Cooled Tilings

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Remind previous lectures

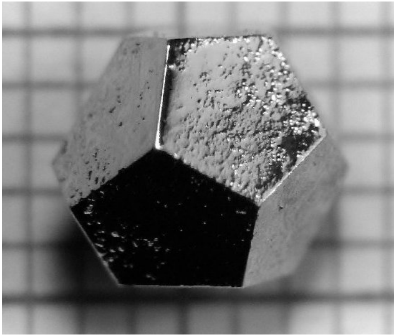
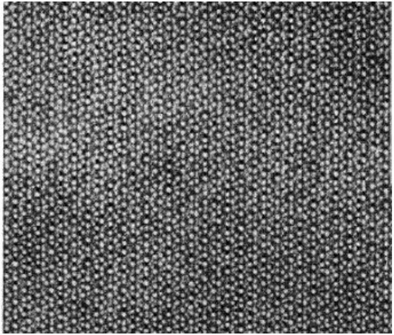
Energy:

Can explain quasiperiodic tiling stability at low temperature.
Mostly fails to explain growth of quasiperiodic tiling.

Entropy:

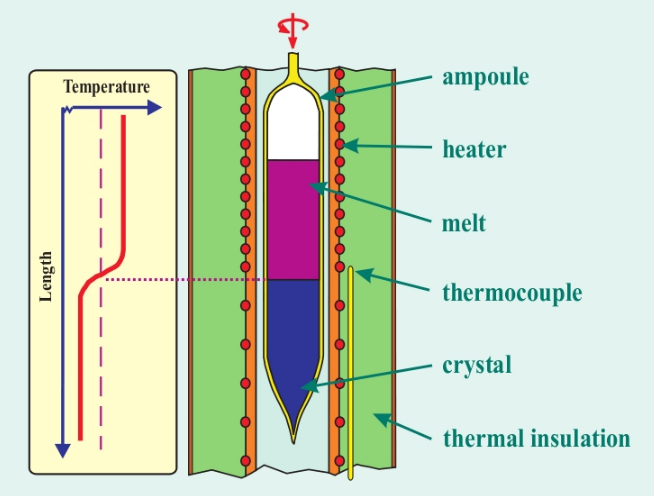
Can explain random tiling stability at high temperature.
Some random tilings may be close to quasiperiodic tilings.

Two problems with random tilings

Real quasicrystals stable at low temperature do exist.
Real quasicrystals which look perfectly quasiperiodic do exist.

The Bridgman-Stockbarger method



A general cooling model (simulated annealing)

Start at high temperature from a maximal entropy random tiling.

Decrease slowly the temperature T up to $T = 0$.

Meanwhile, perform random local transformations step by step.
More precisely: choose at random a location, try to transform:

- with probability 1 if it changes the energy by $\Delta E \leq 0$;
- with probability $\exp(-\Delta E / T)$ otherwise.

Does the process reach a ground state ($E = 0$)? At which rate?

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In what follows, "simplified cooling": allow only $\Delta E \leq 0$.

Formalization

Configuration: word w over $\{1,2\}$ with as many 1 as 2.

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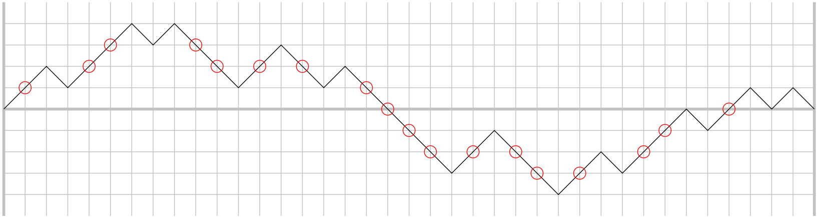

Error: two identical consecutive letters. Counted by $E(w)$

Formalization

Flip: local transformation $12 \leftrightarrow 21$.

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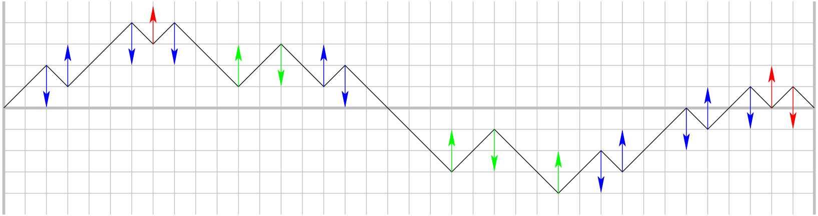
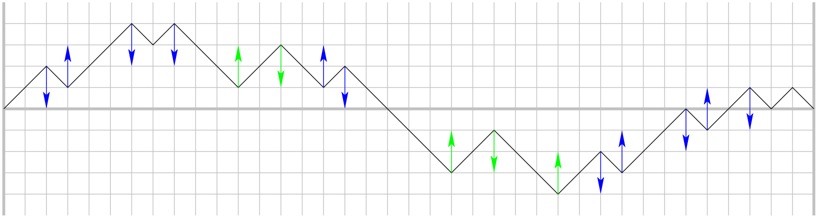


Flips can delete, shift or create errors.

Formalization

Process: $w_t \longrightarrow w_{t+1}$ by a random flip which does not create errors.



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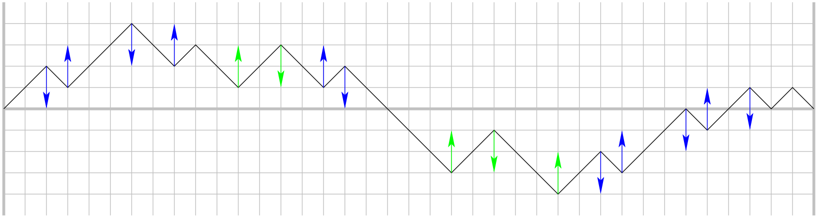

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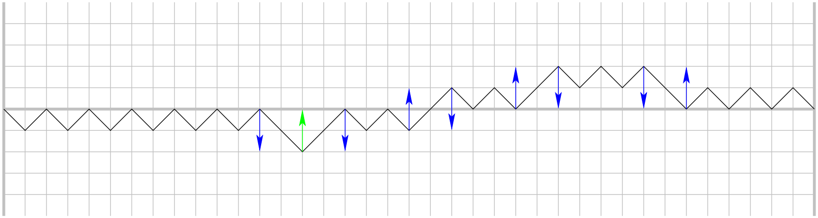

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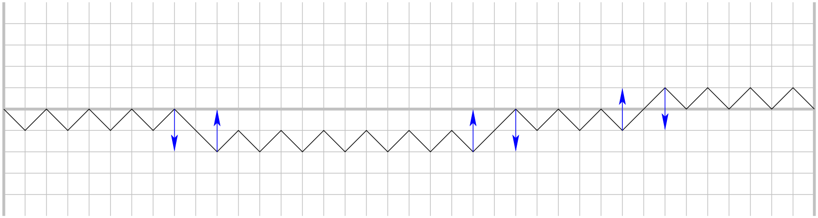

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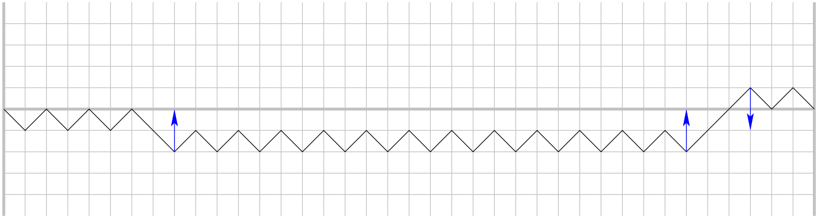

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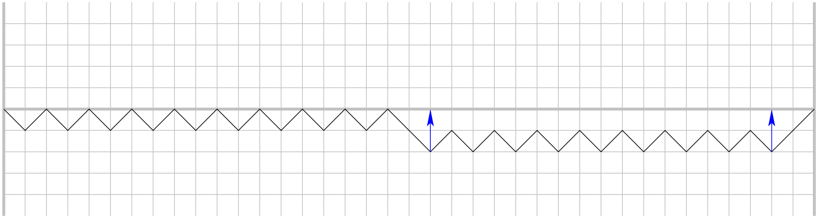

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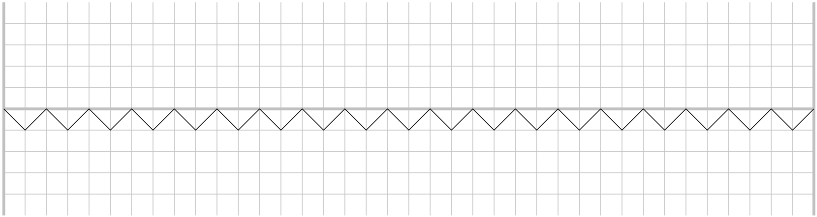

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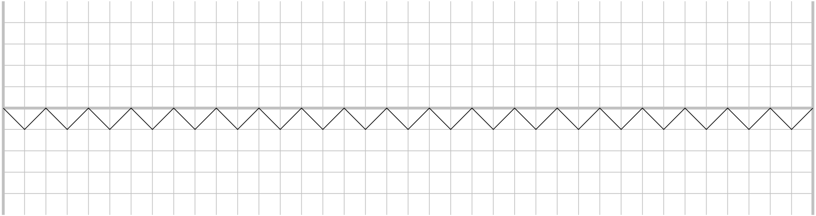


Convergence time: $T(w_0) := \min\{t \geq 0 \mid E(w_t) = 0\}$.

