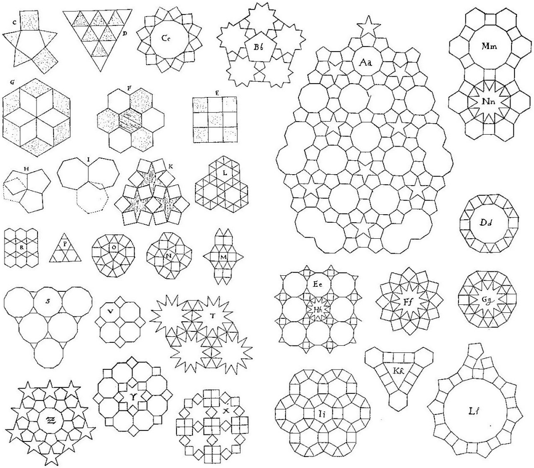
Penrose Tilings

Thomas Fernique

Moscow, Spring 2011

The trouble with Kepler tilings



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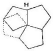
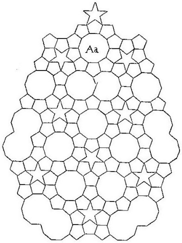
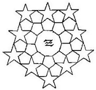
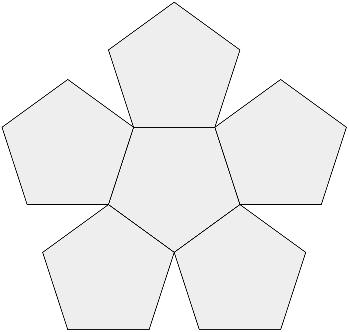




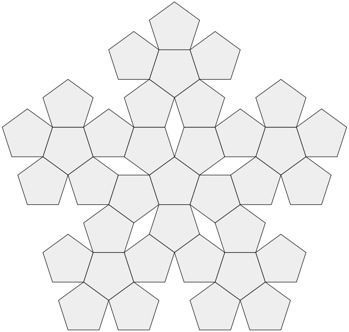


Tilings by pentagons, diamonds, boats and stars



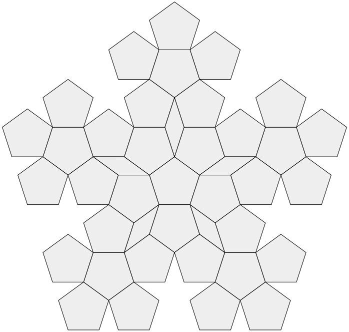
Regular pentagons *almost* tile a bigger pentagon.

Tilings by pentagons, diamonds, boats and stars



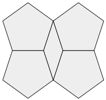
Each pentagon can in turn be tiled by smaller pentagons.

Tilings by pentagons, diamonds, boats and stars



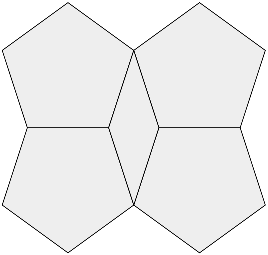
Holes can be filled by *diamonds*.

Tilings by pentagons, diamonds, boats and stars



Consider such a diamond with its neighborhood.

Tilings by pentagons, diamonds, boats and stars



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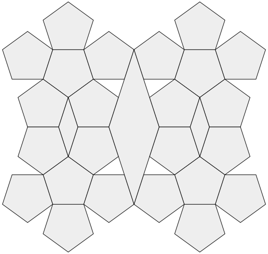
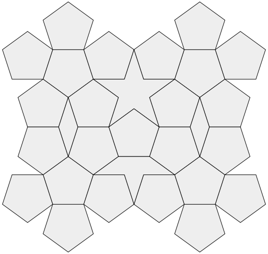


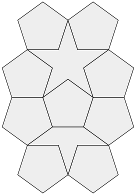
Tile pentagons by smaller pentagons and fill diamond holes.

Tilings by pentagons, diamonds, boats and stars



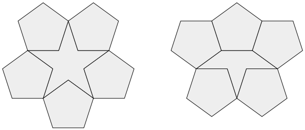
The rest can be tiled with a star, a boat and a pentagon.

Tilings by pentagons, diamonds, boats and stars



Consider the star and the boat, with their neighborhood.

Tilings by pentagons, diamonds, boats and stars



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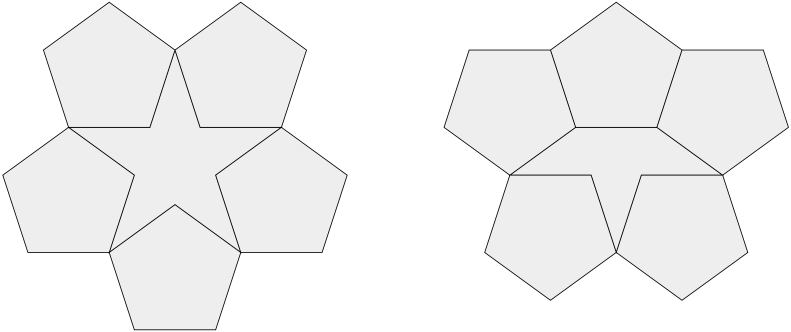

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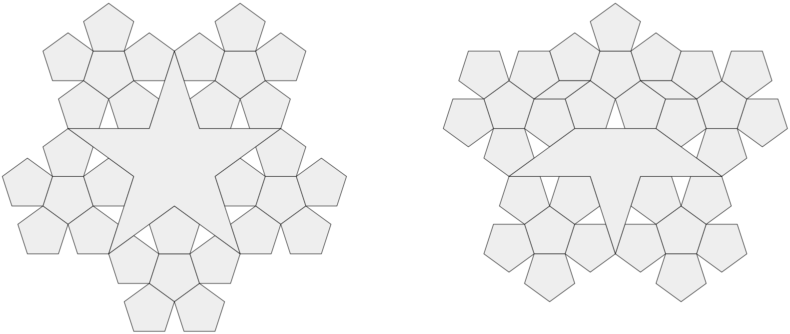
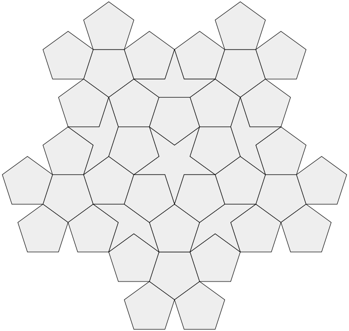
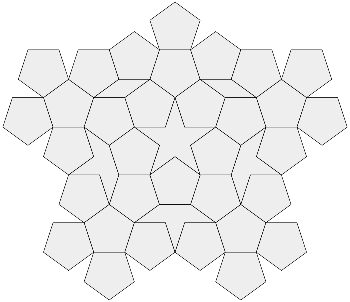


Tile pentagons by smaller pentagons and fill diamond holes.

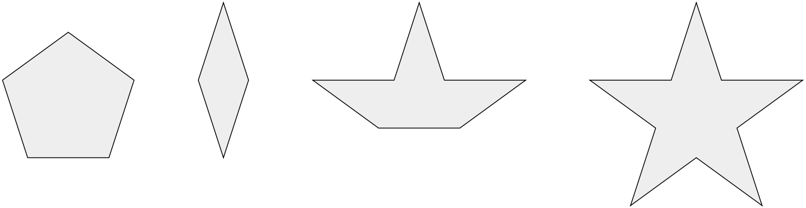
Tilings by pentagons, diamonds, boats and stars





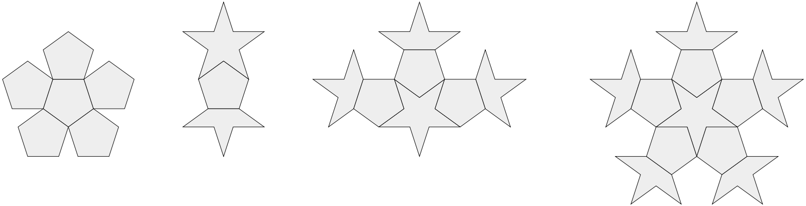
The rest can be tiled with the same tiles (and neighborhood)

Tilings by pentagons, diamonds, boats and stars



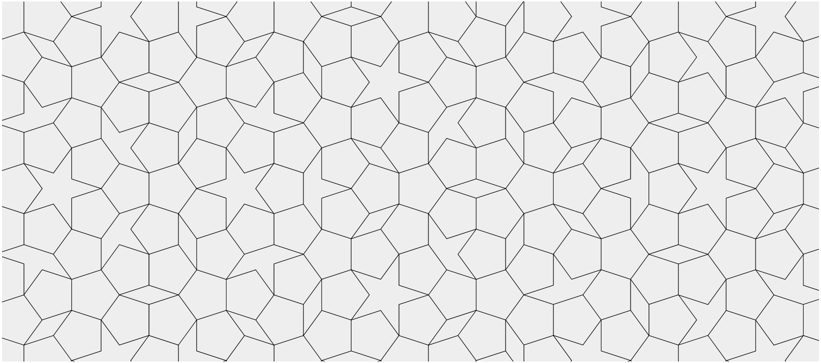
$\rightsquigarrow$ tiling of the plane by pentagons, diamonds, boats and stars.

