Hierarchical Tilings

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Moscow, Spring 2011

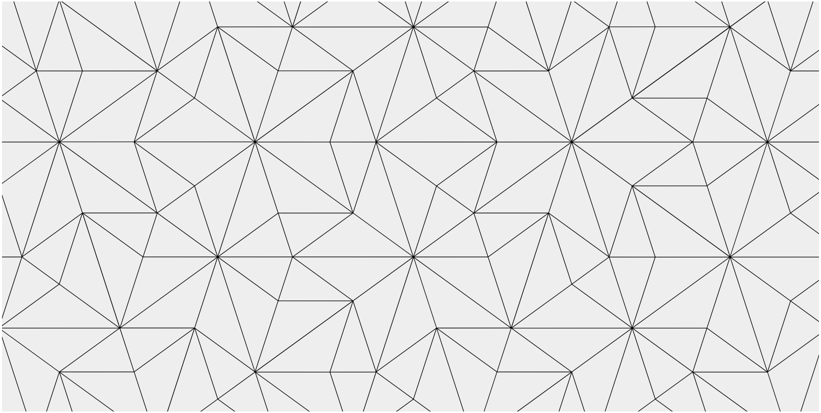
1. Formalism
2. General construction
3. Step-by-step example

1 Formalism

2 General construction

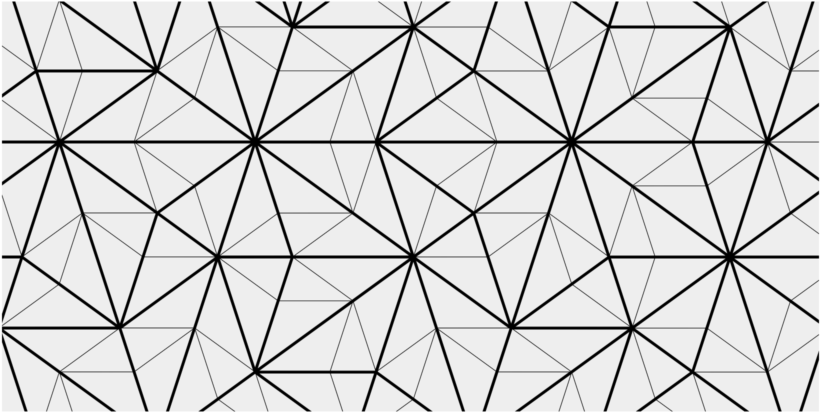
3 Step-by-step example

Back to Penrose



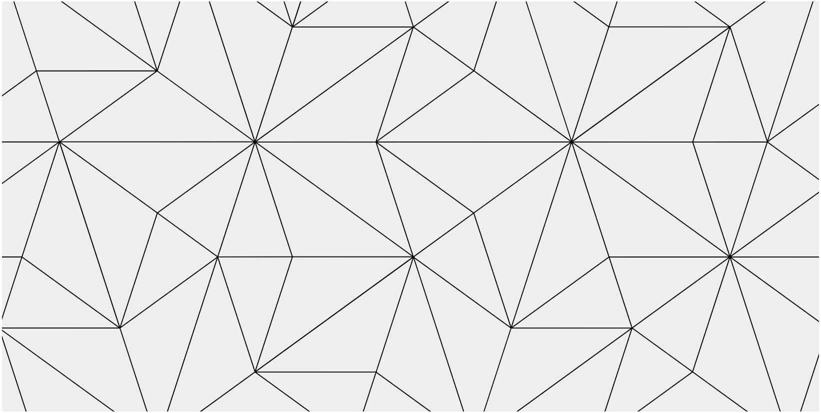
Remind Penrose tilings by Robinson triangles.

Back to Penrose



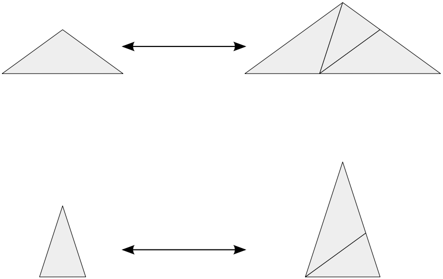
Robinson triangles can be grouped into Robinson macro-triangle.

Back to Penrose



This yields a new tiling by Robinson triangles (up to deflating).

Back to Penrose



Macro-tiles can thus be substituted to tiles (and conversely).

Combinatorial substitution

Definition (Combinatorial substitution)

A *combinatorial substitution* is a finite set of *rules* (P, Q, γ), where P is a tile, Q is a finite tiling called *macro-tile*, and $\gamma : \partial P \to \partial Q$ maps facets of P to disjoint facet sets of Q, called *macro-facet*.

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Let $\sigma = \{(P_i, Q_i, \gamma_i)_i\}$ be a combinatorial substitution.

Definition (Preimage)

Let T be a tiling by the P_i's and T' be a macro-tiling by the Q_i's. If there is a bijection between tiles of T and macro-tiles of T' preserving the *combinatorial structure*, then T is a *preimage* of T'.

Consistency: any macro-tiling by the Q_i's admits a preimage.

Limit set and non-periodicity

Definition (Limit set)

The limit set of a combinatorial substitution σ is the set Λ_σ of the tilings which admit an infinite sequence of preimages under σ.

Proposition

If tilings in Λ_σ have each a unique preimage, then none is periodic.