Conjectures and results

Expected number of flips to get an error-free word:

Worst case (initial word 1^n2^n or 2^n1^n):

- $\Theta(n^3)$ conjectured (simulations);
- $O(n^3)$ proven (next slides).

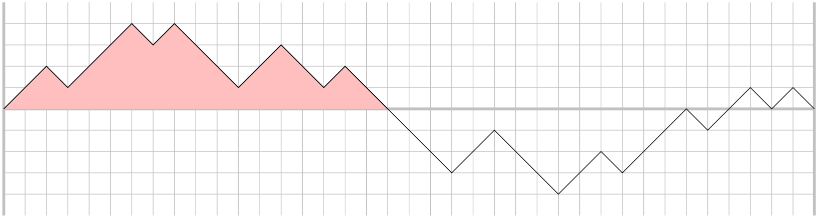
Average case (initial word drawn uniformly at random):

- $\Theta(n^2\sqrt{n})$ conjectured (simulations);
- $O(n^2\sqrt{n}\log n)$ proven.

$\rightsquigarrow$ Polynomial mixing time in any case.

Weighted Dyck factors

We introduce Dyck factor:



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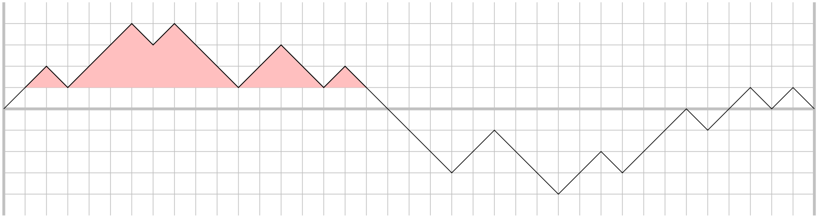

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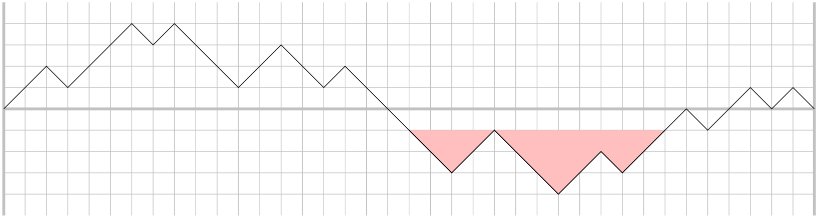

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Definition

Let $0 < \alpha < 1$. Let $DF(w)$ be the Dyck factors of w. One sets:

$$\psi_\alpha(w) := \sum_{v \in DF(w)} (1 + |v|_1)^\alpha.$$

A cubic upper bound

One proves ($n = |w|$):

1. $(1 + \frac{n}{2})^\alpha \le \psi_\alpha(w) \le n^{1+\alpha}$;
2. $\psi_\alpha(w) > (1 + \frac{n}{2})^\alpha \;\Rightarrow\; \mathbb{E}(\Delta\psi_\alpha(w)|w) \le -\frac{\alpha(1-\alpha)}{2} n^{\alpha-2}$;
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One uses that, for a Markov chain $(w_t)_t$ on Ω and $\psi : \Omega \to \mathbb{R}^+$:

Proposition

If $\mathbb{E}\Delta\psi < -\varepsilon$ for $\psi(w) > 0$, then $\mathbb{E}(\min\{t \mid \psi(w_t) = 0\}) \leq \frac{\max\psi}{\varepsilon}$.

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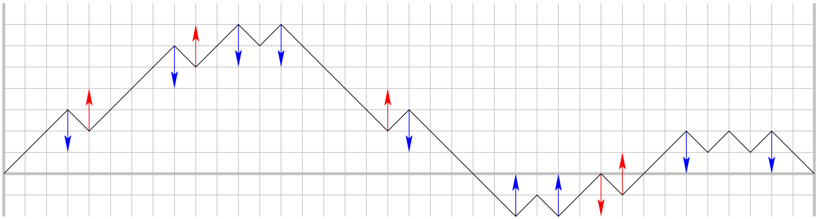
Here, this yields:

Theorem (Bodini-F-Regnault 2010)

The expected convergence time satisfies: $\mathbb{E}(T(w)) \leq \frac{2}{\alpha(1-\alpha)} n^3$.

Proof (sketch)

Main idea ensuring the decrease on expectation (sketch):



A flip can increase (red) or decrease (blue) ψ_α.

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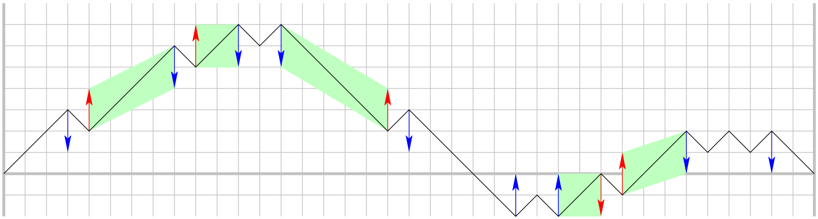

With each red flip is associated a “higher” blue flip.

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Whenever the red flip increases ψ_α by $(p+1)^\alpha - p^\alpha \ldots$

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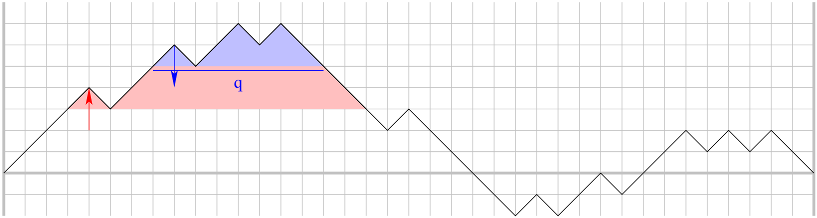

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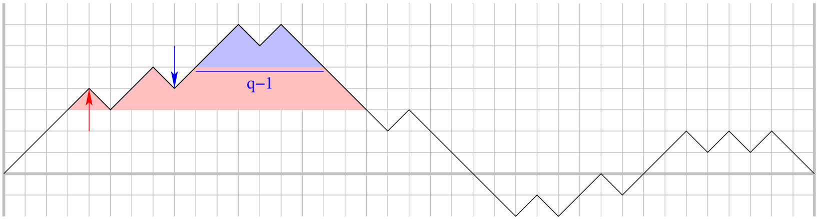