Tilings by pentagons, diamonds, boats and stars



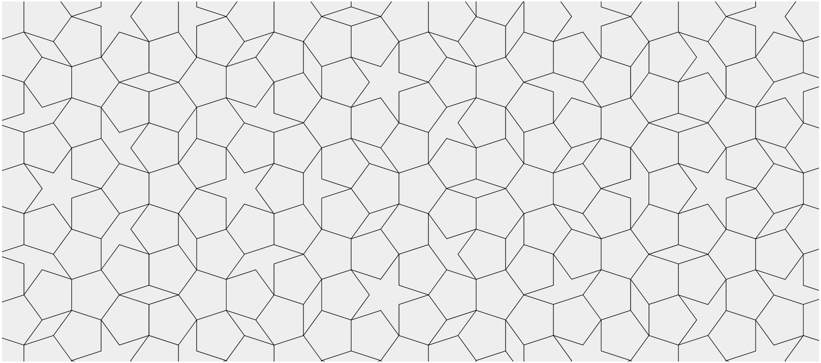
Method: inflate, divide and fill diamond-shaped holes – *ad infinitum*.

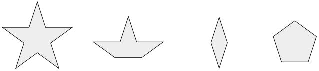
Tilings by pentagons, diamonds, boats and stars



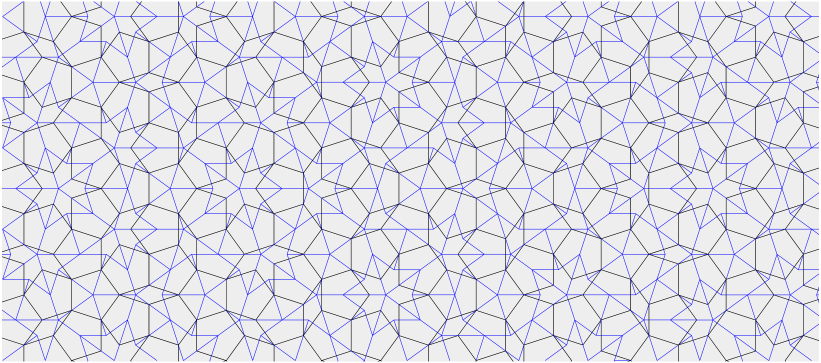
This yields a *hierarchical* tiling of the plane (hence non-periodic).

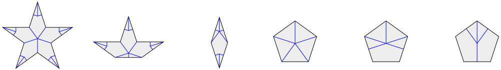
Tilings by kites and darts





Tilings by kites and darts





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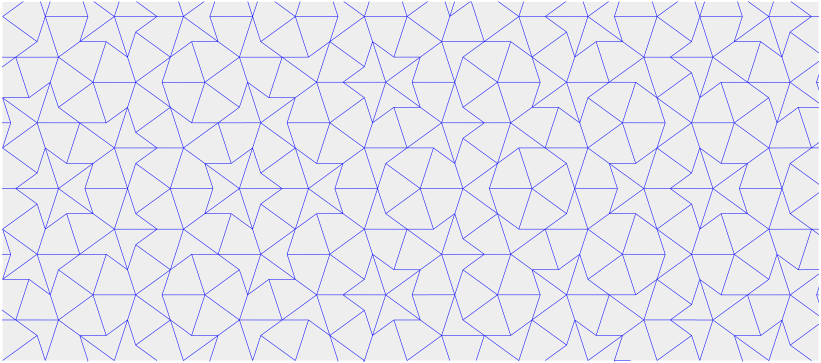

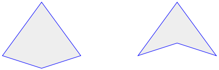

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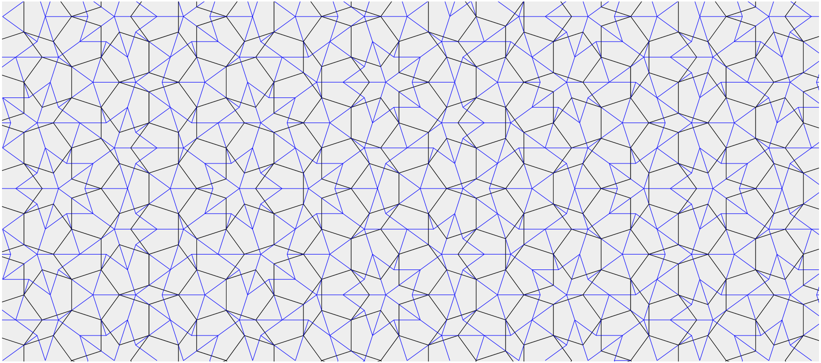

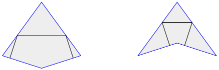

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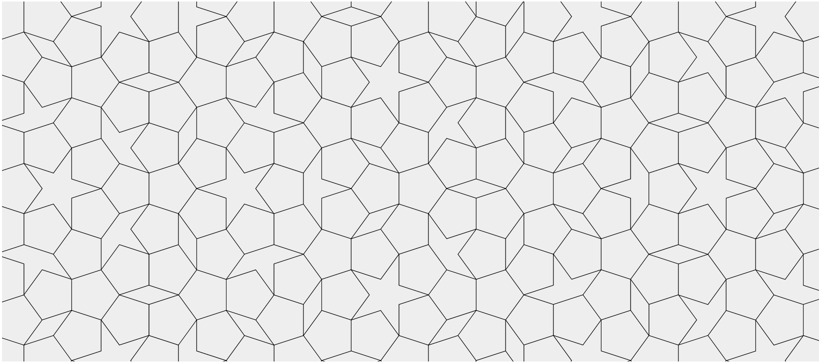
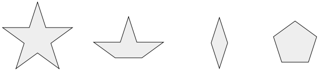




Mutual local derivability

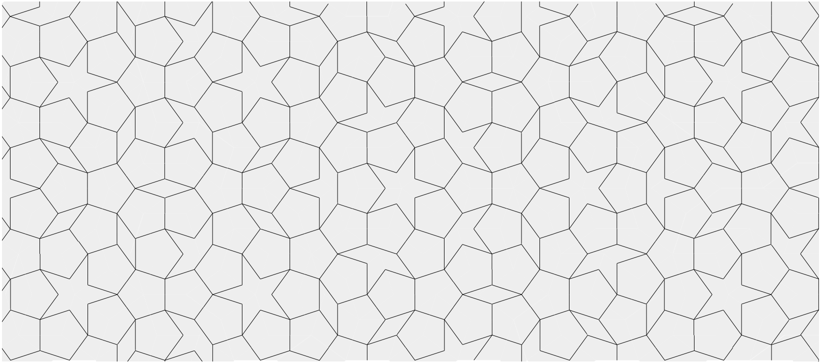
Equivalence relation on tilings:

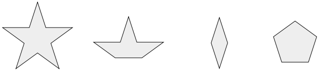
Definition (MLD tilings)

Two tilings are said to be *mutually locally derivable* (MLD) if the one can be obtained from the other by a local map, and *vice versa*.

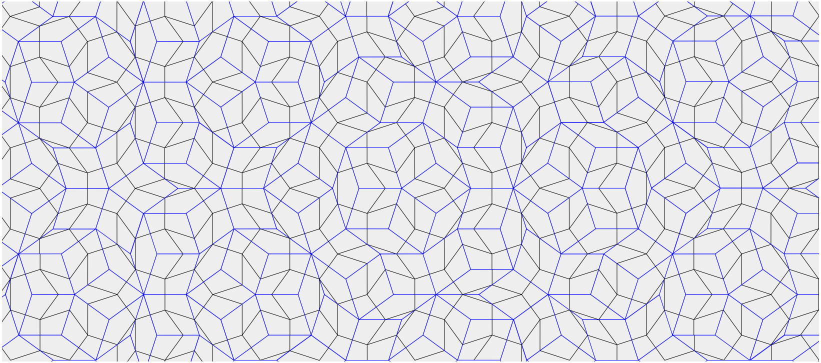
Example: the two previous tilings are MLD.

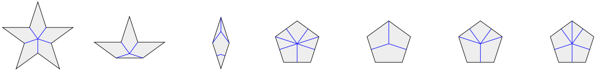
Tilings by thin and fat rhombi





Tilings by thin and fat rhombi





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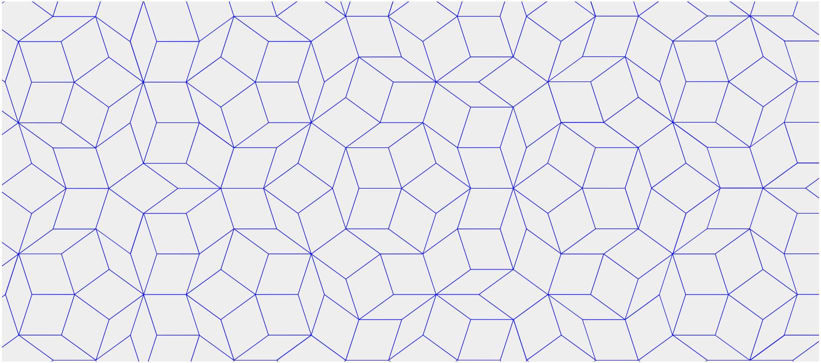

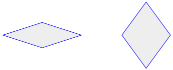

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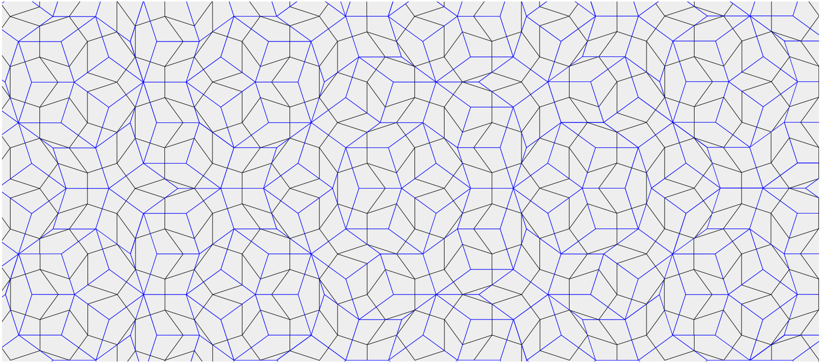

