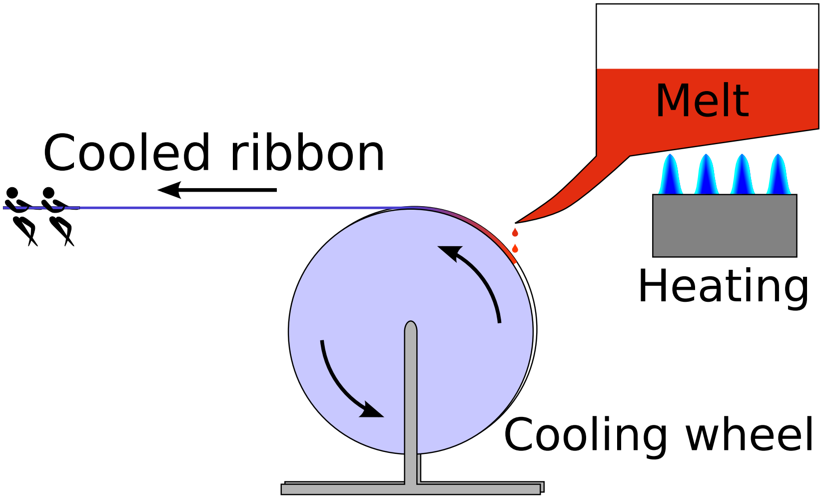
Random Tilings

Thomas Fernique

Moscow, Spring 2011

Quenching (first quasicrystals)



Free energy minimization

Stability: minimal free energy $F = E - TS$

Because of the high temperature in the melt and the rapid cooling:

Minimal free energy $F \Leftrightarrow$ Maximal entropy S.

Matching rules modelling interactions (the energy E): forgotten!

Entropy for tilings?

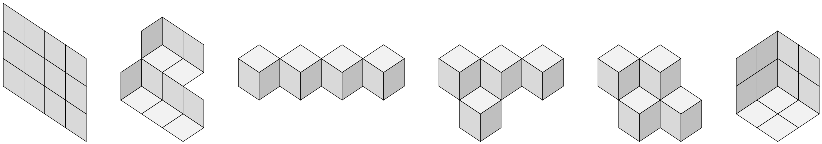
Configuration entropy

Entropy of the tilings of a domain $D \subset \mathbb{R}^2$:

$$S := \log(\text{number of tilings of } D).$$

Entropy per tile of a tiling T of D:

$$s(T) := \frac{S(T)}{|T|}.$$



Example: compute the entropy of these tilings.

Typical tilings of maximal entropy

We are mainly interested in two questions.

1. Which domains do maximize the entropy?
2. Do the tilings of such domains have a "typical look"?

Typical look: properties satisfied by most of large tilings, *e.g.*,

- presence (or not) of some patterns;
- proportions of different tiles (phason strain);
- perp-space fluctuations;
- ...

$\rightsquigarrow$ These are properties that one can expect after quenching!

A Simple case

Let W_n be the set of words of length n over the alphabet $\{1, 2\}$.

Entropy of $w \in W_n$, with $|w|_1 = a$ and $|w|_2 = b$:

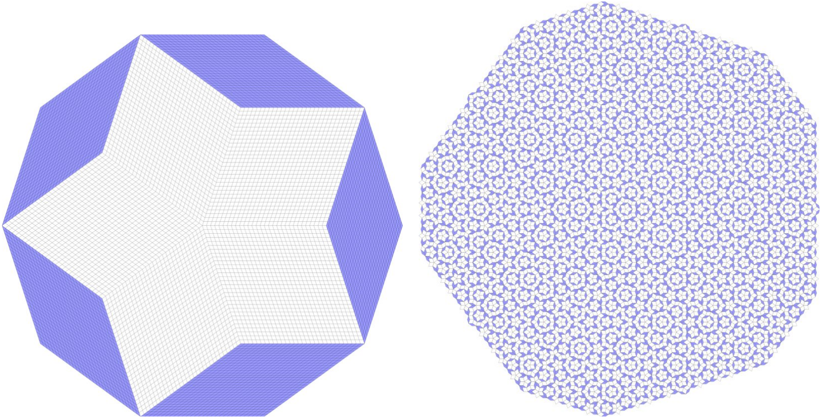
$$s(w) = \frac{1}{n} \log \left(\frac{n!}{a!b!} \right).$$

Words of maximal entropy: $a = b$ (*balanced* words).

Random balanced words are known to have fluctuations in $\sqrt{n}$.

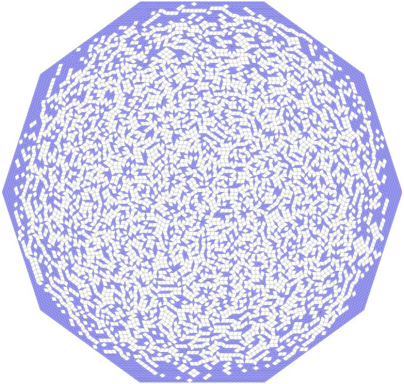
$\rightsquigarrow$ seen as broken lines on $\mathbb{Z}^2$, they stay close to the line $y = x$.

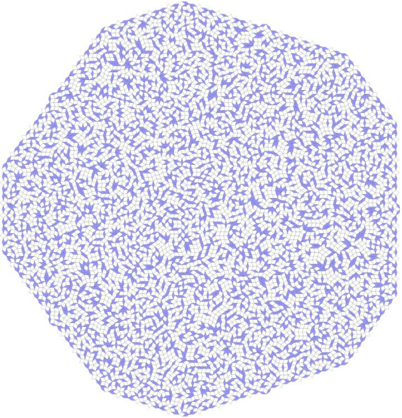
Further cases?