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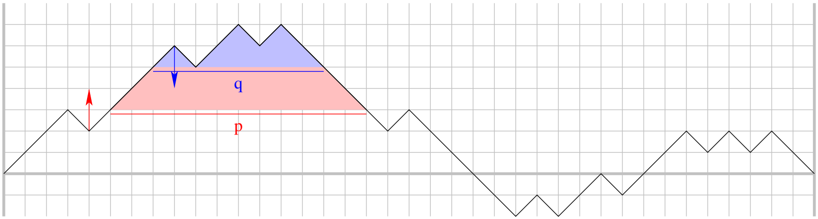
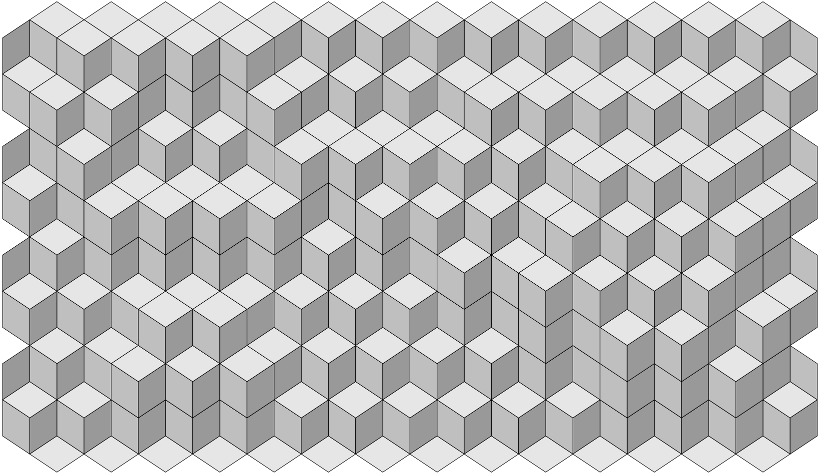


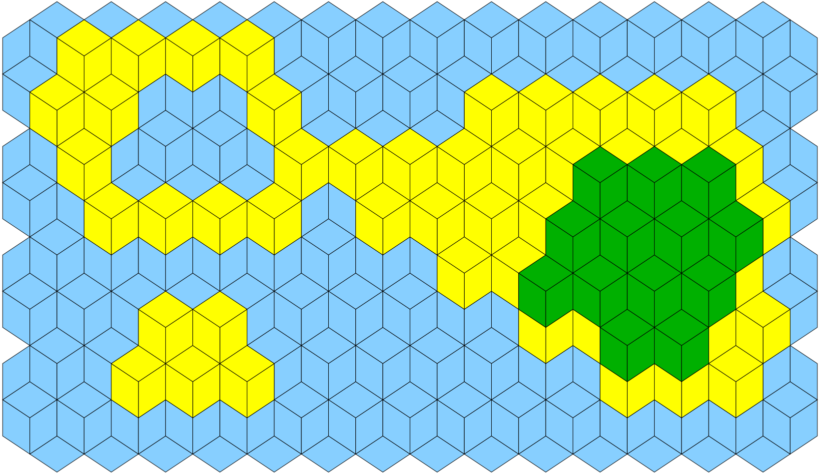
. . . and the concavity of $x \to x^\alpha$ yields a negative total variation.

Formalization



Consider a dimer tiling with *flat boundary*.

Formalization



Height function: color $\sim$ distance of tile center to $x + y + z = 0$.

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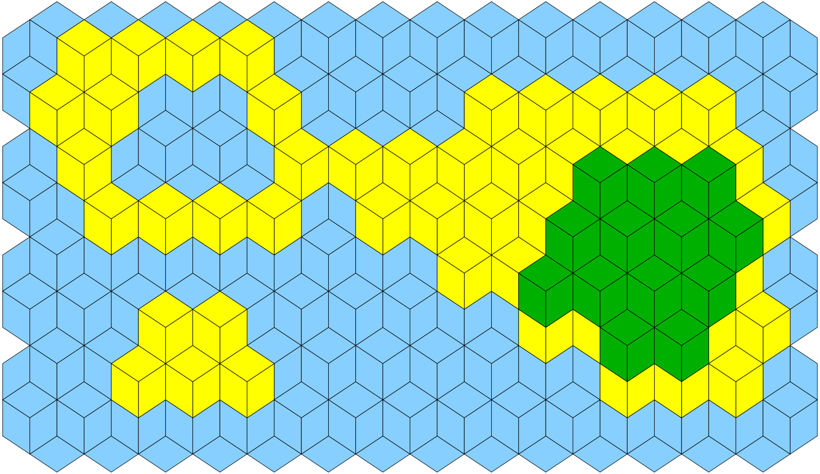

Errors: edges between tiles of different height (*i.e.*, color level set).

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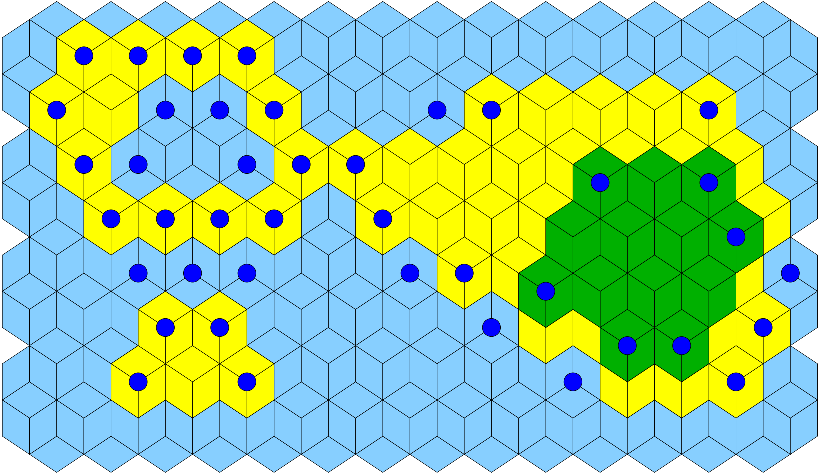
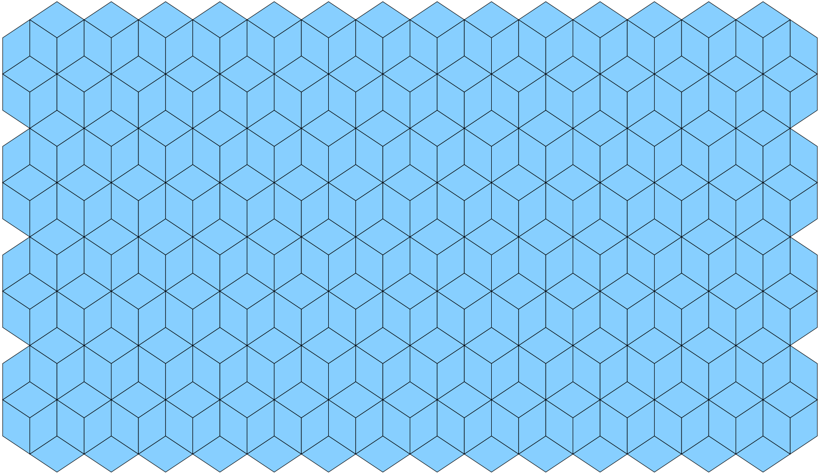


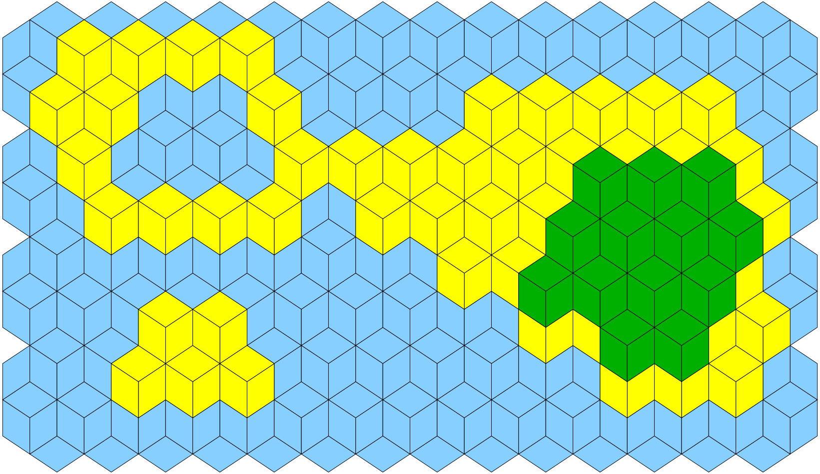
Allowed flips: do not create errors (boundaries do not grow).

Formalization



Error-free tiling (ground state): flat tiling with order 6 symmetry.

Cooling



Cooling: stochastic flips which do not create errors are performed.