Can decorations enforce tilings to have this hierarchical structure?

Self-simulation

Let $\sigma = \{(P_i, Q_i, \gamma_i)_i\}$ be a consistent combinatorial substitution.

Definition

A decorated tile set τ is said to *σ-self-simulate* if there is a set of τ-macro-tiles and a map ϕ from these τ-macro-tiles to τ-tiles s.t.

1. each τ-macro-tile $\mathcal{Q}$ and $\phi(\mathcal{Q})$ form a decorated pair (Q_i, P_i);
2. any τ-tiling can uniquely be seen as a tiling by τ-macro-tiles;
3. each τ-macro-tile $\mathcal{Q}$ is *combinatorially equivalent* to $\phi(\mathcal{Q})$.

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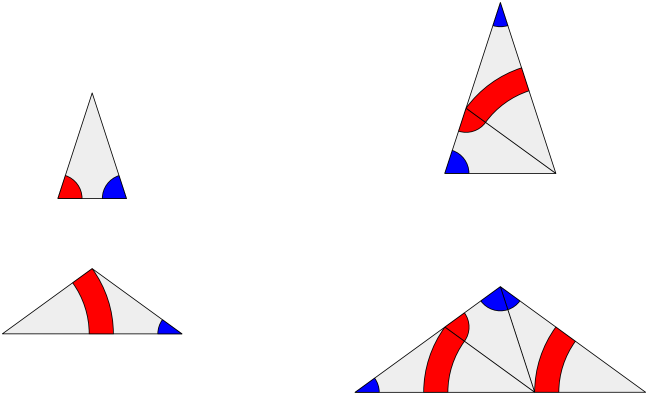
Proposition

If τ is a tile set which σ-self-simulates, then all the τ-tilings are, once undecorated, in the limit set of σ.

Back again to Penrose



Decorated Robinson triangles σ-self-simulates (σ: first slide).

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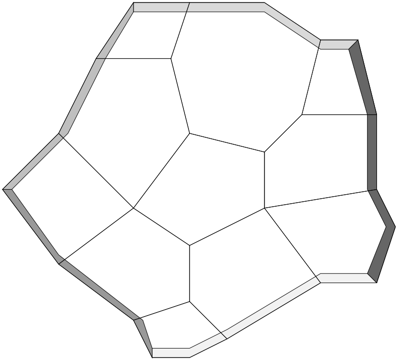
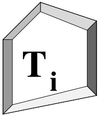
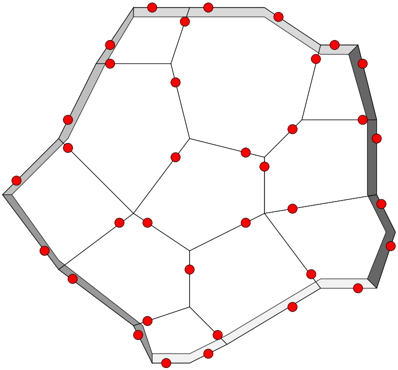
A self-simulating tile set τ

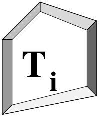




Consider $\sigma = \{(P_i, Q_i, \gamma_i)_i\}$, where each P_i appears in some Q_j.
Let $T_1, \ldots, T_n$ denote all the tiles which appear in the Q_i's.

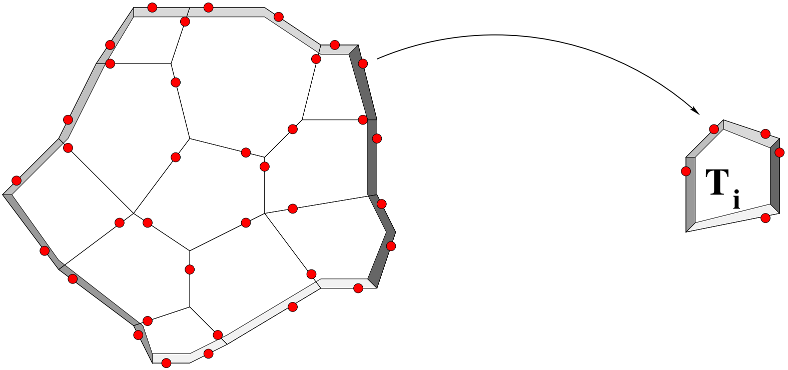
A self-simulating tile set τ





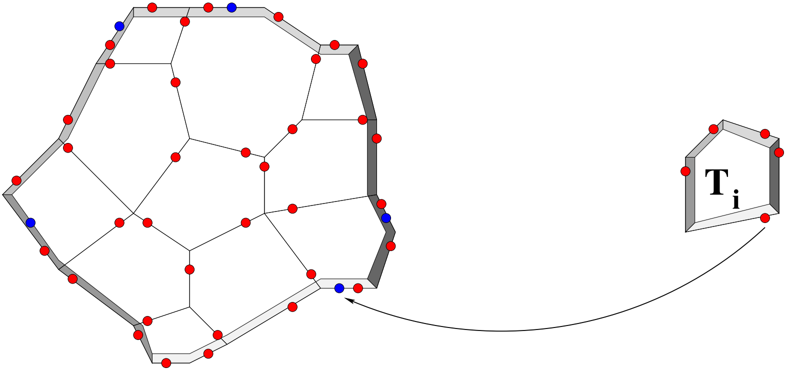
To enforce τ-tilings to be τ-macro-tilings: decorations specify tile neighbors within macro-tiles and mark macro-facets.

