1)^*)} + 3\mathrm{H}_{\pi_Y((4x_1)^*)},$$
$$\mathrm{H}_{-2,-1} = \mathrm{H}_{\pi_Y(x_1^*)} - 11\mathrm{H}_{\pi_Y((2x_1)^*)} + 31\mathrm{H}_{\pi_Y((3x_1)^*)} - 33\mathrm{H}_{\pi_Y((4x_1)^*)} + 12\mathrm{H}_{\pi_Y((5x_1)^*)},$$
$$\mathrm{H}_{-1,-2} = \mathrm{H}_{\pi_Y(x_1^*)} - 9\mathrm{H}_{\pi_Y((2x_1)^*)} + 23\mathrm{H}_{\pi_Y((3x_1)^*)} - 23\mathrm{H}_{\pi_Y((4x_1)^*)} + 8\mathrm{H}_{\pi_Y((5x_1)^*)}.$$

Therefore, $\zeta_{\sqcup\!\sqcup}(-1,-1) = 0$, $\zeta_{\sqcup\!\sqcup}(-2,-1) = -1$, $\zeta_{\sqcup\!\sqcup}(-1,-2) = 0$, and

$$\gamma_{-1,-1} = -\Gamma^{-1}(2) + 5\Gamma^{-1}(3) - 7\Gamma^{-1}(4) + 3\Gamma^{-1}(5) = 11/24,$$
$$\gamma_{-2,-1} = \Gamma^{-1}(2) - 11\Gamma^{-1}(3) + 31\Gamma^{-1}(4) - 33\Gamma^{-1}(5) + 12\Gamma^{-1}(6) = -73/120,$$
$$\gamma_{-1,-2} = \Gamma^{-1}(2) - 9\Gamma^{-1}(3) + 23\Gamma^{-1}(4) - 23\Gamma^{-1}(5) + 8\Gamma^{-1}(6) = -67/120.$$

Candidates for associators with rational coefficients

Since the extension $R_\bullet : (\mathbb{C}[x_0]\langle Y_0\rangle, \top) \to (\mathbb{C}[x_0][x_1^*], ⧢)$ is bijective then

$\Upsilon := ((\mathrm{H}_\bullet \circ \pi_Y \circ R_\bullet) \otimes \mathrm{Id})\mathcal{D}_Y$ and $\Lambda := ((\mathrm{Li}_\bullet \circ R_\bullet \circ \hat{\pi}_Y) \otimes \mathrm{Id})\mathcal{D}_X$,

$Z_\gamma^- := ((\gamma_\bullet \circ \pi_Y \circ R_\bullet) \otimes \mathrm{Id})\mathcal{D}_Y$ and $Z_⧢^- := ((\zeta_⧢ \circ R_\bullet \circ \hat{\pi}_Y) \otimes \mathrm{Id})\mathcal{D}_X$,

where, the morphism of algebras $\hat{\pi}_Y$ is defined, over an algebraic basis, by $\hat{\pi}_Y(x_0) = x_0$ (s.t. $\mathrm{Li}_{R_{\hat{\pi}_Y x_0}}(z) = \log(z)$ and then $\zeta(R_{\hat{\pi}_Y x_0}) = 0$) and, for any $l \in \mathcal{L}ynX - \{x_0\}, \hat{\pi}_Y S_l = \pi_Y S_l$. Hence, $Z_\gamma^- \in \mathbb{Q}\langle\langle Y\rangle\rangle$ and $Z_⧢^- \in \mathbb{Z}\langle\langle X\rangle\rangle$.

In particular, $\langle Z_\gamma^- | y_1\rangle = -1/2$ and $\langle Z_⧢^- | x_1\rangle = \langle Z_⧢^- | x_0\rangle = 0$.

Theorem (associators with rational coefficients)

$$\Delta_{⧺}(\Upsilon) = \Upsilon \otimes \Upsilon \quad \textit{and} \quad \Delta_⧢(\Lambda) = \Lambda \otimes \Lambda,$$
$$\Delta_{⧺}(Z_\gamma^-) = Z_\gamma^- \otimes Z_\gamma^- \quad \textit{and} \quad \Delta_⧢(Z_⧢^-) = Z_⧢^- \otimes Z_⧢^-,$$

and all constant terms are 1*. It follows then*

$$\Upsilon = \prod_{l \in \mathcal{L}ynY}^{\searrow} e^{\mathrm{H}_{\pi_Y R_{\Sigma_l}} \Pi_l} \quad \textit{and} \quad \Lambda = \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\mathrm{Li}_{R_{\hat{\pi}_Y S_l}} P_l} \sim_0 e^{x_0 \log(z)},$$

$$Z_\gamma^- = \prod_{l \in \mathcal{L}ynY}^{\searrow} e^{\gamma_{\pi_Y R_{\Sigma_l}} \Pi_l} \quad \textit{and} \quad Z_⧢^- = \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\zeta_⧢ (R_{\hat{\pi}_Y S_l}) P_l}.$$

Moreover, $\Lambda \in (\mathrm{span}_{\mathbb{C}}\{\mathrm{Li}_S\}_{S \in \mathbb{C}\langle X\rangle ⧢ \mathbb{C}_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle X\rangle\rangle}, \theta_0, \iota_0, \theta_1, \iota_1)\langle\langle X\rangle\rangle$ and, for any $g \in \mathcal{G}$, there exists a letter substitution, σ_g, and a Lie series, $C \in \mathcal{L}ie_{\mathbb{C}}\langle\langle X\rangle\rangle$, such that $\Lambda(g) = \sigma_g(\Lambda)e^C$.

THANK YOU FOR YOUR ATTENTION