Which of these tilings by Penrose rhombi has the largest entropy?

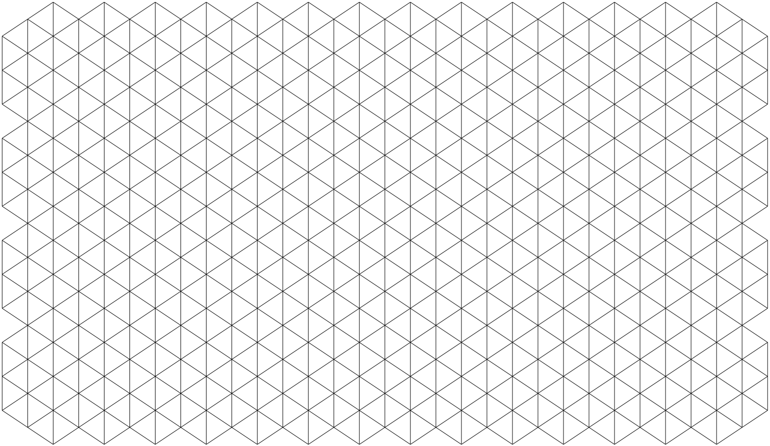
Further cases?





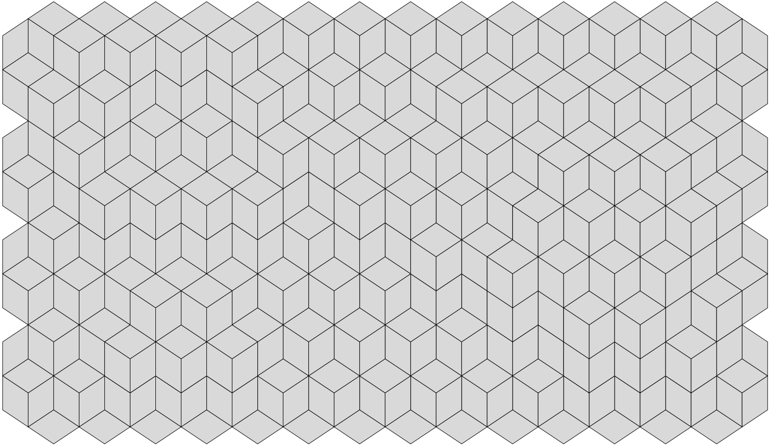
How close to a Penrose tiling can be expected to be a random tiling?

Dimer tilings



Dimer tiling: pairing of adjacent cells of a part of the triangular grid.

Dimer tilings



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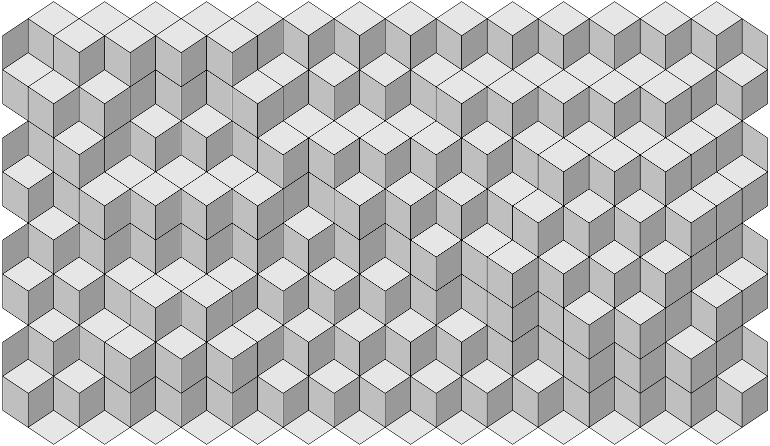
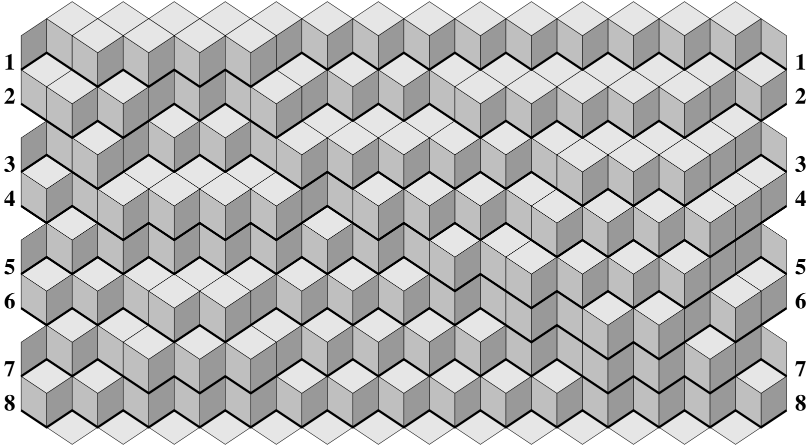


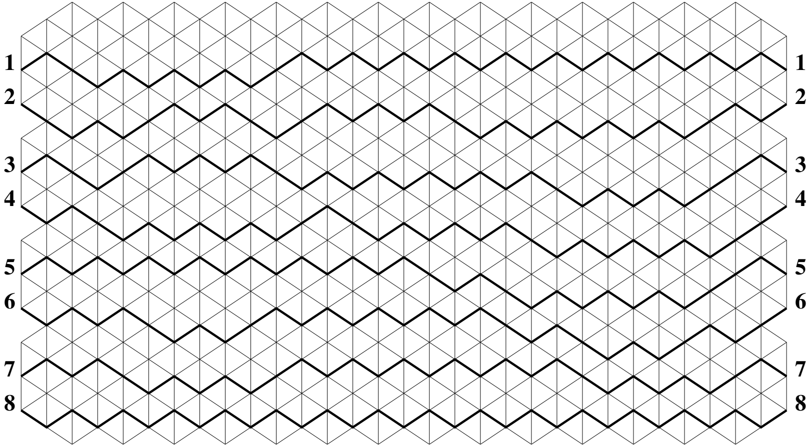
3-dimensional viewpoint: remind the shadows of rhombus tilings!