A self-simulating tile set τ



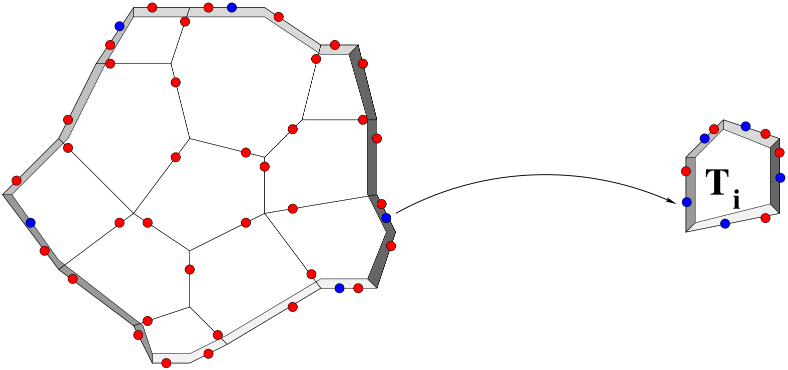
This yields so-called macro-indices on tile facets.

A self-simulating tile set τ



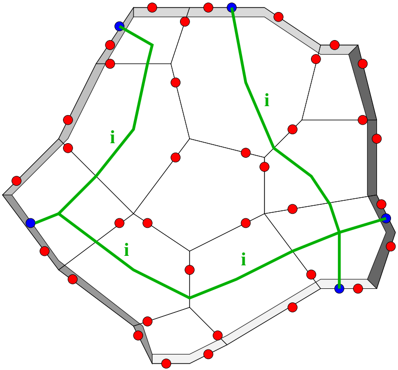
The macro-indices of facets of a τ-tile must then be encoded on the corresponding macro-facets of its simulating τ-macro-tile.

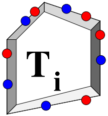
A self-simulating tile set τ



This yields so-called neighbor-indices on tile facets.

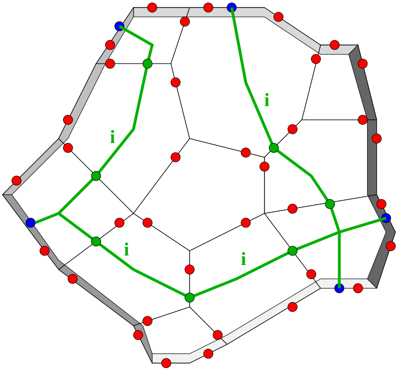
A self-simulating tile set τ

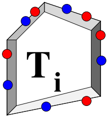




We force these neighbor-indices to come from the same tile T_i, called parent-tile, by carrying its index i between macro-facets, where it is converted into the suitable neighbor-index.

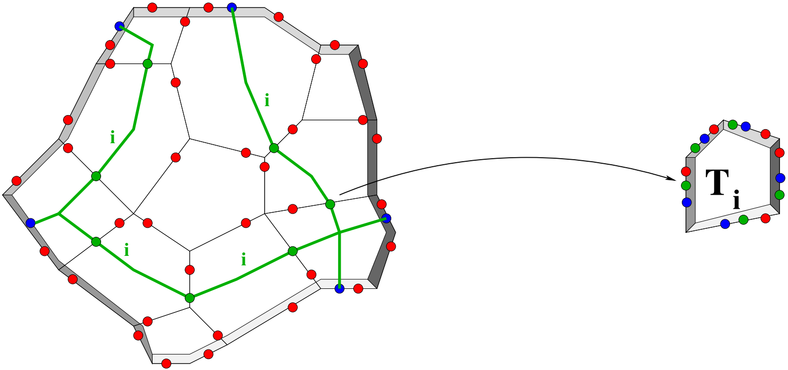
A self-simulating tile set τ





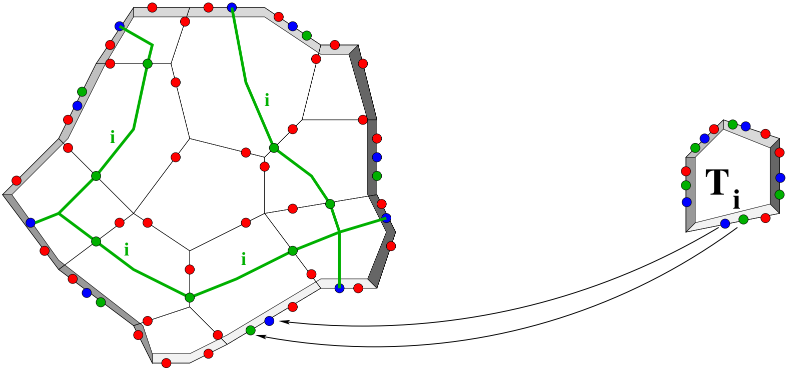
Such tile indices are encoded on facets by so-called parent-index.

A self-simulating tile set τ



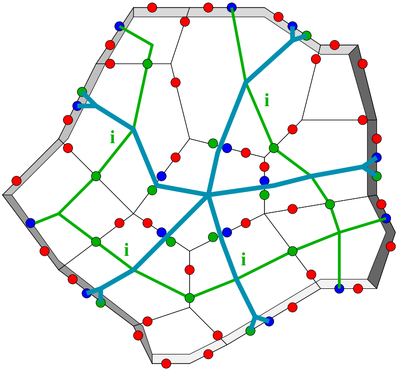
This yields, once again, a new index on each tile facets. . .