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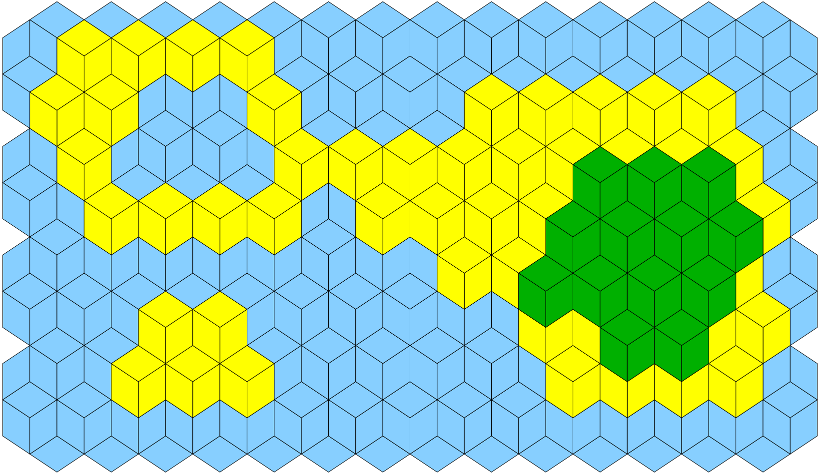

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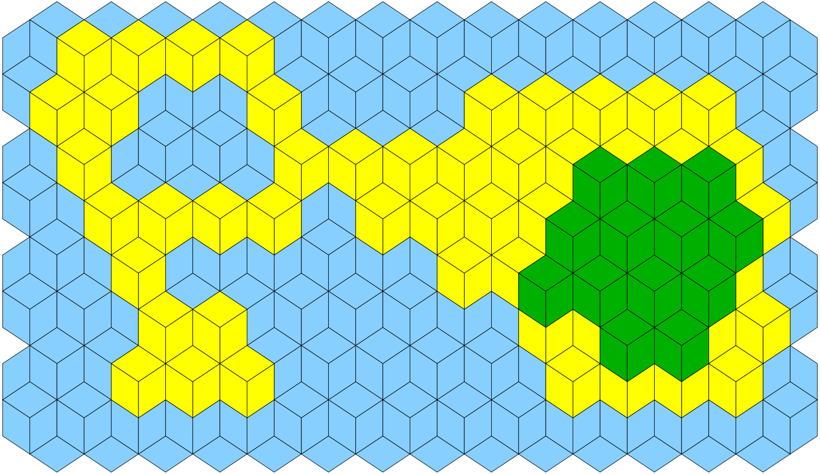

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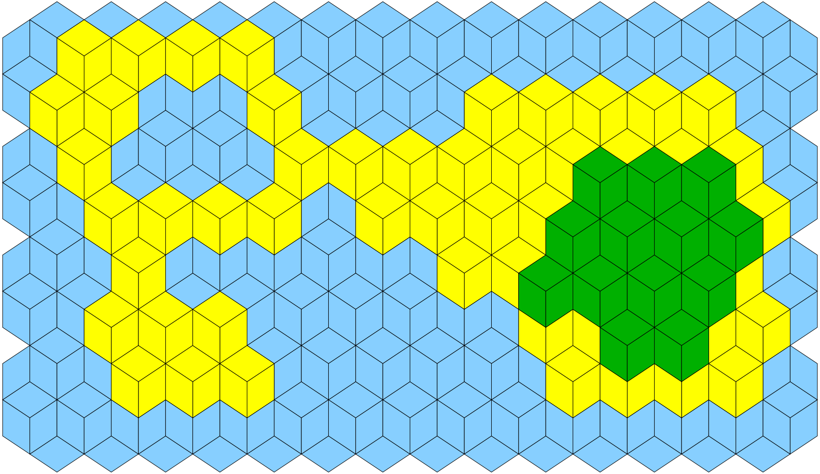

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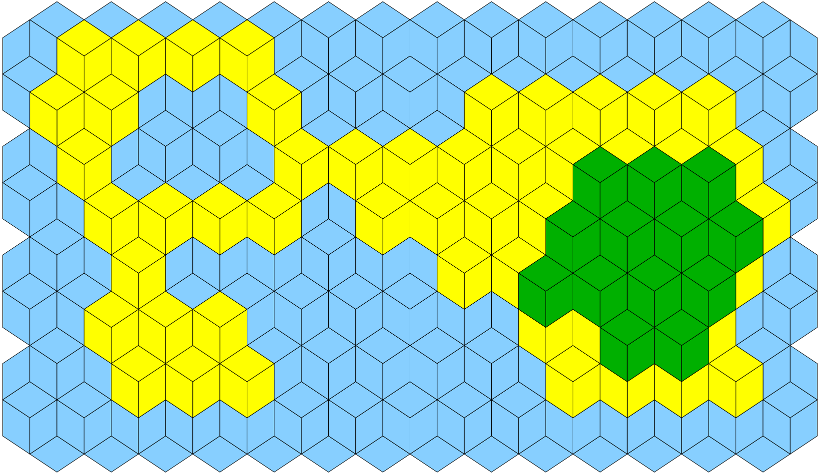

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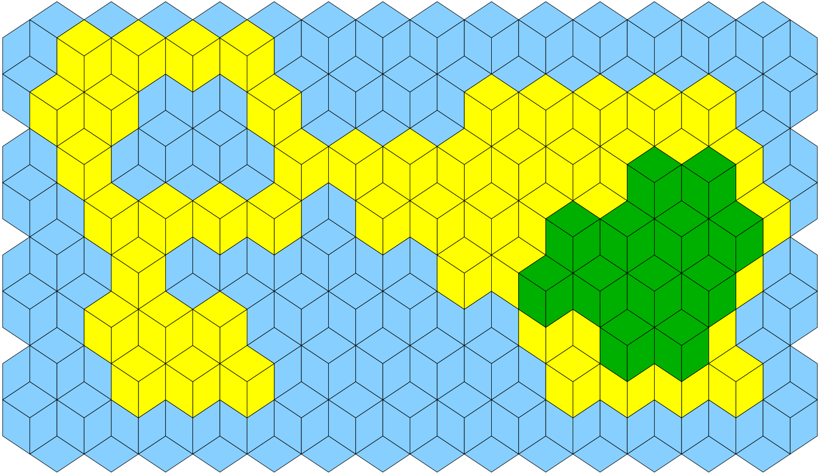

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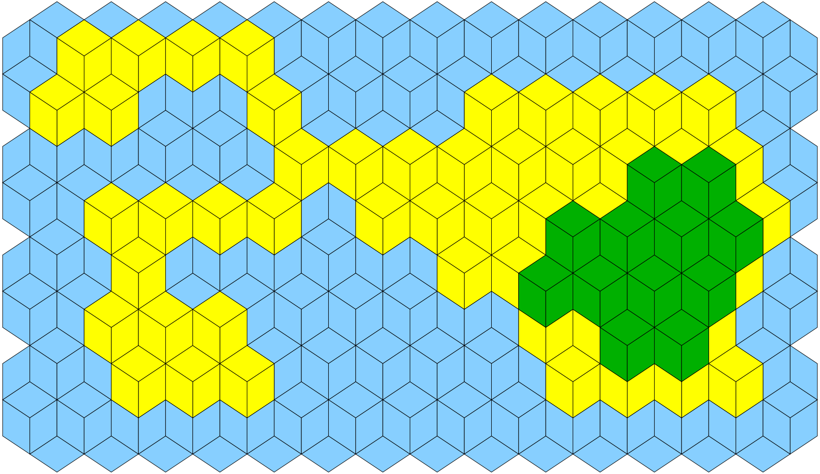

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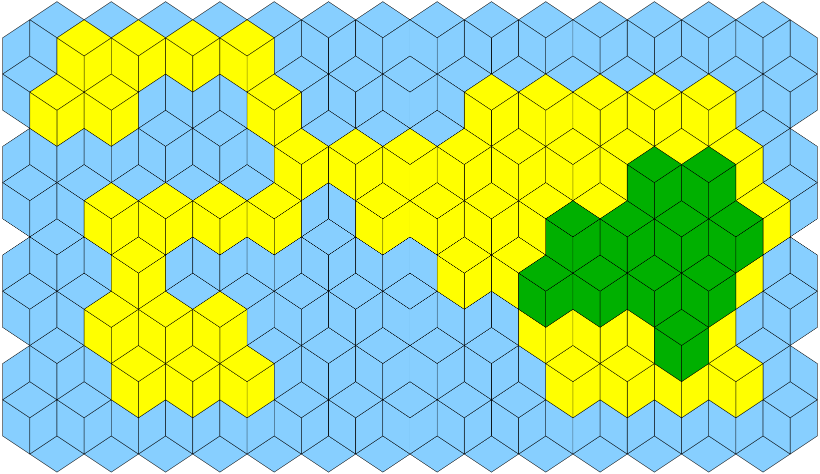