Counting with determinants



Bottom edges of vertical tiles $\rightsquigarrow$ family of non-intersecting paths.

Counting with determinants



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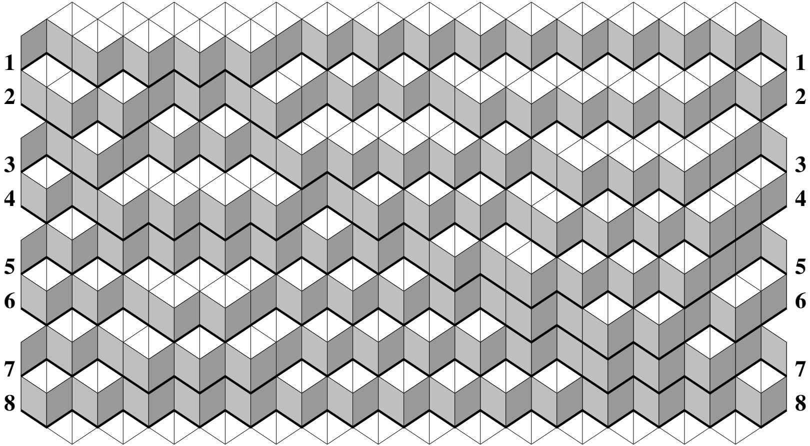
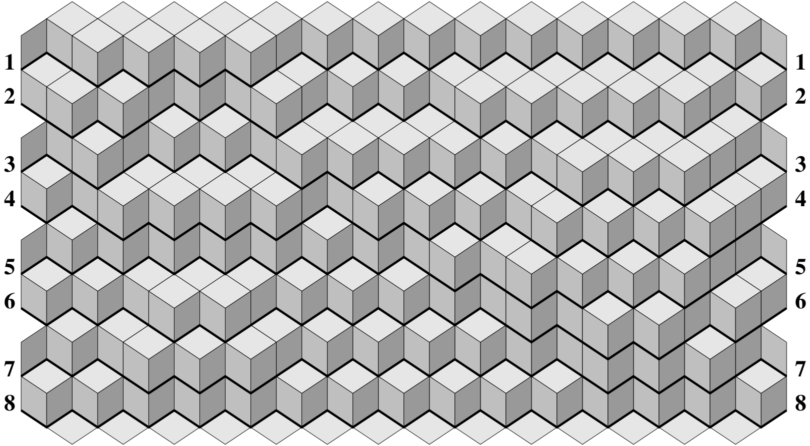


Conversely: family of non-intersecting paths $\rightsquigarrow$ unique dimer tiling.

Counting with determinants



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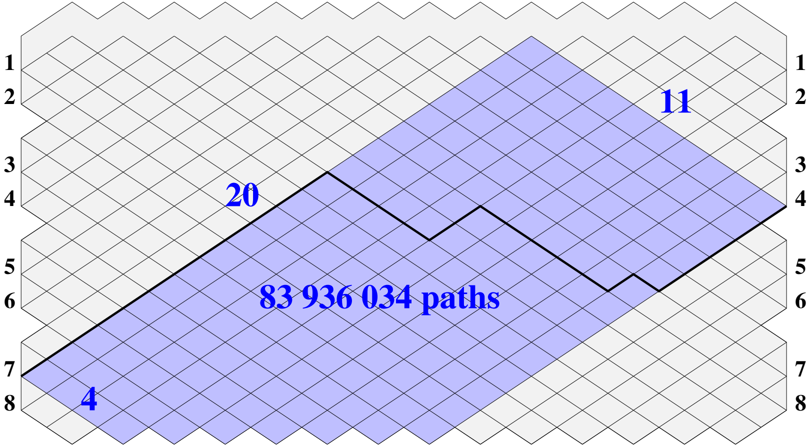
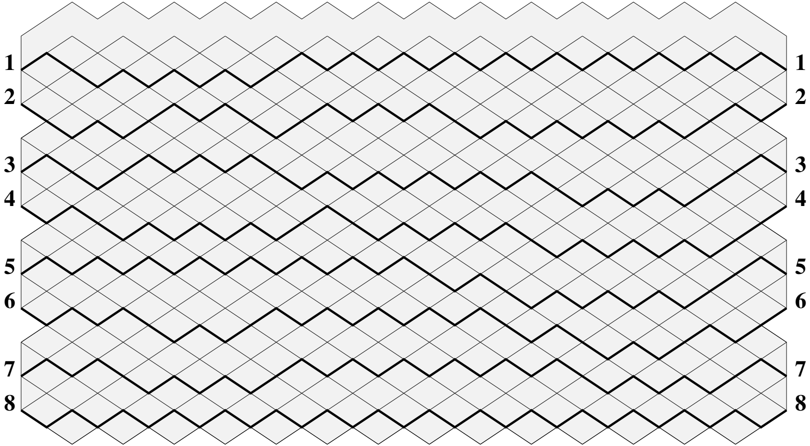