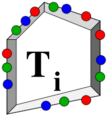
A self-simulating tile set τ



But the trick is that the neighbor-indices and parent-indices of facets of a τ-tile can be encoded on the corresponding big enough macro-facets of the equivalent τ-macro-tile without any new index!

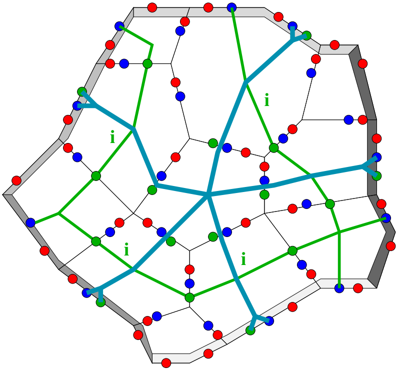
A self-simulating tile set τ

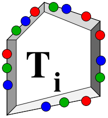




In big enough macro-tiles, we can then carry these pairs of neighbor/parent indices up to a central tile along a star-like network.

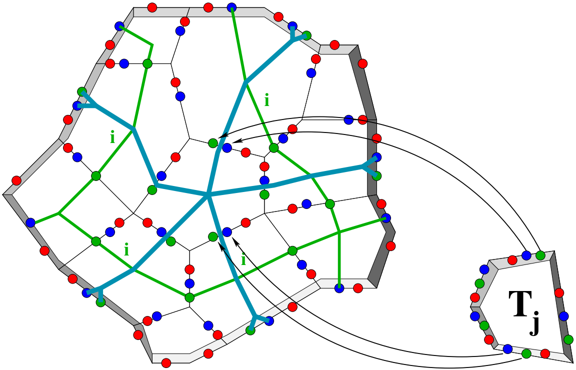
A self-simulating tile set τ





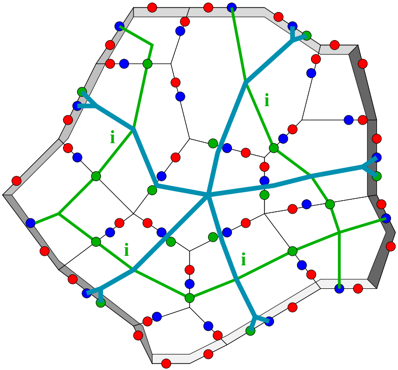
On internal facets not crossed by this network, we copy the macro-index on the neighbor-index (this redundancy is later used).

A self-simulating tile set τ



The pairs on a central τ-tile can be those of any non-central τ-tile (from which the central τ-tile is said to derive).

A self-simulating tile set τ



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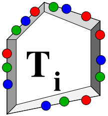
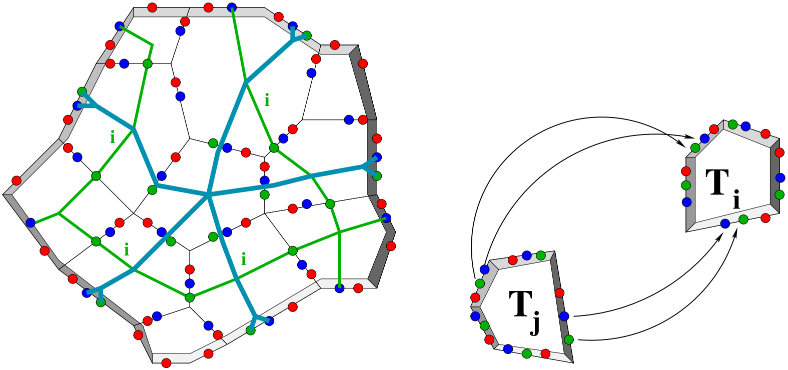


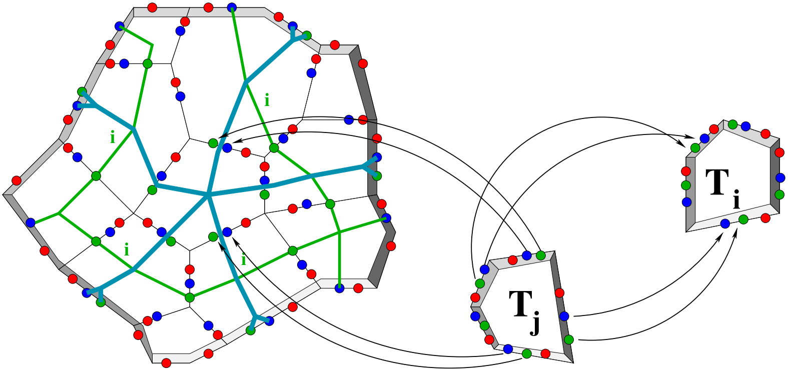
The τ-macro-tile with parent-index i is combinatorially equivalent to T_i endowed with the pairs of the central τ-tile. But is it a τ-tile?