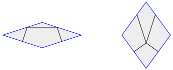

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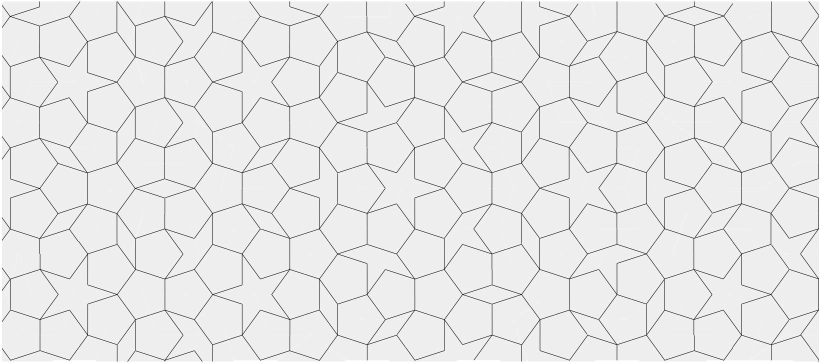
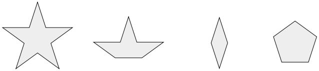
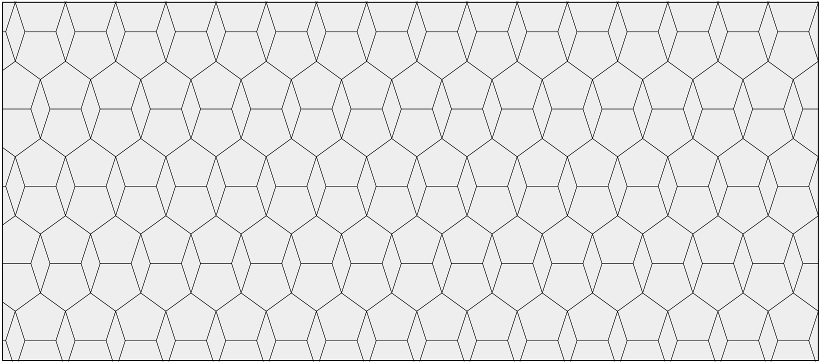




The trouble with Penrose tilings



The previous tilesets also admit periodic tilings.

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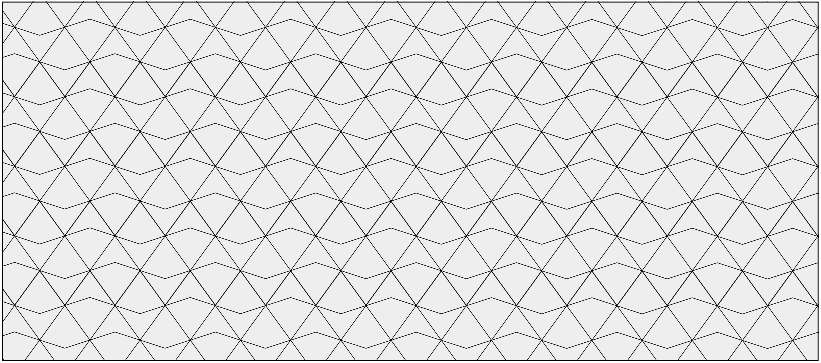

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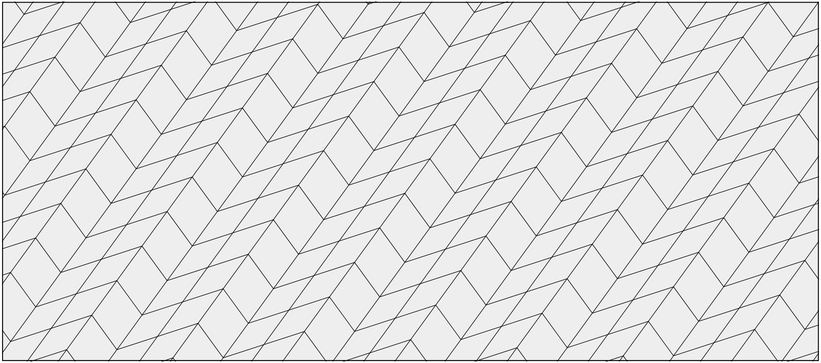
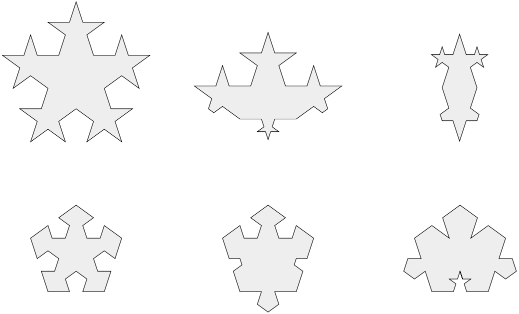
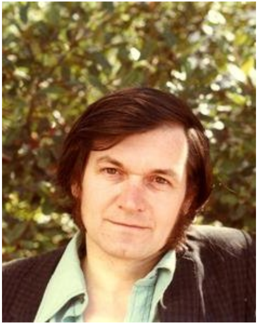


The previous tilesets also admit periodic tilings.

Three aperiodic tilesets (Penrose, 1974)

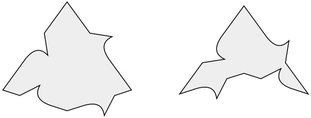


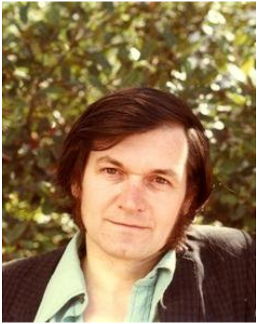


Penrose's trick: notch edges to enforce the hierarchical structure.

(tiles up to rotation)

Three aperiodic tilesets (Penrose, 1974)





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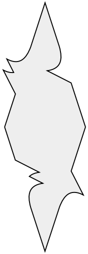
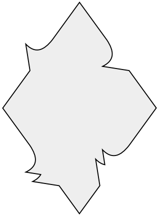
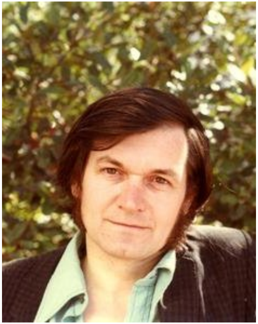






Penrose's trick: notch edges to enforce the hierarchical structure.

(tiles up to rotation)

Penrose tilings (formal)

Definition (Equivalence)

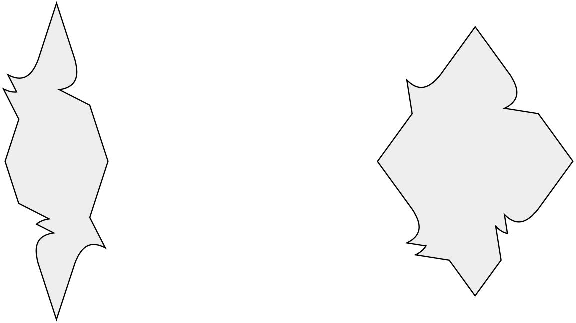
Two tilesets are *equivalent* if any tiling by one tileset is mutually locally derivable from a tiling by the other tileset.

Exercise: prove the equivalence of Penrose tilesets.

Definition (Penrose tilings)

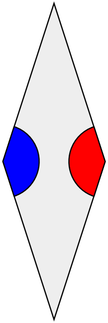
A *Penrose tiling* is a tiling mutually locally derivable from a tiling of the whole plane by one of the three previous Penrose tilesets.

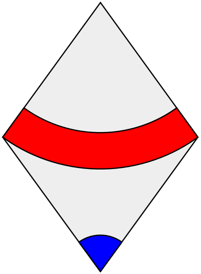
Robinson triangles



Notched rhombi are equivalent to colored rhombi.

Robinson triangles





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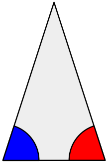
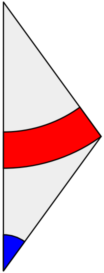
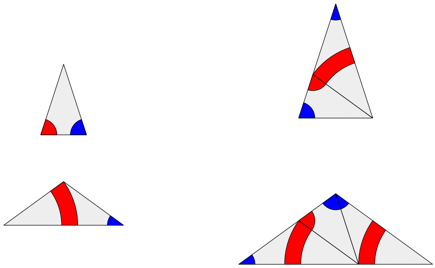




Colored rhombi are equivalent to *Robinson triangles*.

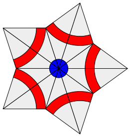
(tiles up to rotations and reflections)

Robinson macro-triangles



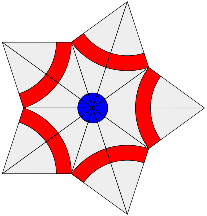
Robinson macro-triangle: homothetic unions of Robinson triangles.
Up to this homothety, triangles and macro-triangles are equivalent.

Inflate and divide



Pattern + inflate/divide *ad infinitum* $\rightsquigarrow$ tiling of the plane (Kőnig).