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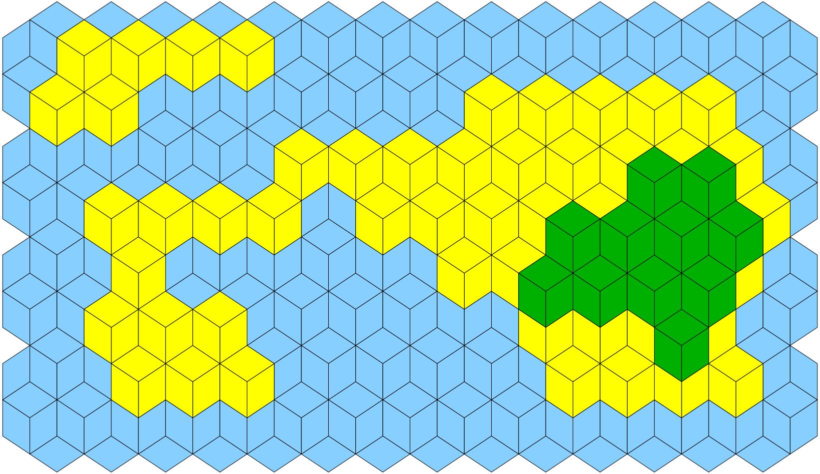

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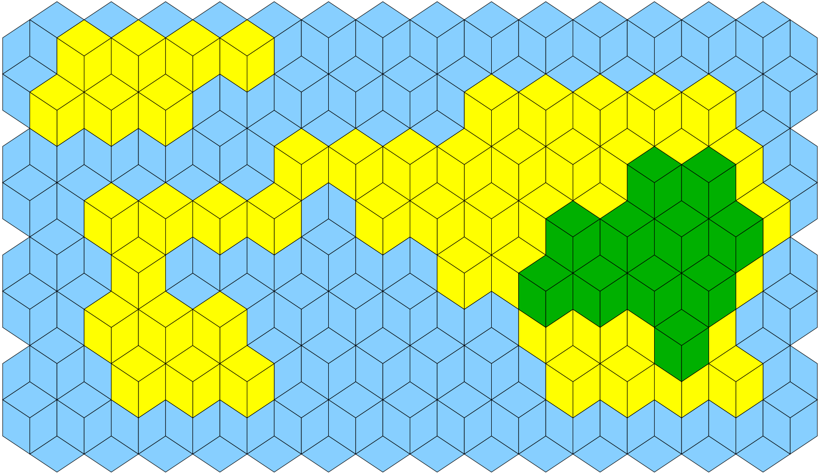

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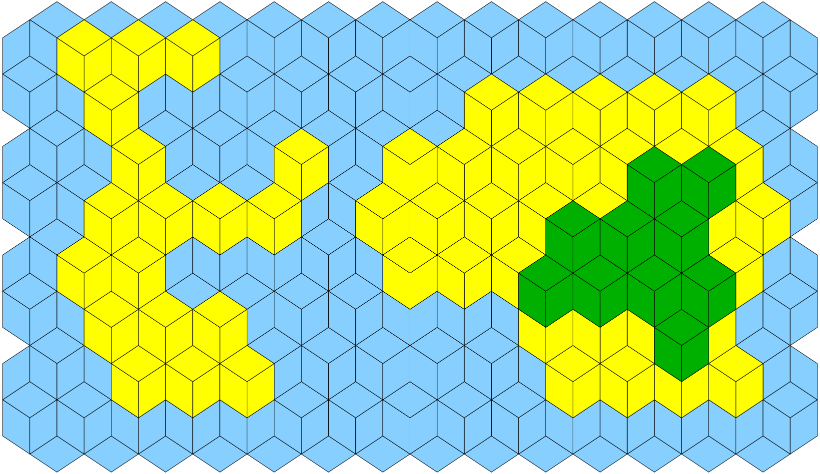

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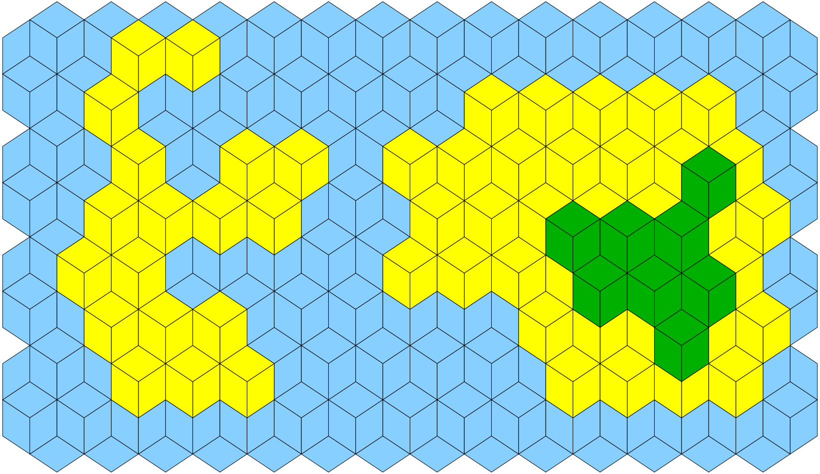

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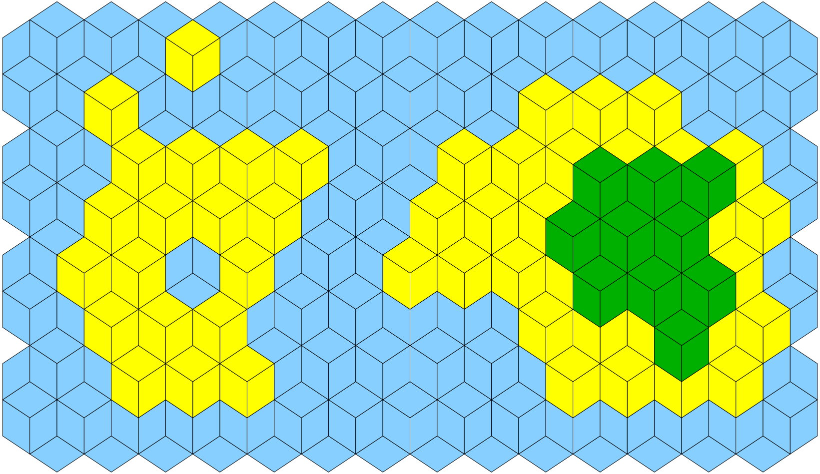

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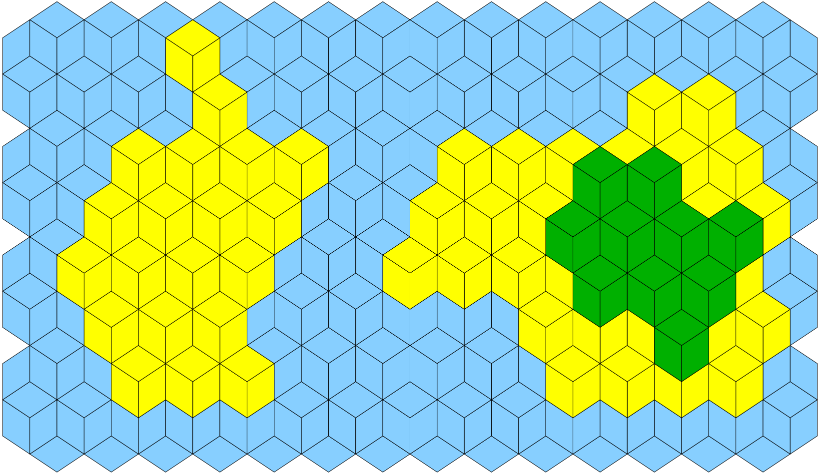