A self-simulating tile set τ



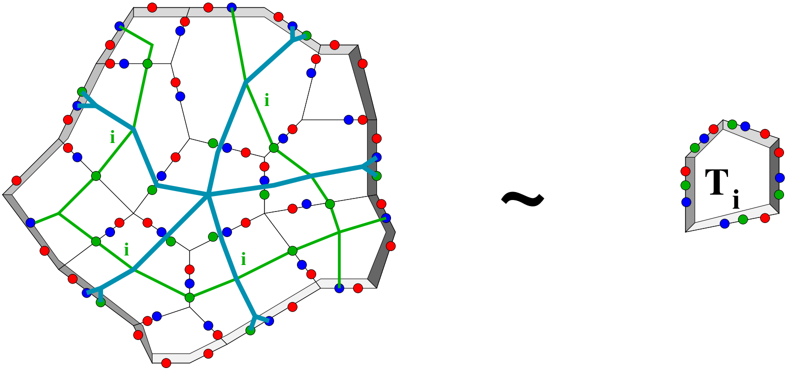
If T_i is a central tile, then its pairs can be derived from any non-central τ-tile (as for any central tile)...

A self-simulating tile set τ



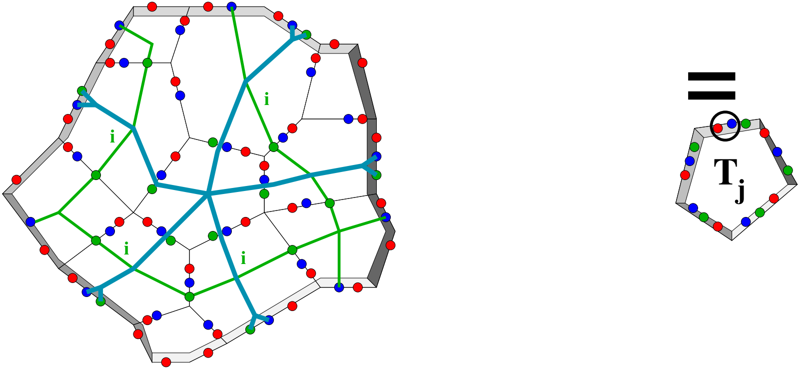
... in particular from the non-central τ-tile from which are also derived the pairs of the central τ-tile of our τ-macro-tile.

A self-simulating tile set τ



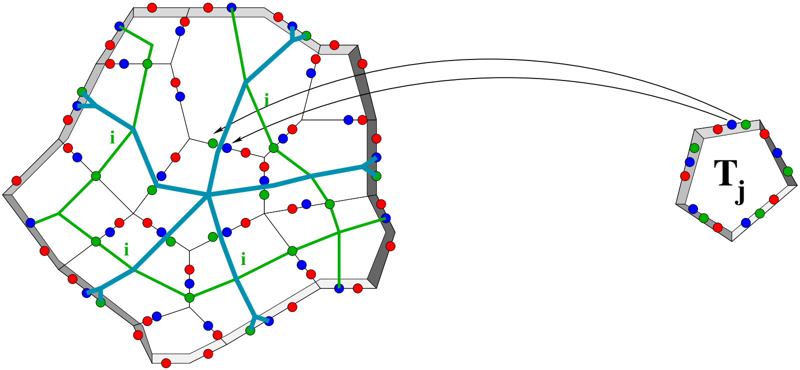
In this case, the equivalent decorated T_i is a derived central τ-tile.

A self-simulating tile set τ



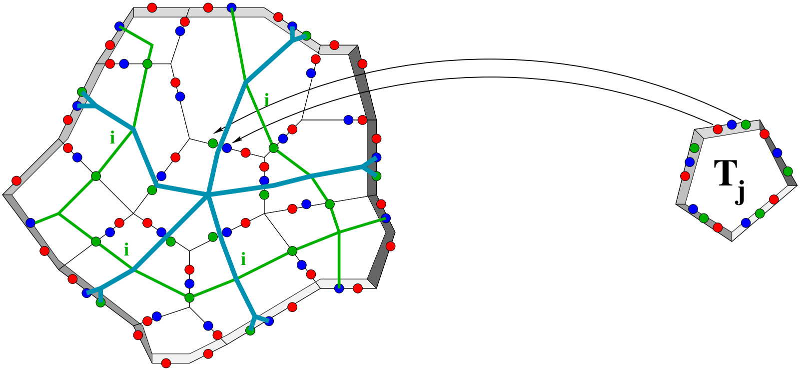
Otherwise, consider the non-central τ-tile from which derives our central τ-tile; at least one facet is internal and not crossed by a network: its neighbor and macro indices are equal (by redundancy).

A self-simulating tile set τ



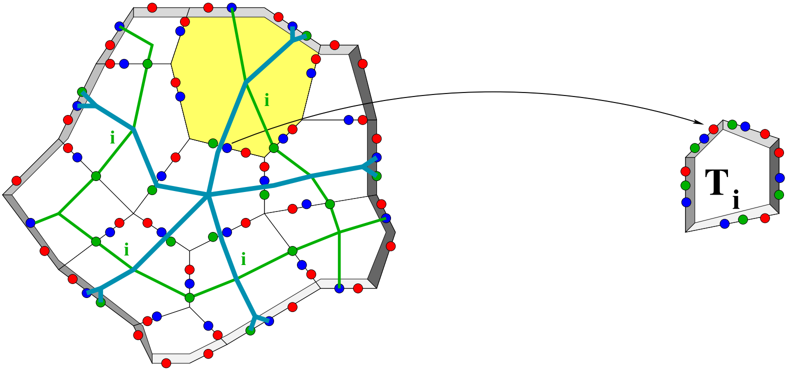
Thus, by copying the neighbor and parent indices (derivation)...