Inflate and divide



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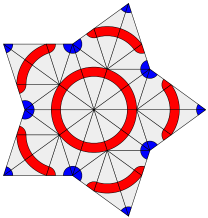

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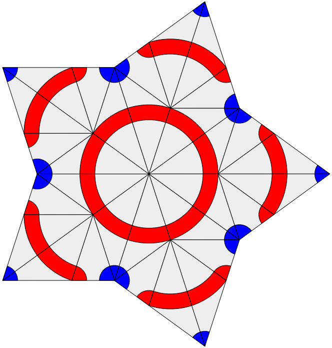

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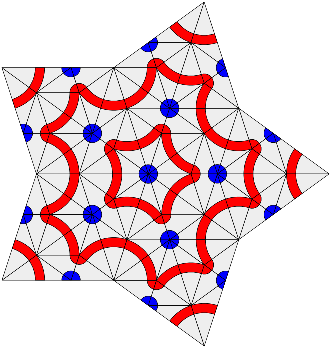

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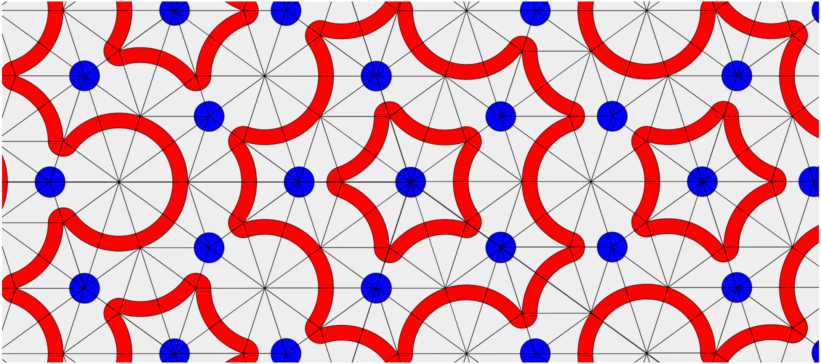
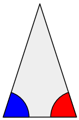


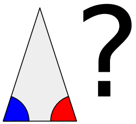
Pattern + inflate/divide *ad infinitum* $\rightsquigarrow$ tiling of the plane (Kőnig).

Group and deflate



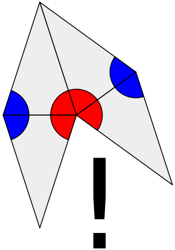
Conversely, fix a tiling of the plane. Consider a thin triangle (if any).

Group and deflate



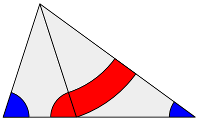
What can be its *red neighbor*?

Group and deflate



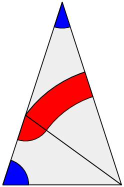
A thin triangle would yield an uncompletable vertex.

Group and deflate



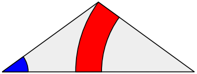
This is thus a fat triangle.

Group and deflate



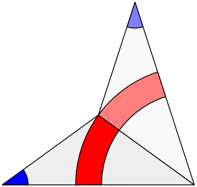
We can group both to form a thin macro-triangle.

Group and deflate



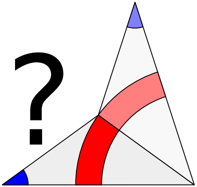
Consider a remaining fat triangle (if any).

Group and deflate



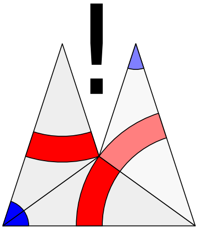
Its red neighbor is fat (otherwise it would be already grouped).

Group and deflate



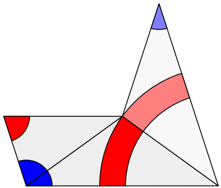
What can be its *blue neighbor*?

Group and deflate



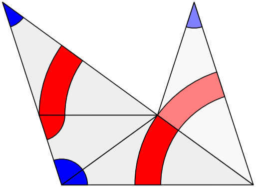
A fat triangle would yield an uncompletable vertex.

Group and deflate



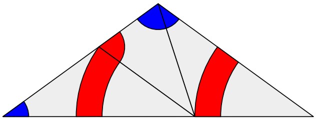
This is thus a thin triangle .

Group and deflate



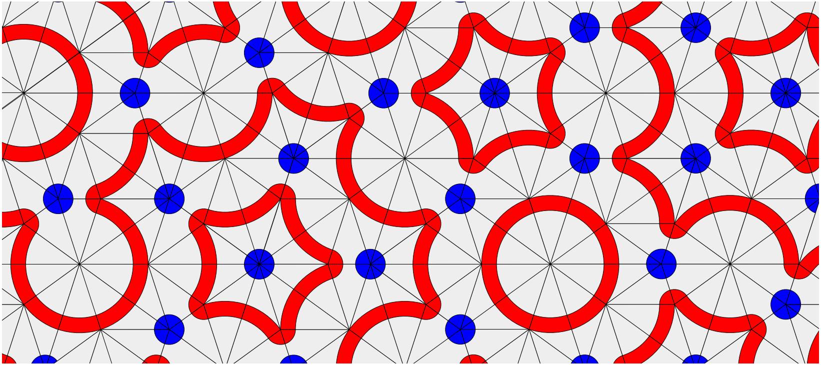
This is thus a thin triangle grouped into a thin macro-triangle.

Group and deflate



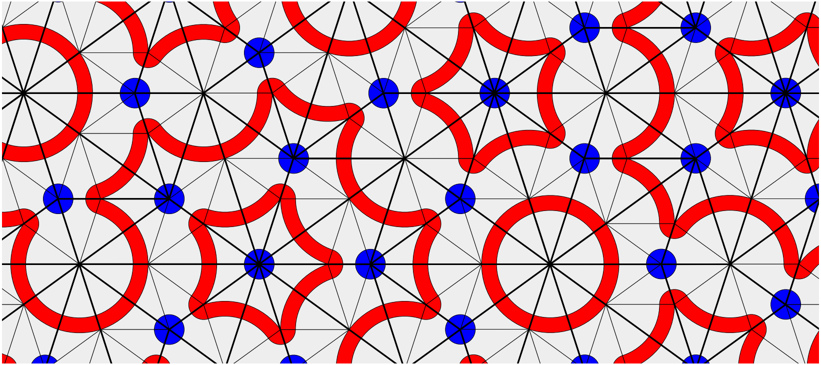
We can group the three triangles to form a fat macro-triangle.

Group and deflate



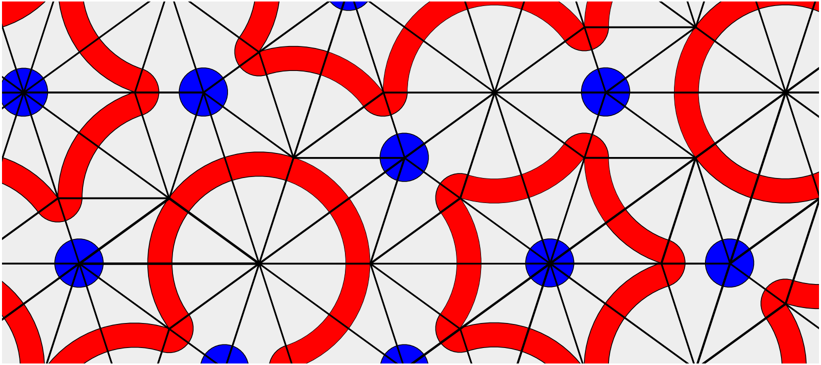
Hence, any tiling by Robinson triangles. . .

Group and deflate



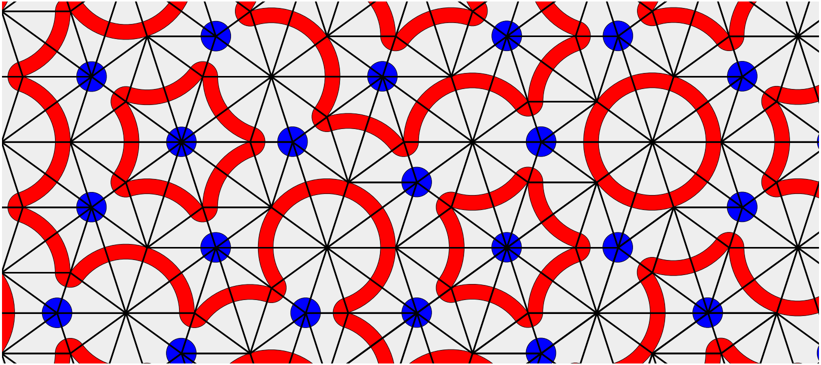
. . . can be uniquely seen as a tiling by Robinson macro-triangles.

Group and deflate



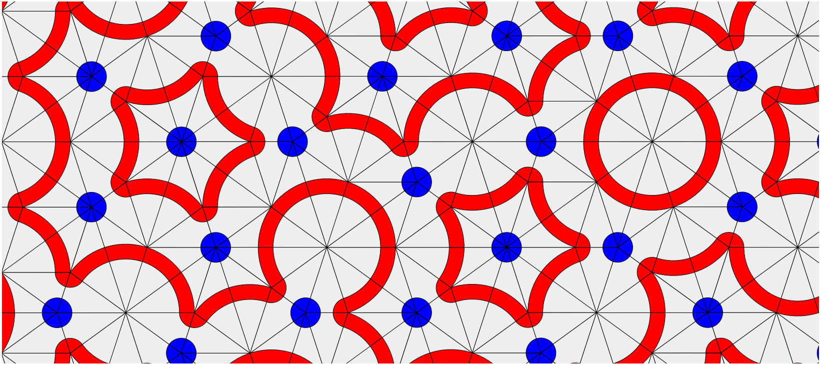
Macro-triangles can be consistently replaced by triangles. . .