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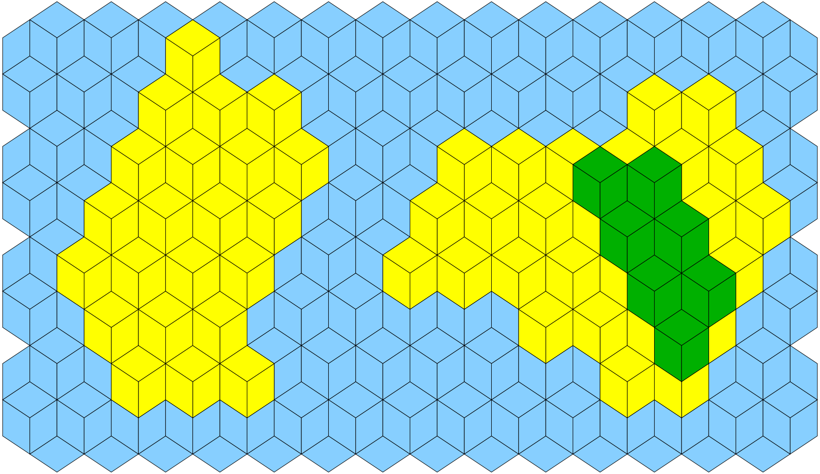

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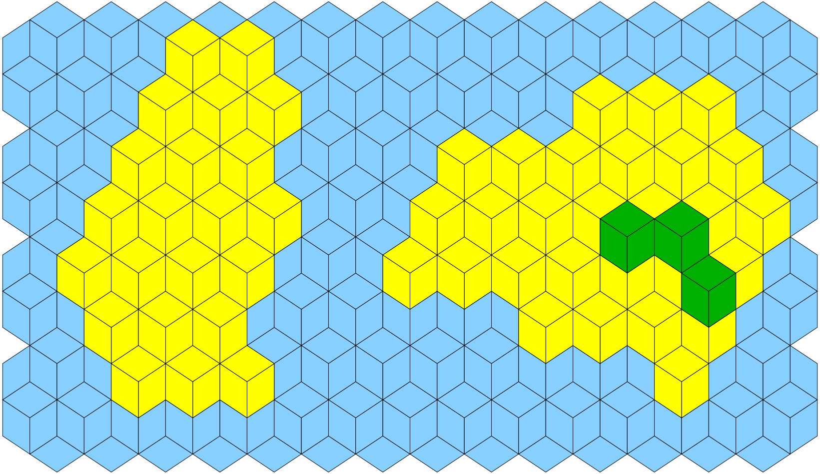

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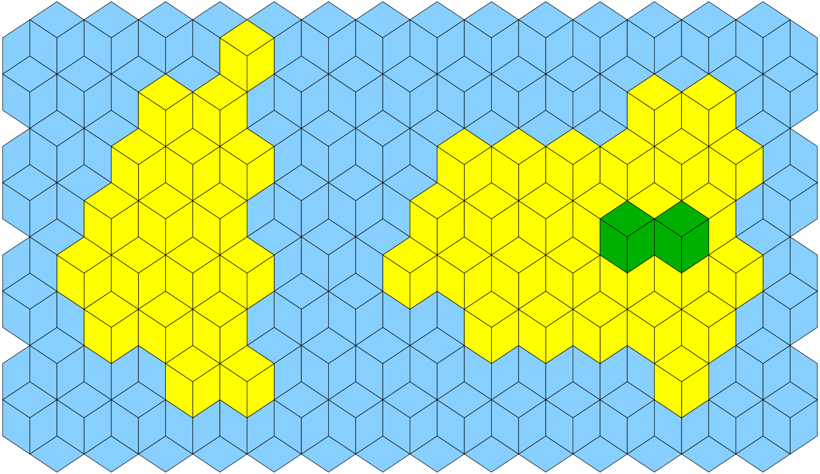

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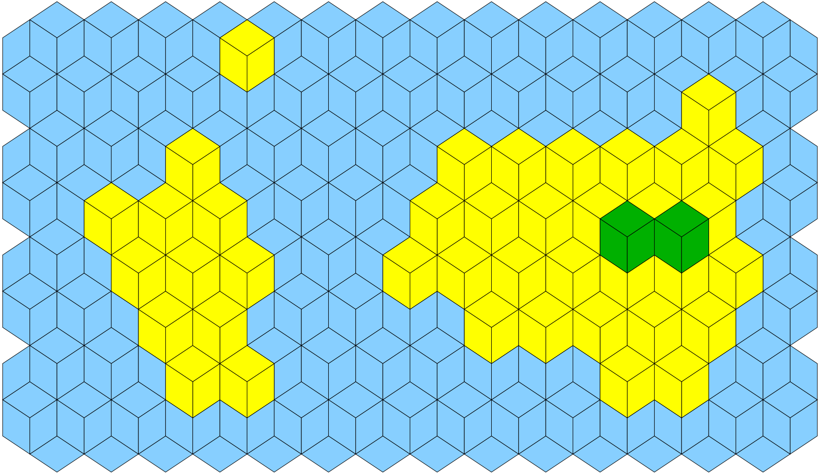

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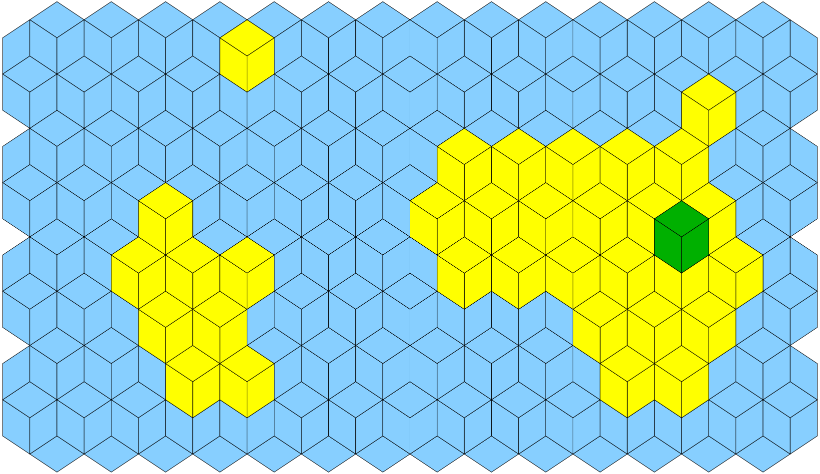

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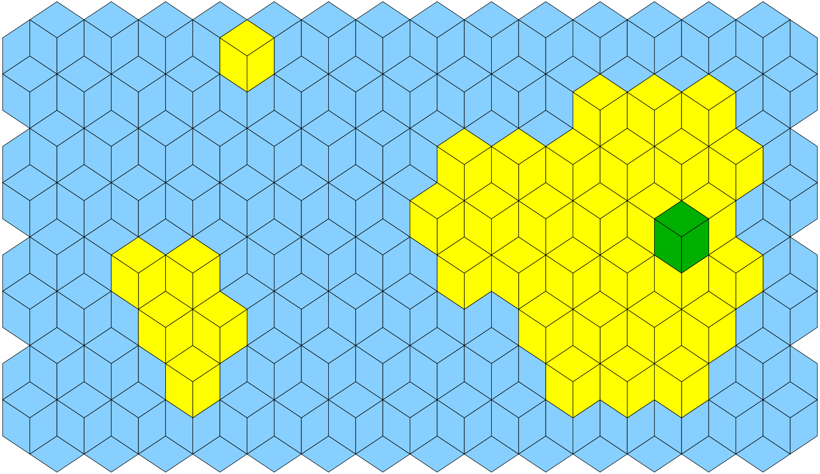

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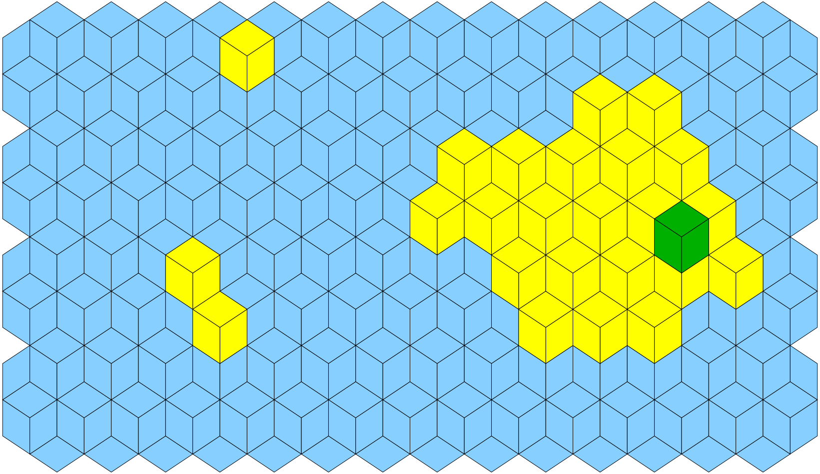