A self-simulating tile set τ



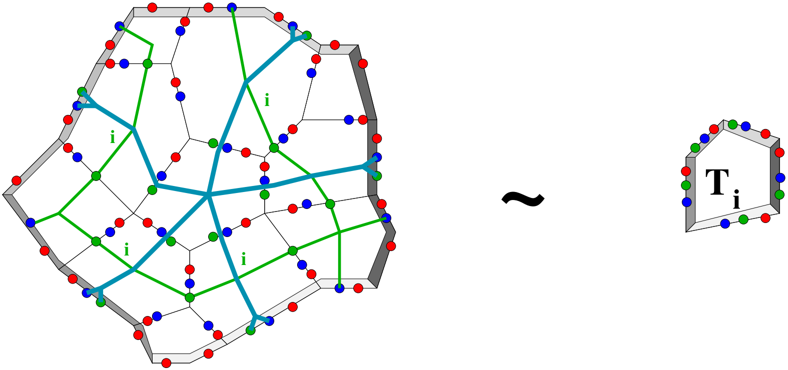
. . . one copies a macro-index on our central τ-tile, and thus on the whole corresponding network branch.

A self-simulating tile set τ



A tile on this k-th branch which also knows the parent-index i can then force this macro-index to be the one on the k-th facet of a decorated T_i (recall that all the decorated T_i have the same one).

A self-simulating tile set τ



In this case, the equivalent decorated T_i is the non-central τ-tile from which derives the central τ-tile of our τ-macro-tile.

Back to the limit set

τ self-simulates for $\sigma \rightsquigarrow$ the τ-tilings are (once undecorated) in Λ_σ.

Conversely, any tiling in Λ_σ can be decorated into a τ-tiling (easy).

Theorem

The limit set of a (suitable) combinatorial substitution is sofic.

Note 1: tilings in the limit set may be periodic... but not τ-tilings.

Note 2: such a general construction easily yields thousands of tiles!

1 Formalism

2 General construction

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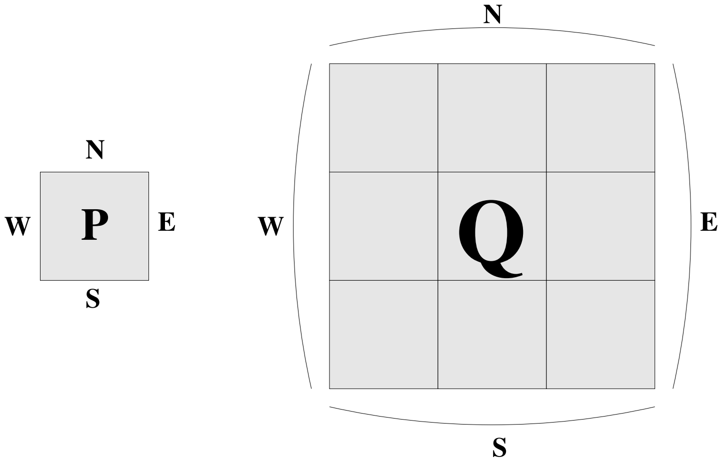
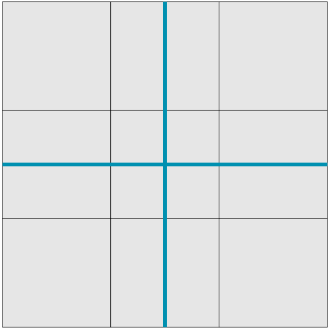
Settings



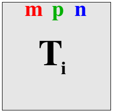
Consider this combinatorial substitution with only one rule.

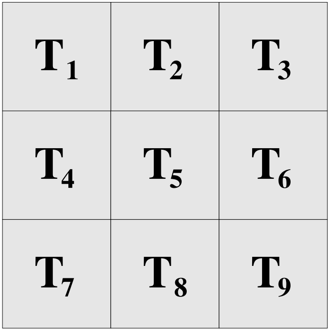
Settings



The network is a cross connecting external facets called *ports*.

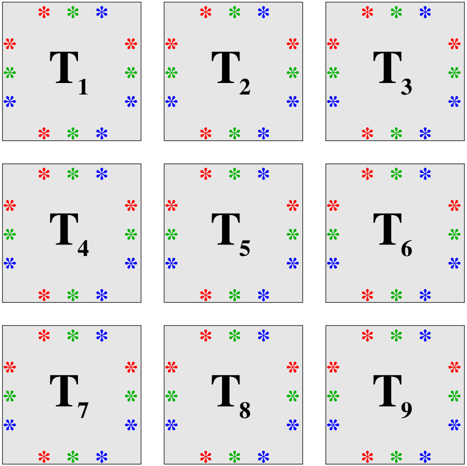
Settings





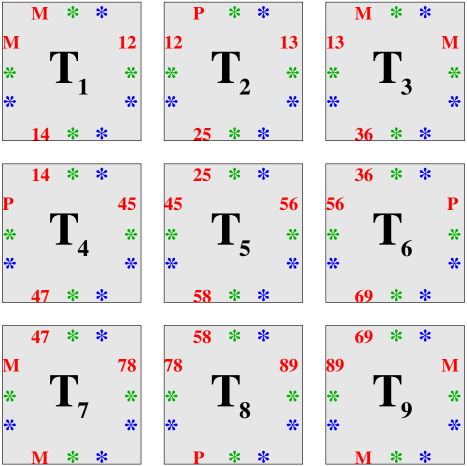
Tiles are numbered; on facets: macro-, parent- and neighbor-indices.