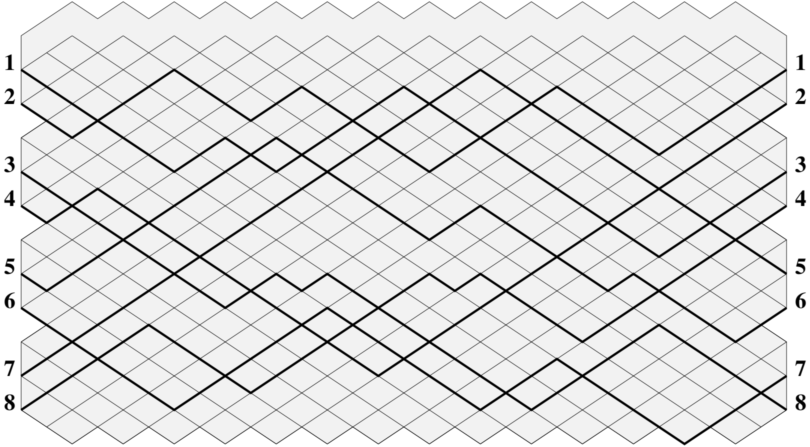
Let $N := \det(\lambda_{i,j})$, where $\lambda_{i,j}$ is the number of paths from i to j.

Counting with determinants



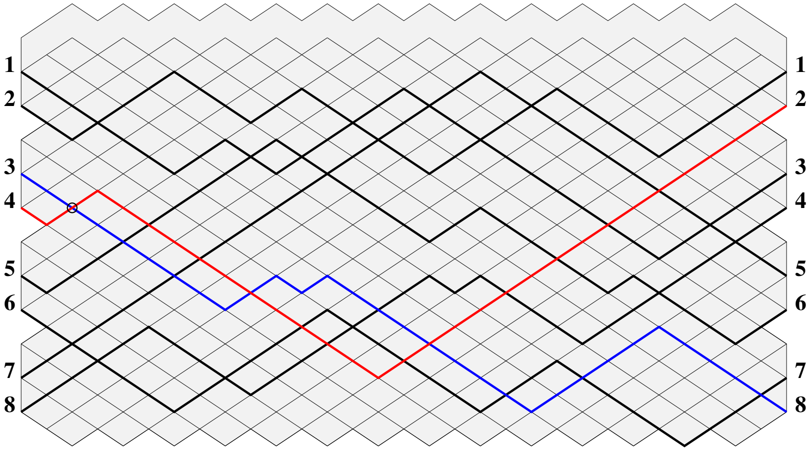
Theorem: there are N distinct families of non-intersecting paths.

Counting with determinants



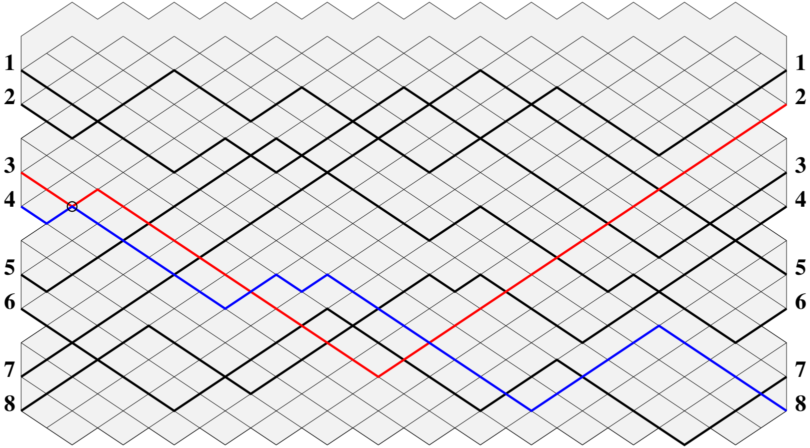
$$\sum_{\sigma \in \mathcal{S}_n} \varepsilon(\sigma)\lambda_{1,\sigma(1)} \cdots \lambda_{n,\sigma(n)}$$: weighted sum of all the path families.

Counting with determinants



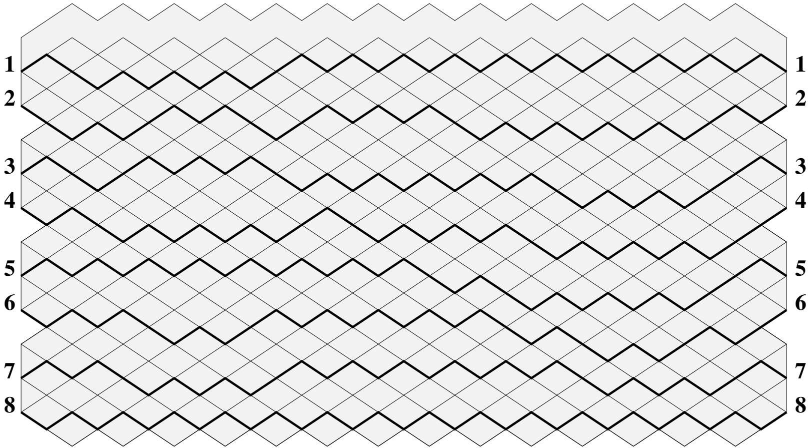
Order grid vertices. Consider the first intersection of a "bad" family.