Group and deflate



. . . and by deflating we get a new Penrose tiling.

Group and deflate



Group/deflate *ad infinitum* $\rightsquigarrow$ aperiodicity of Robinson triangles.

Uncountability

Proposition

Penrose tilesets admit uncountably many tilings of the plane.

Proof:

- track a tile in the tiling hierarchy $\rightsquigarrow$ infinite tile sequence;
- tiles of isometric tilings $\rightsquigarrow$ sequences with the same tail ends;
- infinite tile sequence $\rightsquigarrow$ tiling;
- countably many sequences with the same tail end;
- uncountably many infinite tile sequences.

Quasiperiodicity

Pattern of size r (in a tiling): tiles lying in a closed ball of radius r.

Definition (Quasiperiodic tiling)

A tiling is said to be *quasiperiodic* if, for any $r > 0$, there is $R > 0$ such that any pattern of size r appears in any pattern of size R.

Quasiperiodic $\equiv$ bounded gap $\equiv$ repetitive $\equiv$ uniformly recurrent.

Any periodic tiling is also quasiperiodic (take $R = r + \mathrm{period}$).

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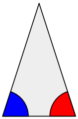
Any periodic tiling is also quasiperiodic (take $R = r + \mathrm{period}$).

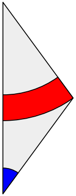
Proposition

Penrose tilings are quasiperiodic.

Proof: first check for $r = \mathrm{Diam}(\mathrm{tile})$, then group/deflate.

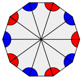
Rotational symmetry

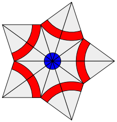


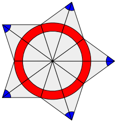


Tile angles multiple of $\frac{\pi}{5}$ $\rightsquigarrow$ only 5- or 10-fold symmetries (or 2-fold).

Rotational symmetry

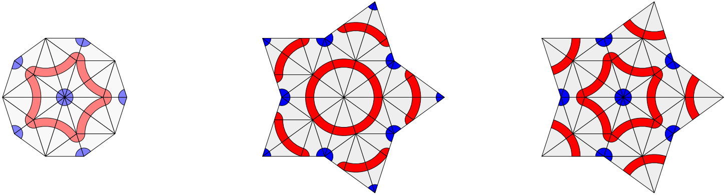






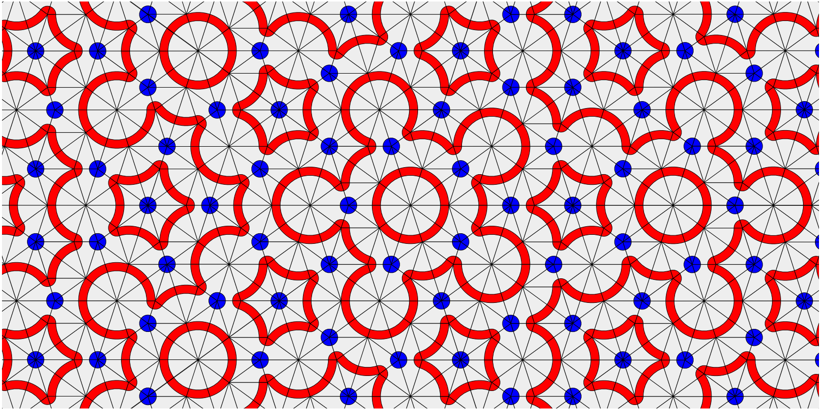
Up to isometry, three 5-fold elementary patterns.

Rotational symmetry



Inflate/divide $\rightsquigarrow$ two 5-fold Penrose tilings of the plane.

Rotational symmetry



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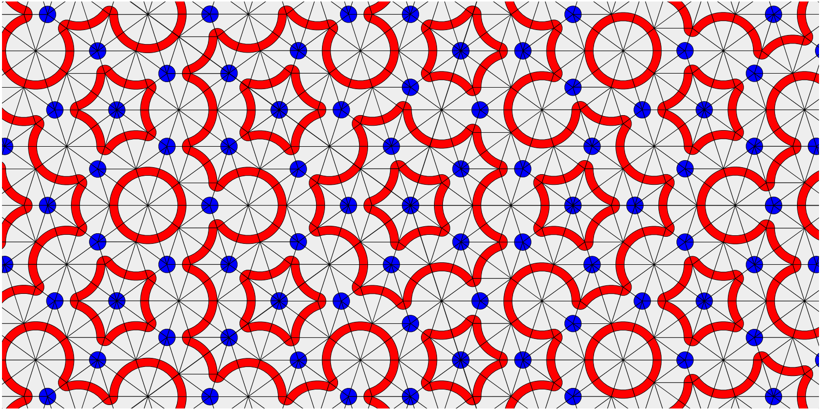

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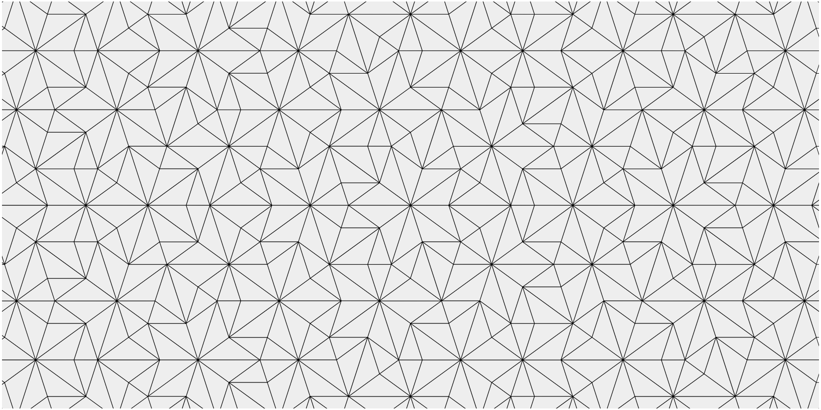
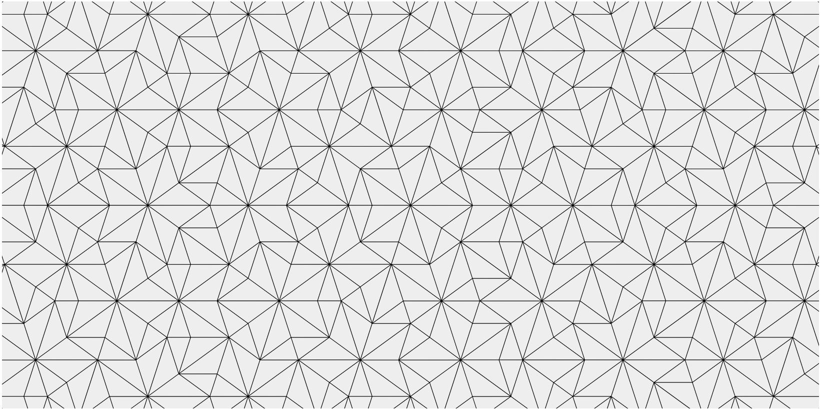


They are different even up to decorations.

Rotational symmetry



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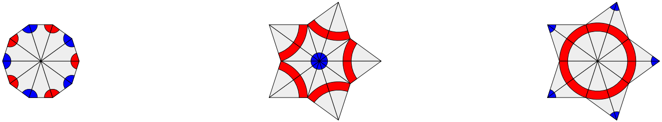
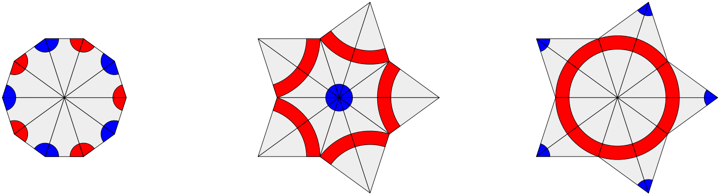


Conversely, a symmetry center must live in the whole hierarchy.

Rotational symmetry



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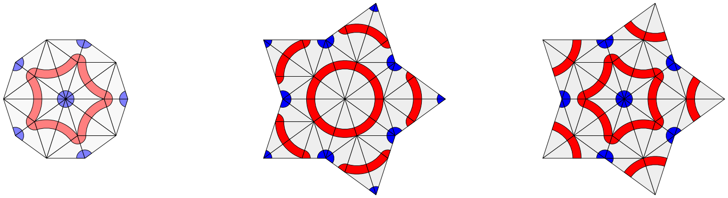

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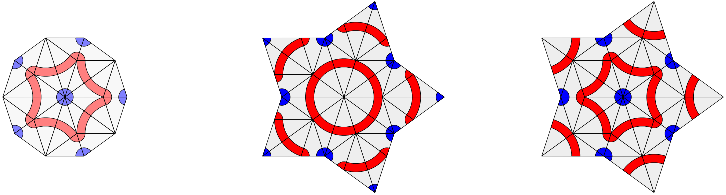
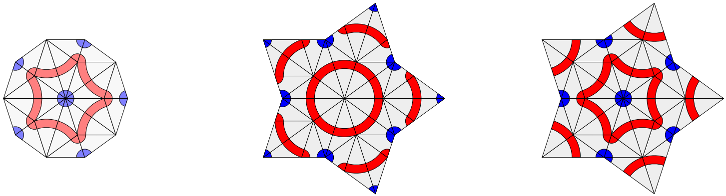


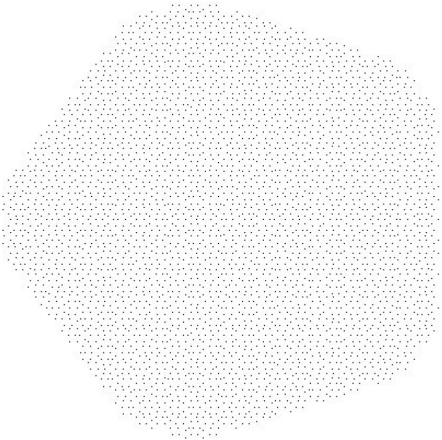
There is thus only two 5-fold Penrose tilings of the plane.