Step 1



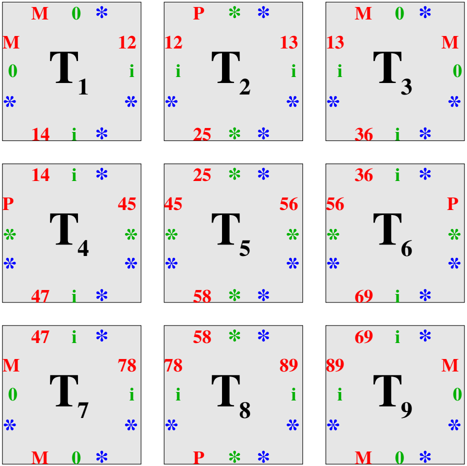
Indices are initially all undefined.

Step 1



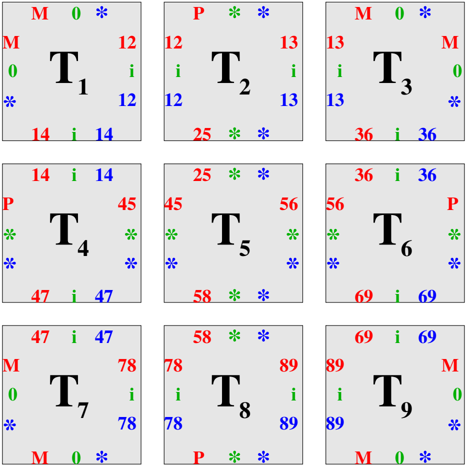
Macro-indices: enforce any tilings to be a tiling by macro-tiles.

Step 2



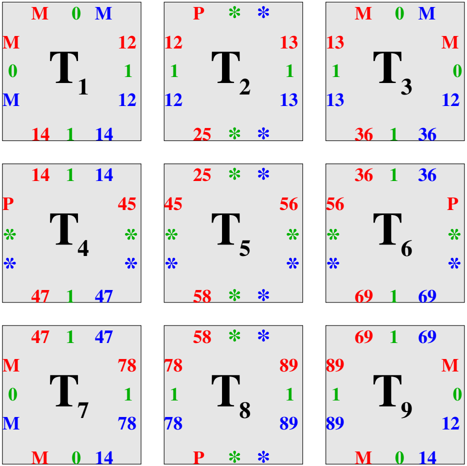
Parent-indices outside the network $\rightsquigarrow$ parent-tile T_i $(i = 1, \dots, 9)$.

Step 3



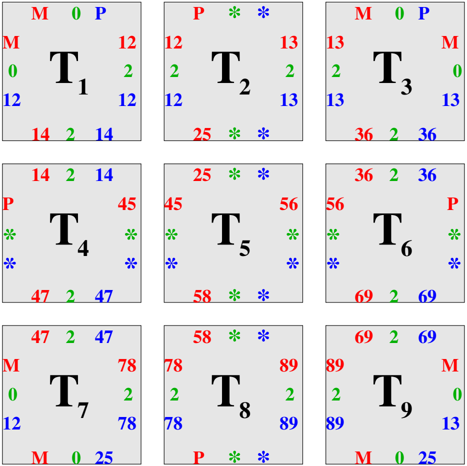
Neighbor-index of an internal facet not on the network: macro-index.

Step 3



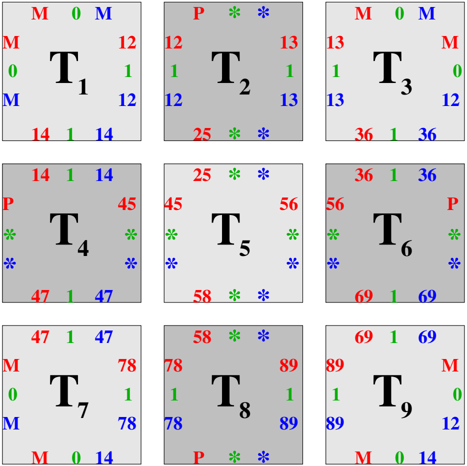
Parent-tile $\rightsquigarrow$ neighbor-index of an external facet not on the network.

Step 3



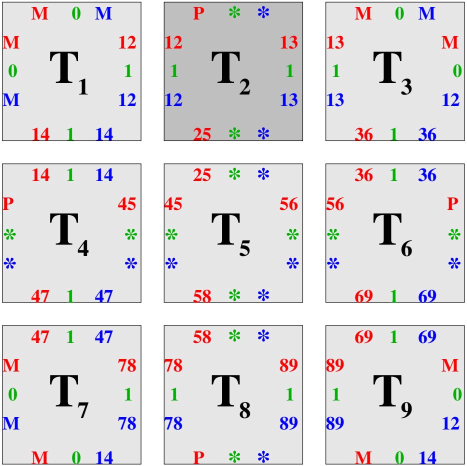
Parent-tile $\rightsquigarrow$ neighbor-index of an external facet not on the network.