Counting with determinants



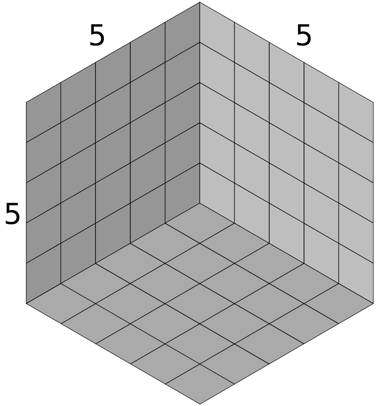
Uncross paths at this vertex $\rightsquigarrow$ pairing with another bad family.

Counting with determinants



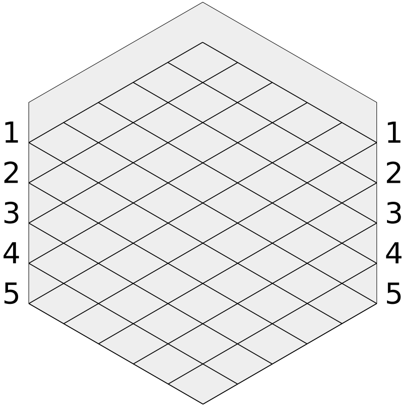
Paired families have opposite signature $\rightsquigarrow$ N counts the good ones!

Entropy of boxed tilings (MacMahon 1916, Elser 1984)



Entropy of this size 75 tiling?

Entropy of boxed tilings (MacMahon 1916, Elser 1984)



$$N = \left| \quad \right|$$

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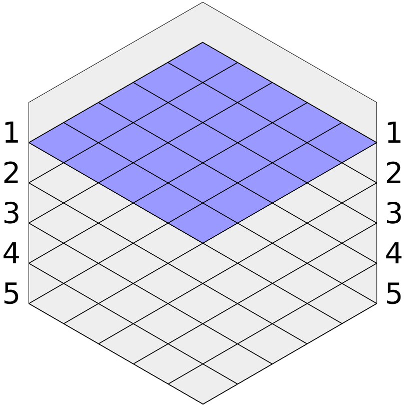

$$N = \begin{vmatrix} 252 & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{vmatrix}$$

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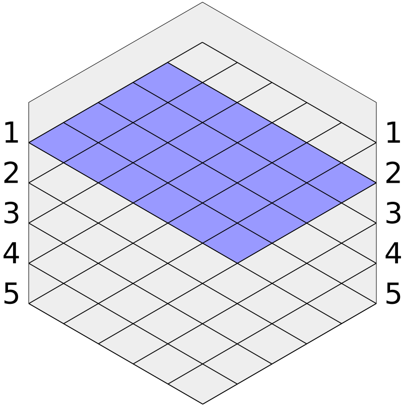

$$N = \begin{vmatrix} 252 & 210 & & & \\ & & & & \\ & & & & \\ & & & & \end{vmatrix}$$

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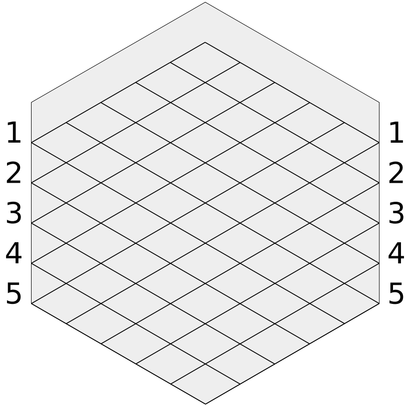

$$N = \begin{vmatrix} 252 & 210 & 120 & 45 & 10 \\ 210 & 252 & 210 & 120 & 45 \\ 120 & 210 & 252 & 210 & 120 \\ 45 & 120 & 210 & 252 & 210 \\ 10 & 45 & 120 & 210 & 252 \end{vmatrix}$$

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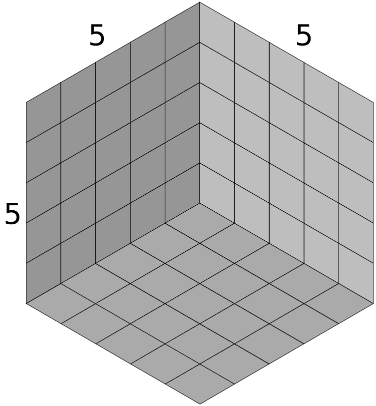

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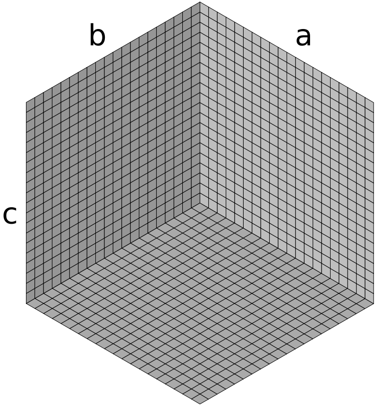
Entropy of this size 75 tiling? $s = \frac{\log(N)}{75} = \frac{\log(267\,227\,532)}{75} \simeq 0.259.$

Entropy of boxed tilings (MacMahon 1916, Elser 1984)

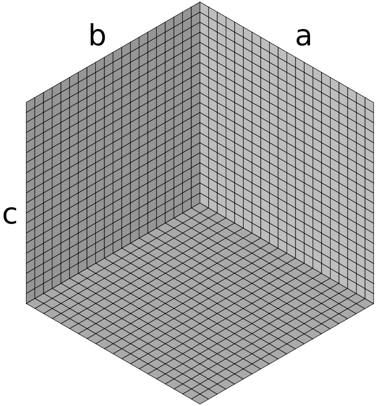


For $a \geq b \geq c$:

$$N_{a,b,c} = \left| \binom{a+b}{a+i-j}_{1 \leq i,j \leq c} \right|$$

Entropy of this size $3n^2$ tiling?