Rotational symmetry



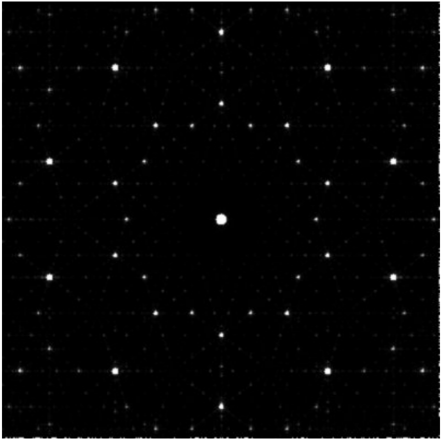
The uncountably many others have only *local* 5-fold symmetry.

Remind quasicrystals



Point-holes at vertices of a Penrose tiling $\rightsquigarrow$ 5-fold diffractogram.

Remind quasicrystals



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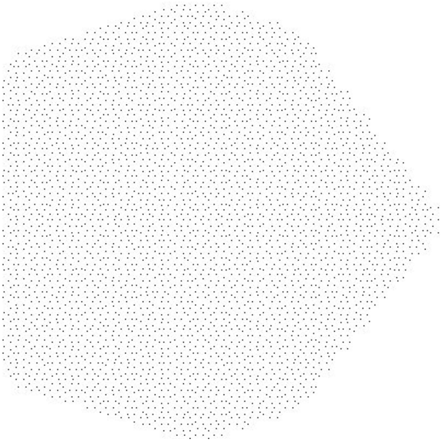

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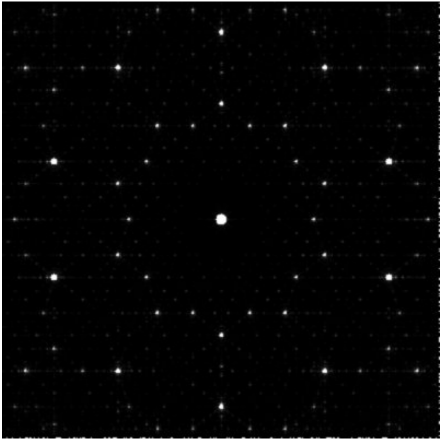
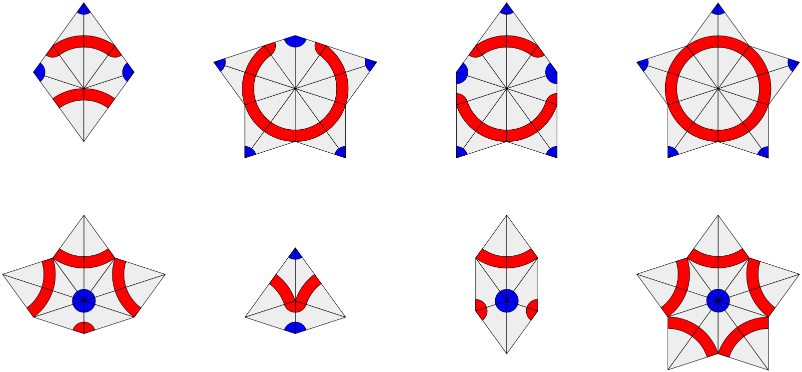


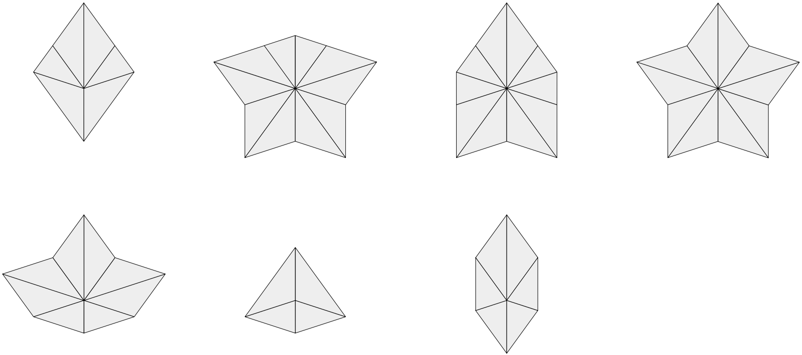
Point-holes at vertices of a Penrose tiling $\rightsquigarrow$ 5-fold diffractogram.

Vertex atlas



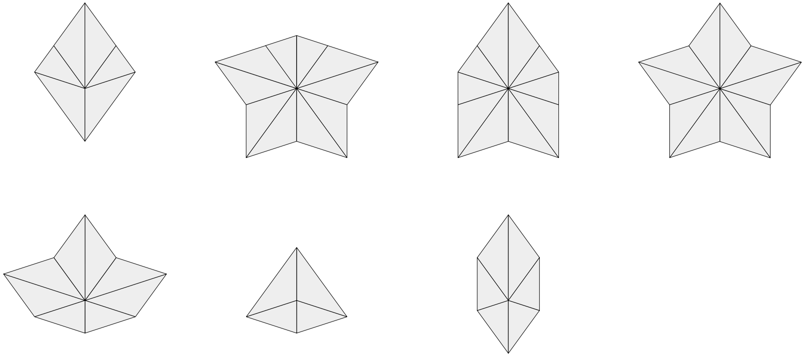
In Penrose tilings, Robinson triangles fit in 8 ways around a vertex.

Vertex atlas



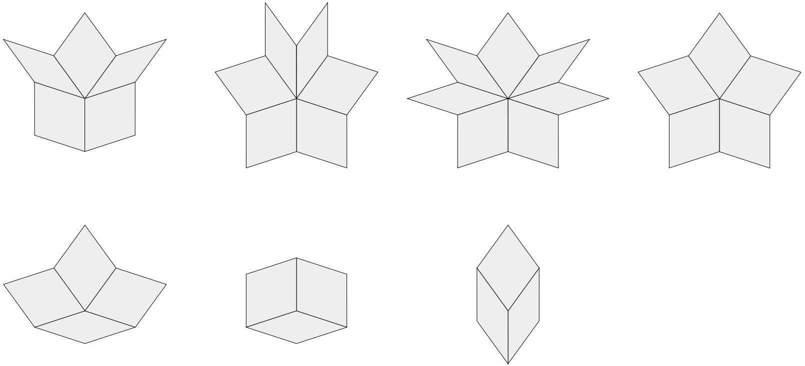
Up to decorations, this yields a *vertex atlas* of size 7.

Vertex atlas



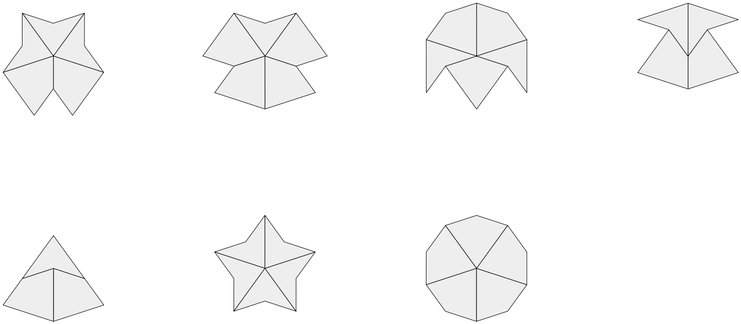
Proposition: any tiling with this vertex atlas is a Penrose tiling.

Vertex atlas



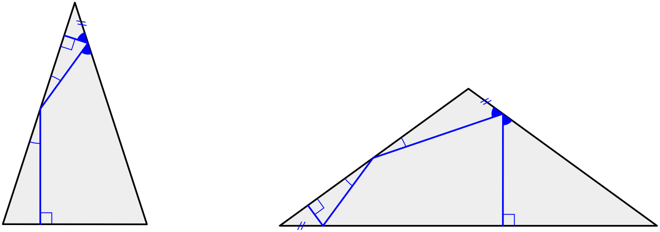
A similar vertex atlas for tilings by thin and fat rhombi.

Vertex atlas



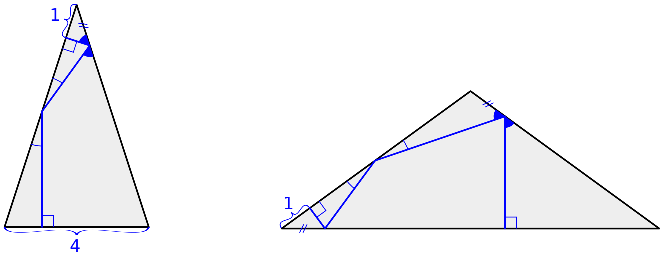
A similar vertex atlas for tilings by kites and darts.