Step 4



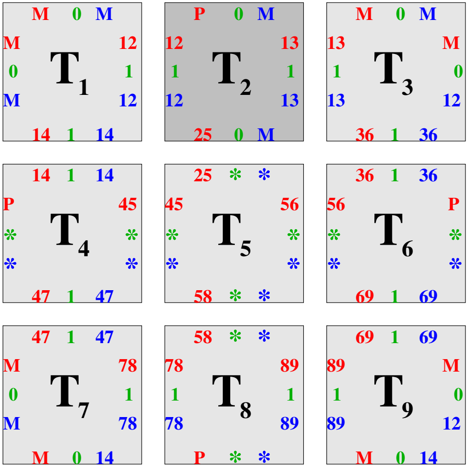
Consider the non-central tiles on the network: T_2, T_4, T_6 and T_8.

Step 4



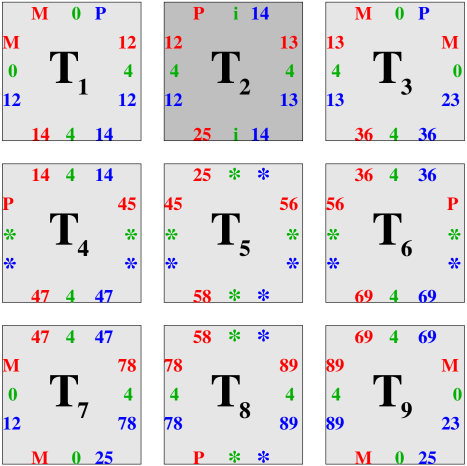
Consider, *e.g.*, T_2. Parent/neighbor pair on the North/South facets: any pair allowed on the North facet of the parent-tile.

Step 4



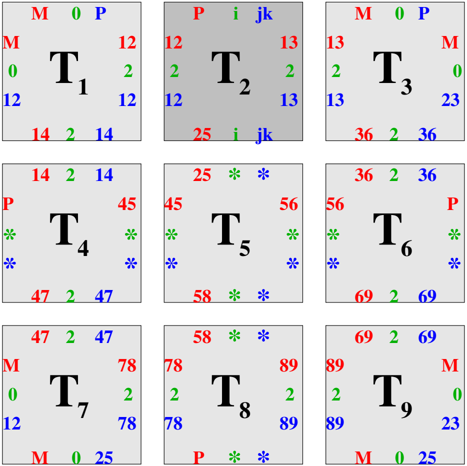
Parent-tiles T_1 or T_3: only one decorated tile for each.

Step 4



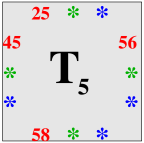
Parent-tiles T_4, T_6, T_7 or T_9: 9 decorated tiles each.

Step 4



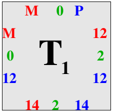
Parent-tiles T_2 or T_8: no restriction $\rightsquigarrow$ 80 decorated tiles each.

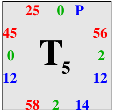
Step 5



Parent/neighbor pairs of the central tile T_5: those of any non-central tile $\rightsquigarrow$ *exactly* twice more decorated tiles (*i.e.*, 1656 at all).

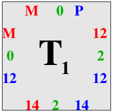
Self-simulation

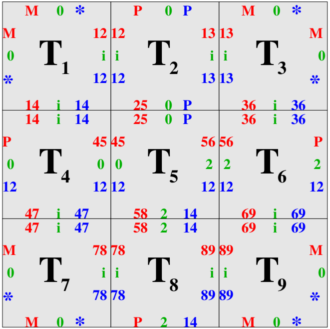




Consider a central tile (right) deriving from, say, a decorated T_1.

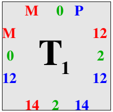
Self-simulation

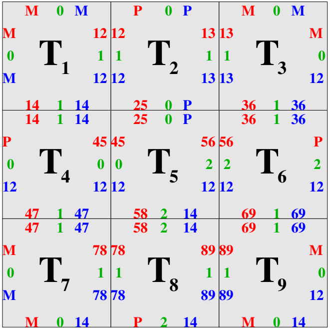




This forces all the decorations of the corresponding macro-tile, with neighbor-indices on the network forcing a parent-tile T_1 or T_5.

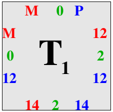
Self-simulation

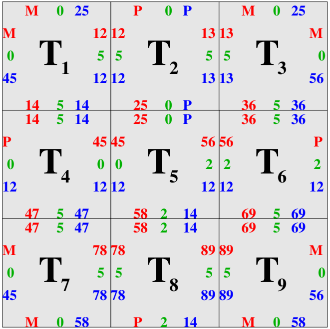




If the parent-tile is T_1, then the macro-tile is combinatorially equivalent to the tile from which derive the pairs.

Self-simulation





Otherwise, it is combinatorially equivalent to its own central-tile.

Some references for this lecture:

- Shahar Mozes, *Tilings, substitution systems and dynamical systems generated by them*, J. Anal. Math. **53** (1989).
- Chaim Goodman-Strauss, *Matching rules and substitution tilings*, Ann. Math. **147** (1998).
- Thomas Fernique, Nicolas Ollinger, *Combinatorial substitutions and sofic tilings*, in proc. JAC'10 (2010).

These slides and the above references can be found there:

`http://www.lif.univ-mrs.fr/~fernique/qc/`