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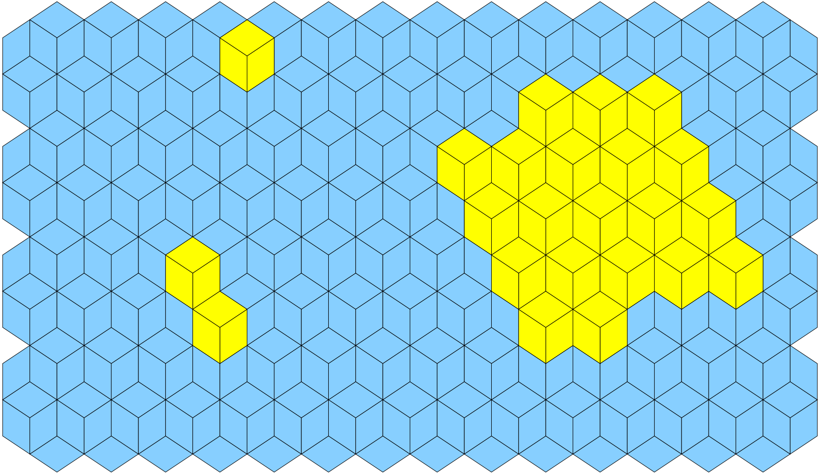

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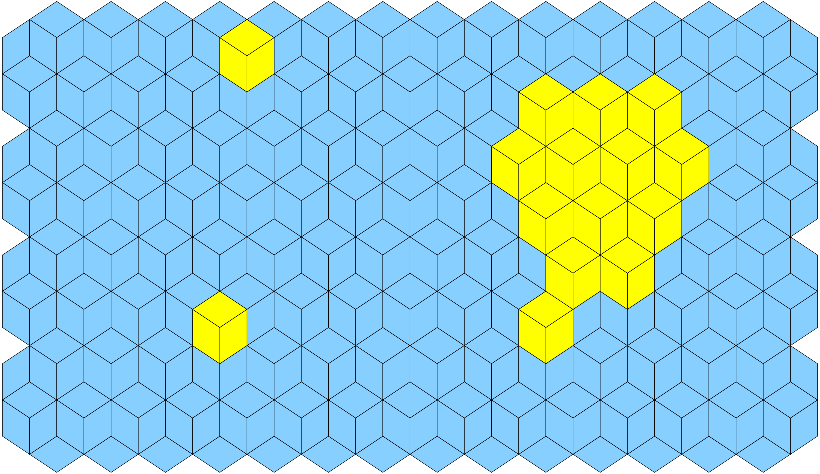

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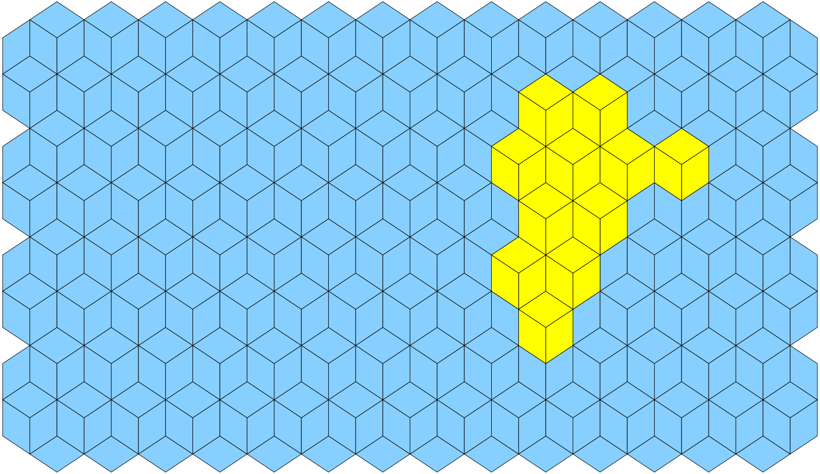

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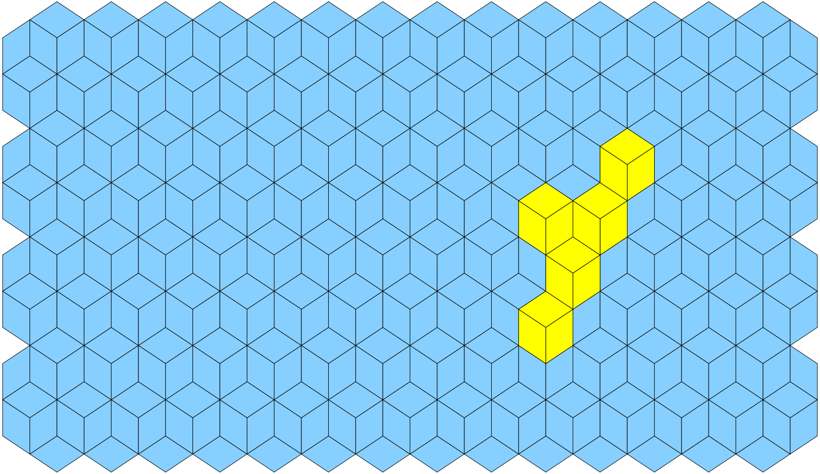

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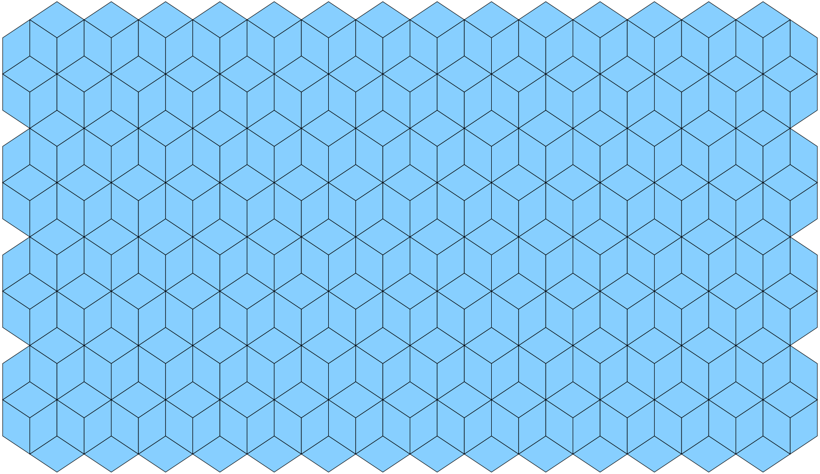


Cooling: stochastic flips which do not create errors are performed.

Conjectures and results

Expected number of flips to get an error-free dimer tiling:

Worst case:

- $\Theta(n^2)$ conjectured (simulations);
- $O(n^2\sqrt{n})$ proven.

Average case (initial tiling drawn uniformly at random):

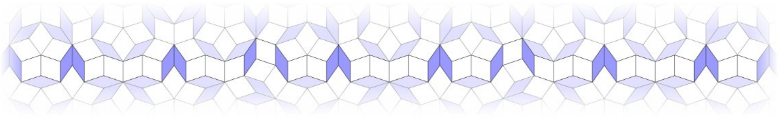
- $\Theta(n\sqrt{n})$ conjectured (simulations);
- $O(n^2 \log n)$ proven, assuming order $\log n$ fluctuations.

$\rightsquigarrow$ Polynomial mixing time in any case.

Formalization

Theorem (Kleman-Pavlovitch 1987)

A tiling of the plane by Penrose rhombi is a generalized Penrose tiling iff, in any ribbon, symmetric thin (resp. fat) tiles alternate.



Error: two consecutive similar thin (resp. fat) tiles in a ribbon.