Entropy of boxed tilings (MacMahon 1916, Elser 1984)



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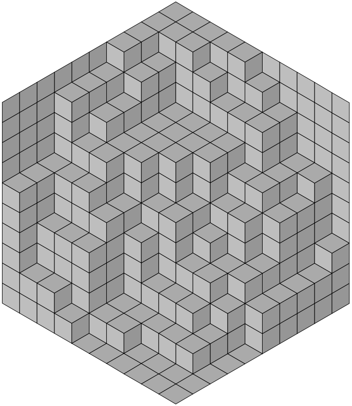
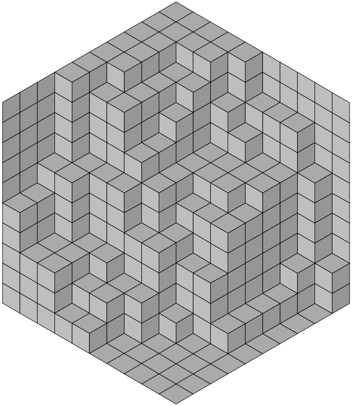
$$N_{a,b,c} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$$

Entropy of this size $3n^2$ tiling? $s_{a,b,c} \leq s_{n,n,n} \underset{n\infty}{\rightarrow} \log \frac{3\sqrt{3}}{4} \simeq 0.262.$

The Arctic circle (Cohn-Larsen-Propp, 1997)





Do the boxed tilings have a typical look?

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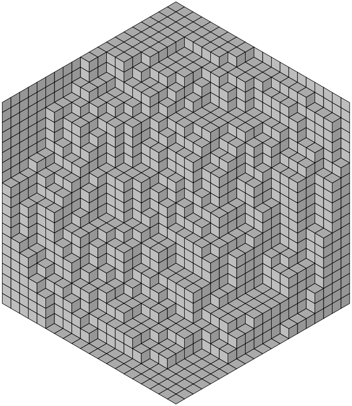

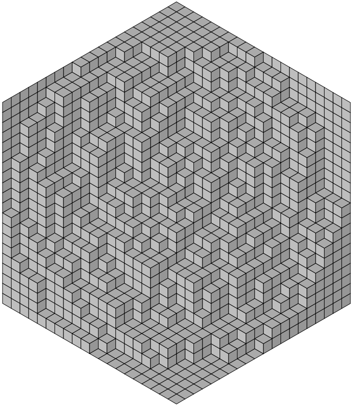

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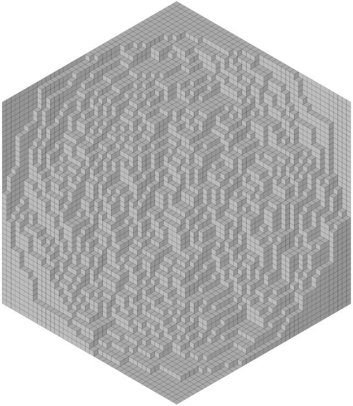

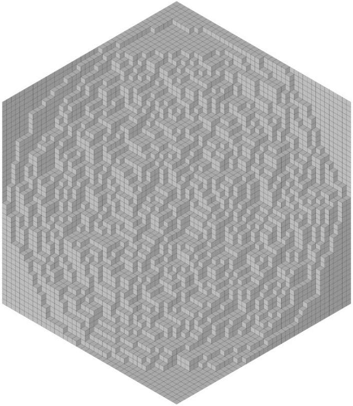

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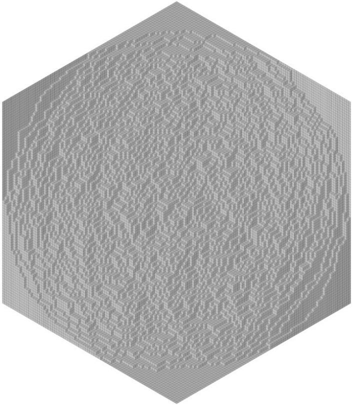

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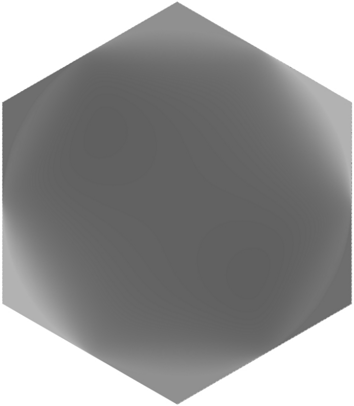

Theorem

Normalized random boxed tilings converge in probability towards a limit surface (exponentially fast). This surface is characterized:

- *flat outside an* arctic circle*;*
- *subtly wavy inside it.*

Do the boxed tilings have a typical look?

A Variational principle (Cohn-Kenyon-Propp, 2001)

Height function of a tiling: distance to the plane $x + y + z = 0$.

Theorem

Let $R \subset \mathbb{R}^2$ be bounded by a piecewise smooth simple closed curve. If, for $n \geq 0$, R_n is a tileable domain which approximates nR, then

$$\lim_{n \to \infty} s(R_n) = \sup_h \frac{1}{|R|} \iint_R \mathrm{ent}\left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right) dx\, dy,$$

where $\mathrm{ent} : \mathbb{R}^2 \to \mathbb{R}$ and h is any 2-Lipschitz real function on R.