Ammann bars



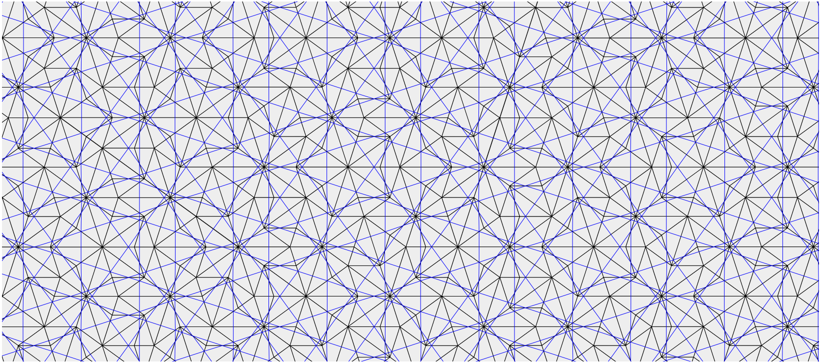
Draw these two particular billiard trajectories in each triangle.

Ammann bars



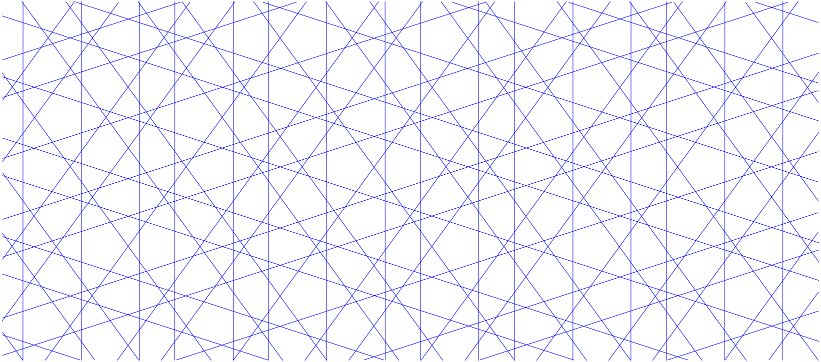
Some trigonometry $\rightsquigarrow$ ratio characterizing the trajectories.

Ammann bars



This draws on Penrose tilings five bundles of parallel lines.

Ammann bars



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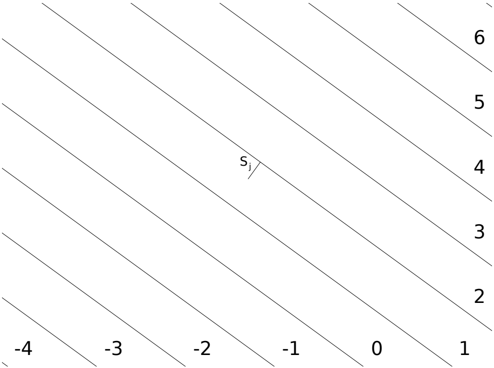
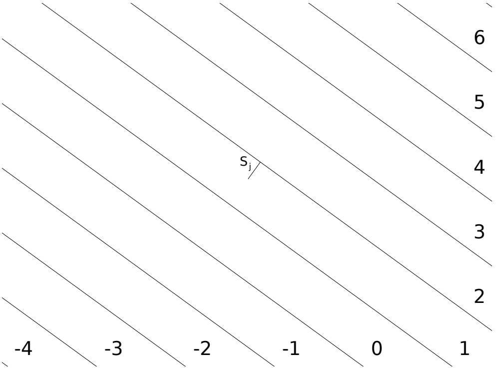
Pentagrid (De Bruijn, 1981)



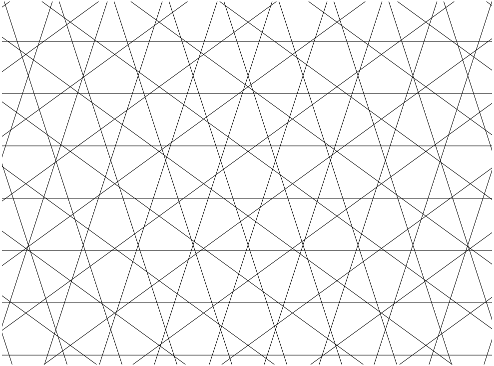
Shift $s_j \in \mathbb{R} \rightsquigarrow$ grid $\{z \in \mathbb{C} \mid \mathrm{Re}(z\zeta^{-j}) + s_j = 0\}$, where $\zeta = e^{2i\pi/5}$.

Pentagrid (De Bruijn, 1981)



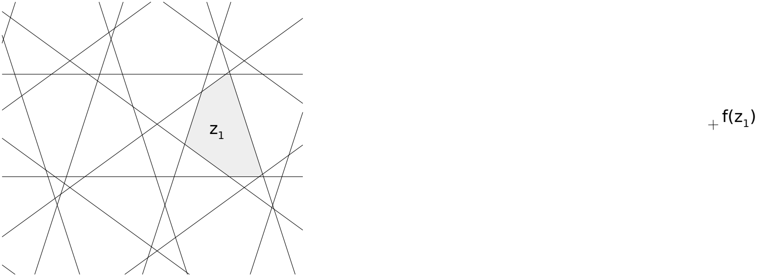
Strip numbering: $K_j(z) = \lceil \mathrm{Re}(z\zeta^{-j}) + s_j \rceil$.

Pentagrid (De Bruijn, 1981)



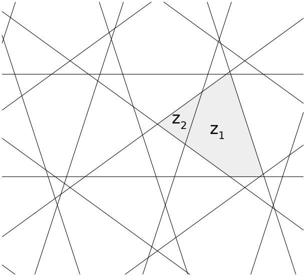
Pentagrid: grids $0, \ldots, 4$, with $s_0 + \ldots + s_4 \in \mathbb{Z}$ and no triple point.

From pentagrids to rhombus tilings



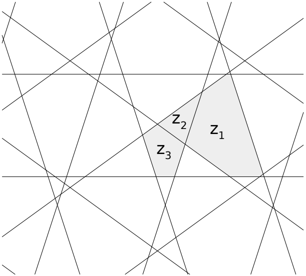
$f(z) := \sum_{0 \leq j \leq 4} K_j(z)\zeta^j$ maps each mesh onto a rhombus vertex.

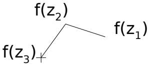
From pentagrids to rhombus tilings



$f(z) := \sum_{0 \leq j \leq 4} K_j(z)\zeta^j$ maps each mesh onto a rhombus vertex.

From pentagrids to rhombus tilings





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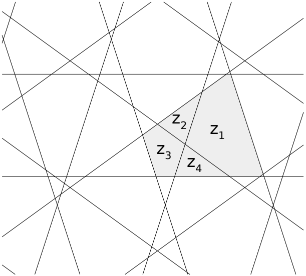

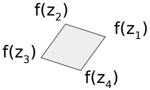

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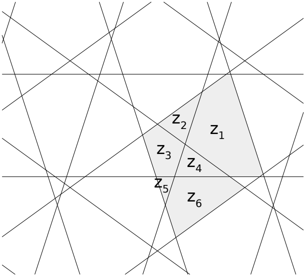
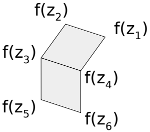
From pentagrids to rhombus tilings





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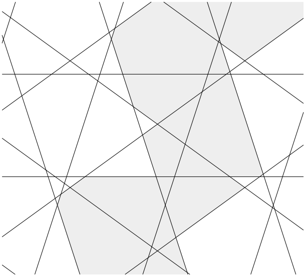

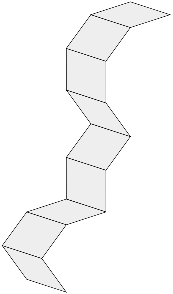

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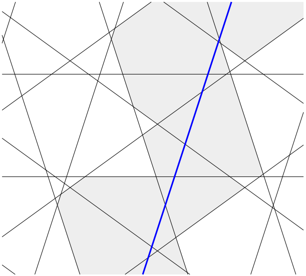
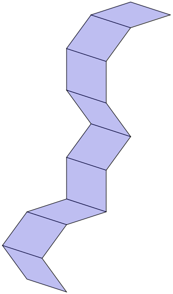
From pentagrids to rhombus tilings





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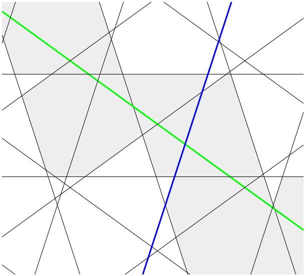
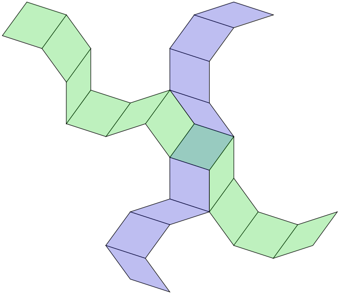
From pentagrids to rhombus tilings





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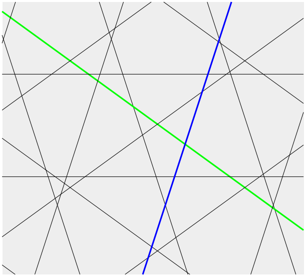
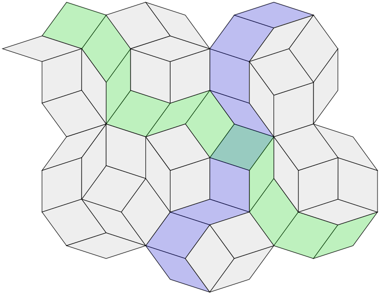
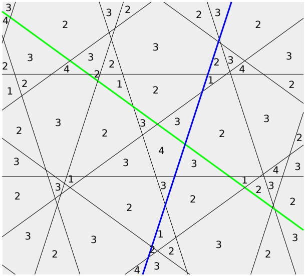
From pentagrids to rhombus tilings

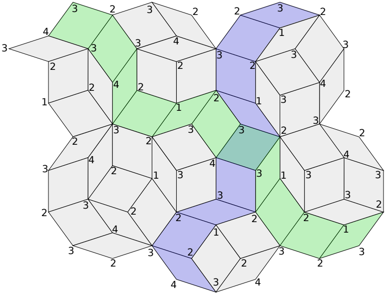




$$f(z) := \sum_{0 \leq j \leq 4} K_j(z)\zeta^j$$ maps each mesh onto a rhombus vertex.