Formalization

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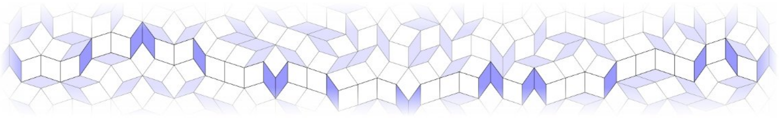
A tiling of the plane by Penrose rhombi is a generalized Penrose tiling iff, in any ribbon, symmetric thin (resp. fat) tiles alternate.



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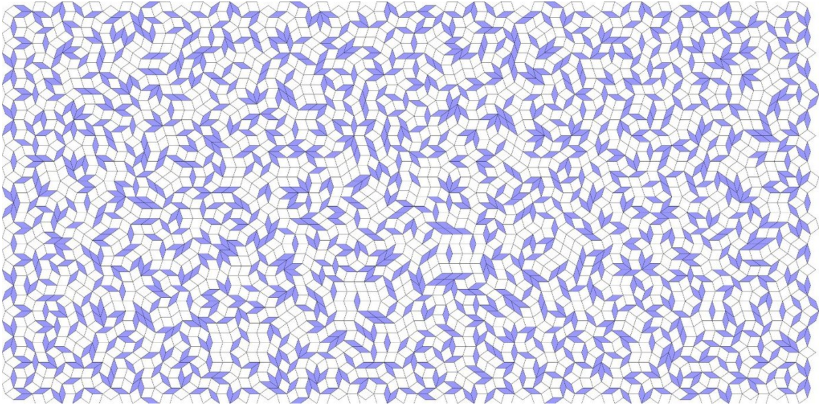
Cooling



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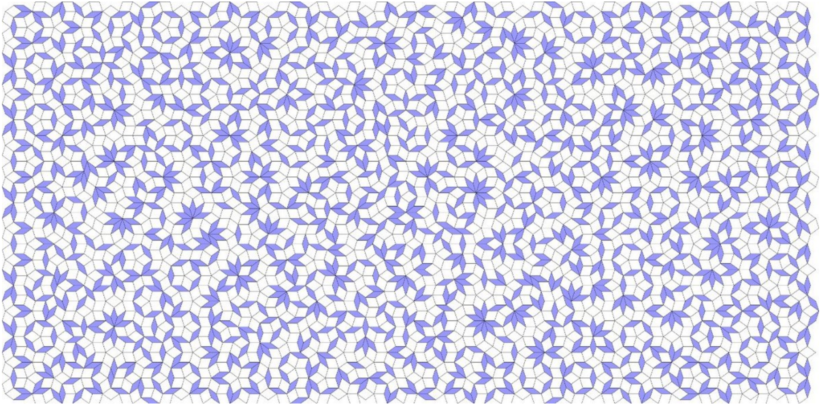
Cooling



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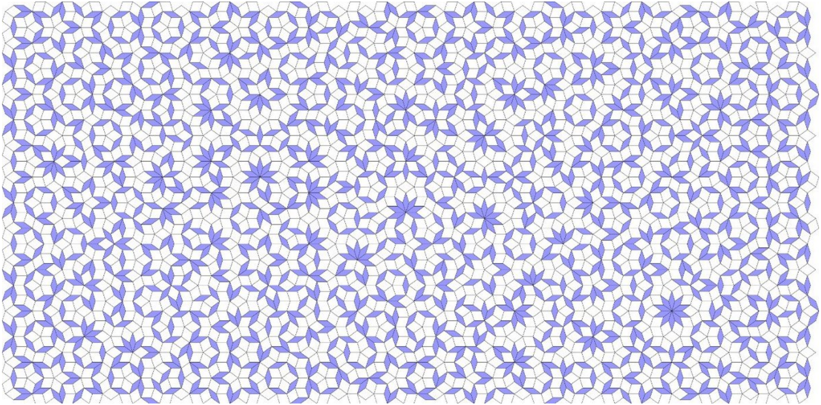
Cooling



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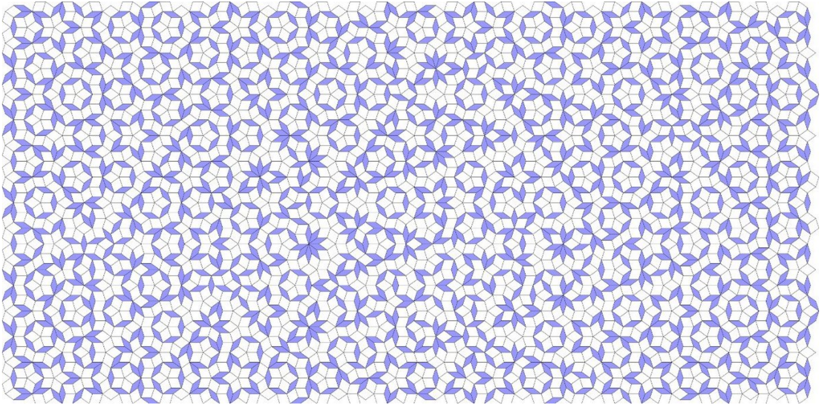
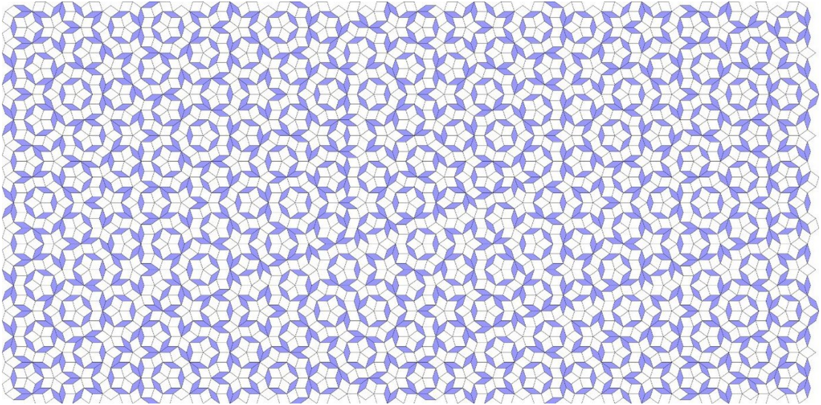


Cooling: stochastic flips which do not create errors are performed.

Cooling



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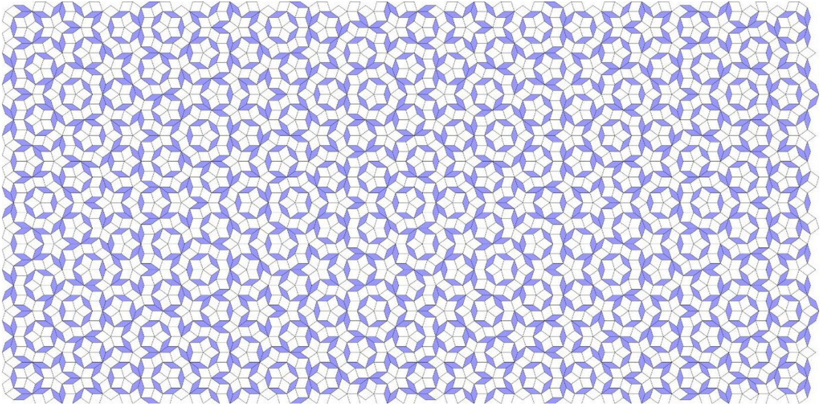
Cooling



Cooling: stochastic flips which do not create errors are performed.

Conjectures and results

Expected number of flips to get an error-free tiling:

Worst case:

- $\Theta(n^2)$ conjectured (simulations);
- nothing proven (even not finiteness!).

Average case (initial tiling drawn uniformly at random):

- $\Theta(n\sqrt{n})$ conjectured (simulations);
- nothing proven.

Further cases? Back to matching rules for rhombus tilings...

Some references for this lecture:

- Olivier Bodini, Thomas Fernique, Damien Regnault, *Crystallization by stochastic flips*, J. Phy.: Conf. ser. **226** (2010).
- Olivier Bodini, Thomas Fernique, Damien Regnault, *Stochastic flips on two-letter words*, in proc. of ANALCO'10 (2010).
- Thomas Fernique, Damien Regnault, *Stochastic flips on dimer tilings*, Disc. Math. Theor. Comput. Sci. (2010).

These slides and the above references can be found there:

`http://www.lif.univ-mrs.fr/~fernique/qc/`