Moreover, the normalized R_n's random height functions converge in probability (exponentially fast) towards the integral-maximizing h.

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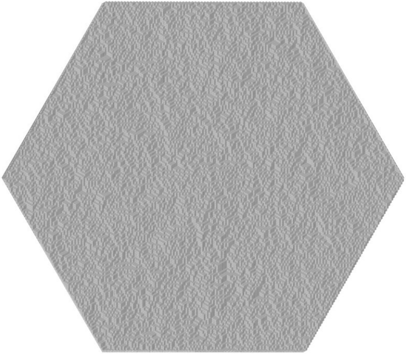
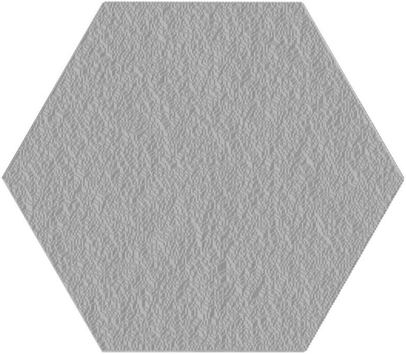
Note: this does not tell anything about the integral-maximizing h. However, one checks that ent is concave with a maximum in $(0, 0)$.

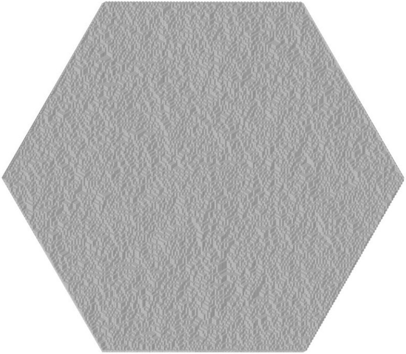
Dimer tilings of maximal entropy

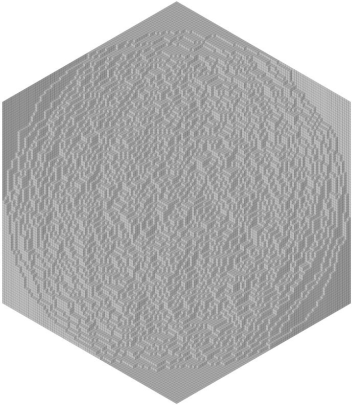




The entropy is thus maximal for flat boundary height functions.
Typical dimer tilings then stay close to the plane $x + y + z = 0$.

Dimer tilings of maximal entropy





Entropy: $s_{\text{flat}} = -\frac{2}{\pi}\int_0^{\frac{\pi}{3}} \log(2\cos(t))\,dt \simeq 0.338 > s_{\text{boxed}} = 0.262.$

Principle

Energy:

Can explain quasiperiodic tiling stability at low temperature.
Mostly fails to explain growth of quasiperiodic tiling.

Entropy:

Can explain random tiling stability at high temperature.
Some random tilings may be close to quasiperiodic tilings.

Question:

Can we explain the growth of random tilings?

A simple case

Growth of a two-letter word via a Bernoulli process of parameter p:

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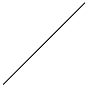

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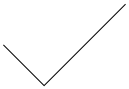

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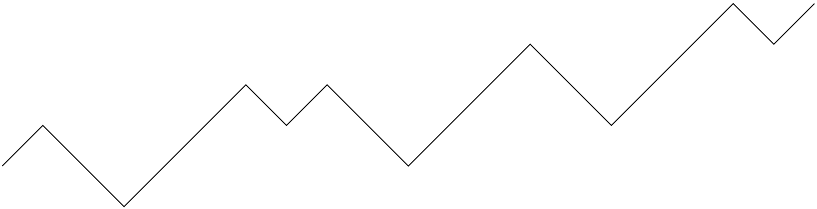

This is a biased random walk which, once normalized, converges in probability (exponentially fast) towards a line segment (slope $\frac{1-p}{p}$).

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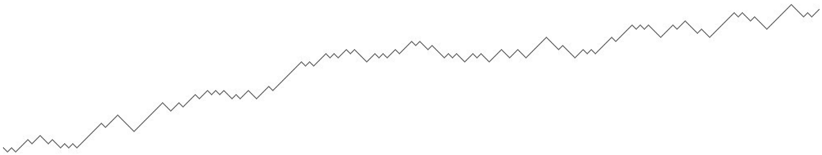

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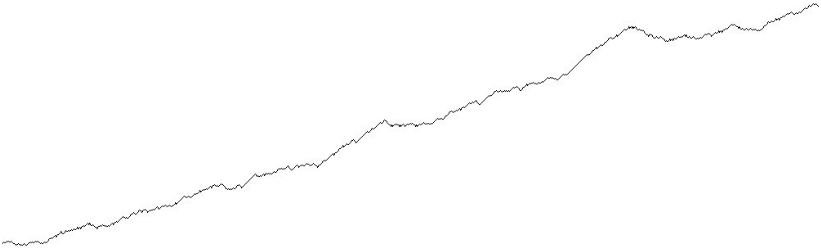