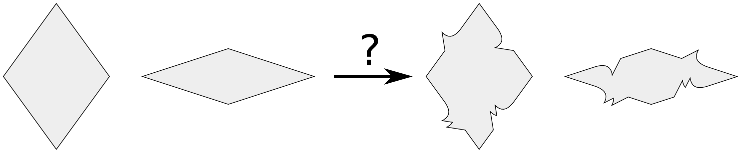
Indices





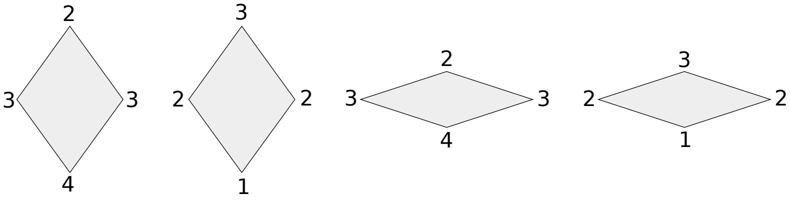
$$i(z) := \sum_{0 \leq j \leq 4} K_j(z) \text{ maps each mesh into } \{1, 2, 3, 4\}. \qquad \text{(check)}$$

From rhombus tilings to Penrose tilings



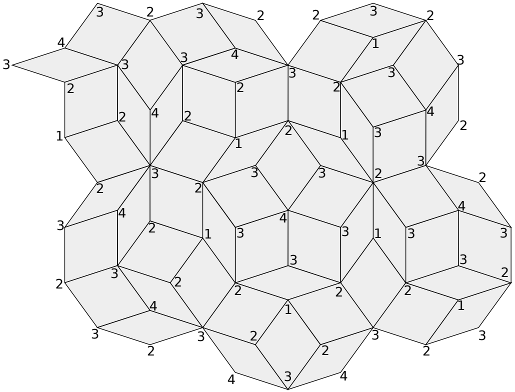
Can the rhombi of such tilings be consistently notched?

From rhombus tilings to Penrose tilings



Lemma: each rhombus can be endowed by indices in only two ways.

From rhombus tilings to Penrose tilings



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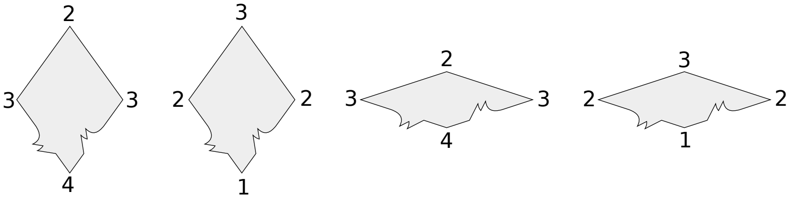
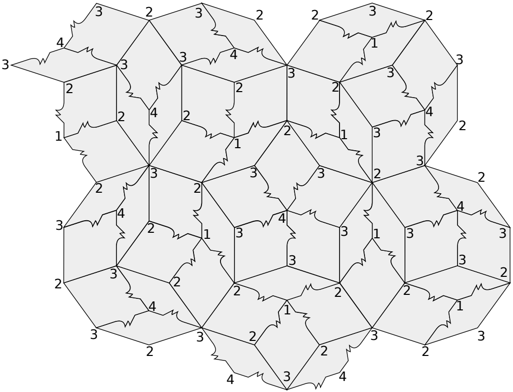
From rhombus tilings to Penrose tilings



Double-arrow notchings from 3 to 4 and from 1 to 2: consistent.

From rhombus tilings to Penrose tilings



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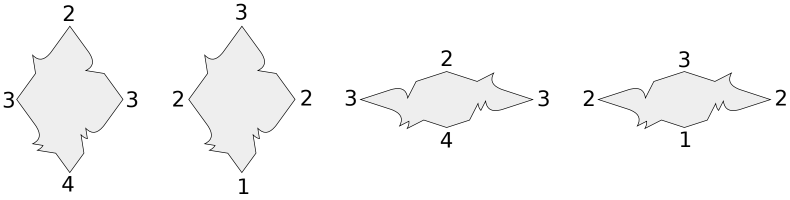
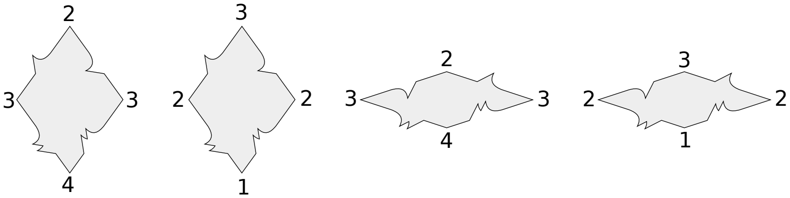
From rhombus tilings to Penrose tilings



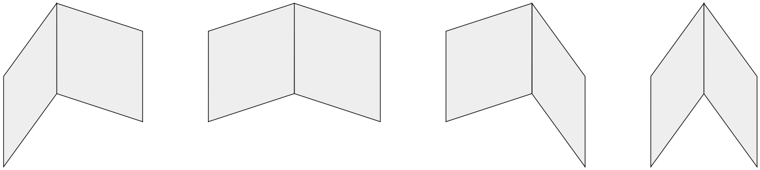
Notchings of remaining edges are forced. Is it consistent?

From rhombus tilings to Penrose tilings



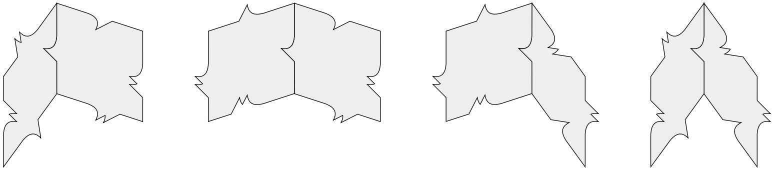
Remark: single-arrow notchings go from acute to obtuse angles.

From rhombus tilings to Penrose tilings



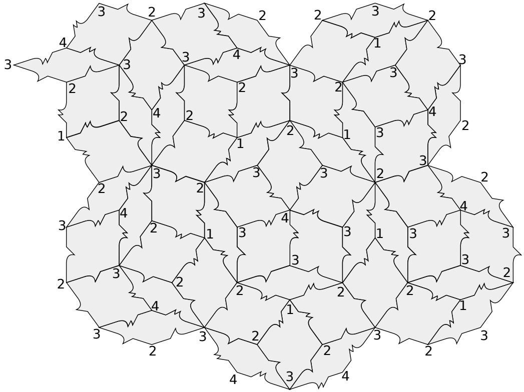
Lemma: along an edge between 2 and 3, acute/obtuse angles match.

From rhombus tilings to Penrose tilings



This yields the consistency of the single-arrow notchings.

From rhombus tilings to Penrose tilings



This yields the consistency of the single-arrow notchings.

From Penrose tilings to pentagrids

Let $\phi_{\vec{s}}$ be the rhombus tiling derived from the pentagrid of shift $\vec{s}$.

Lemma: group/deflate $\phi_{\vec{s}}$ yields $\phi_{g(\vec{s})}$, where $g(\vec{s})_j = s_{j-1} + s_{j+1}$.

From Penrose tilings to pentagrids

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Theorem (de Bruijn, 1981)

The Penrose tilings are exactly the tilings derived from pentagrids.

Proof:

- Fix a Penrose tiling $\phi = \phi^{(0)}$;
- inflate/divide n times $\rightsquigarrow$ Penrose tiling $\phi^{(n)}$;
- find $\vec{s}_n$ such that $\phi_{\vec{s}_n}$ and $\phi^{(n)}$ agree on $B(0,1)$;
- group/deflate n times $\rightsquigarrow$ $\phi_{g^n(\vec{s}_n)}$ and $\phi^{(0)}$ agree on $B(0,\tau^n)$;
- $g^n(\vec{s}_n) \in [0,1)^5 \rightsquigarrow$ accumulation point $t \rightsquigarrow \phi = \phi^{(0)} = \phi_t$.

Application

Remind:

- there are uncountably many Penrose tilings;
- exactly two of them have a global five-fold symmetry.

Proof:

- uncountably many shifts $s_0 + \ldots + s_4 \in \mathbb{Z}$;
- $\vec{0}$ center of symmetry iff $s_j = \frac{2}{5}$ or $s_j = -\frac{2}{5}$.

Some references for this lecture:

- Roger Penrose, *Pentaplexity: a class of non-periodic tilings of the plane*, Eureka **39** (1978).
- Nicolaas Govert de Bruijn, *Algebraic theory of Penrose's non-periodic tilings of the plane*, Indag. Math. **43** (1981).
- Marjorie Senechal, *Quasicrystals and Geometry*, Cambridge University Press, 1995. Chap. 6.

These slides and the above references can be found there:

`http://www.lif.univ-mrs.fr/~fernique/qc/`