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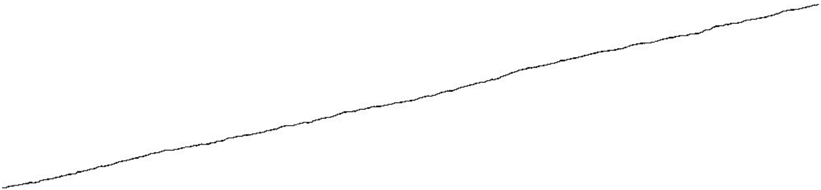

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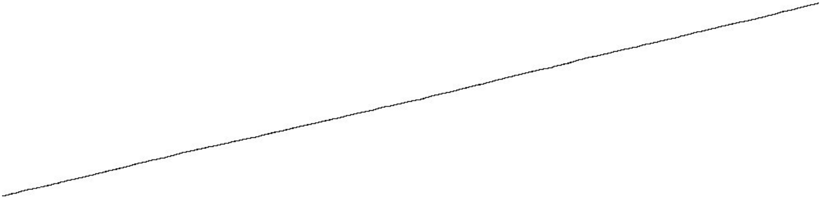


This is a biased random walk which, once normalized, converges in probability (exponentially fast) towards a line segment (slope $\frac{1-p}{p}$).

The growth is local and yields either periodic or non-periodic lines! However, the stability decreases when the slope goes away from 1.

A more complex case (Elser-Joseph 1997)

Chemical potential: parameter $\mu > 0$.

Energy of a tiling ($\sim$ surface tension):

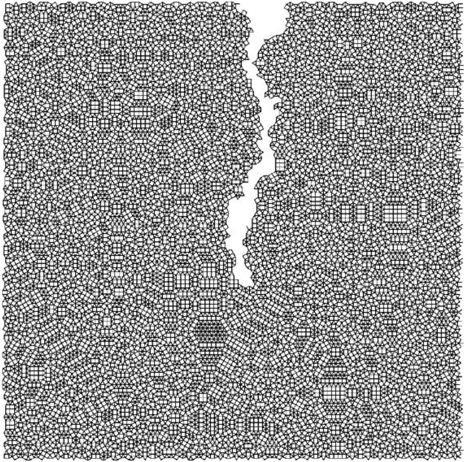
$$E := \sum_{x \text{ vertex}} \mu - \theta(x),$$

where $\theta(x)$ denotes the total angle subtended by tiles around x.

Growth: at each step, add or remove a randomly chosen edge with probability 1 if E decreases or $\exp(-\Delta E/T)$ if E increases by ΔE.

Claim: the growth rate tends to decrease with μ, increase with T.

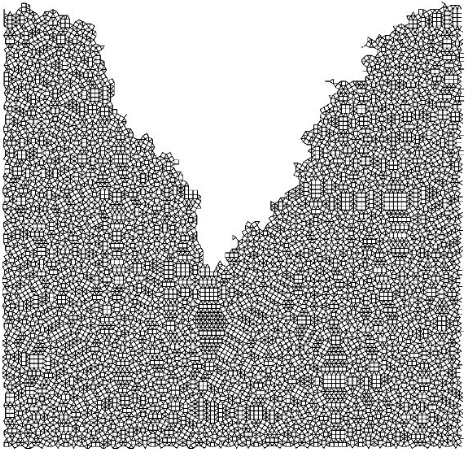
Tears and annealing



(Picture: D. Joseph)

Growth on a cylinder with seed base. A fast growth can create *tears*.

Tears and annealing



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Tears may be healed by melting, *i.e.*, by playing suitably on T and μ.

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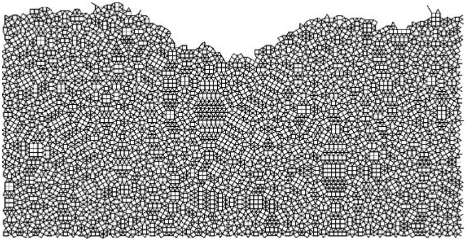

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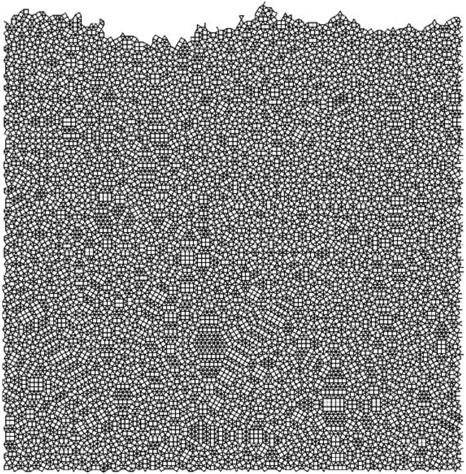
Tears and annealing



(Picture: D. Joseph)

Tears may be healed by melting, *i.e.*, by playing suitably on T and μ.

Some comments

Very handwavy framework, no formal result or even conjecture.

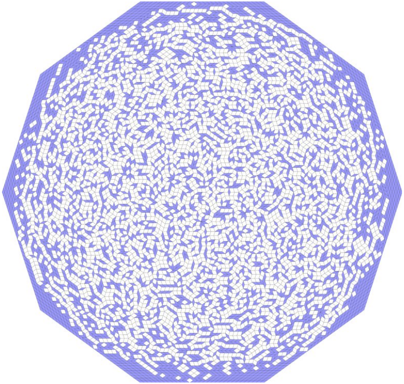
Simulations yield interesting tilings which look like random.
To what extent? Are they entropically stabilized?

Growth governed by energy minimization at positive temperature.
But should not entropy maximization play a role in such a growth?

Entropy could help to explain growth because of its non-locality.
But it is not easy to compute ΔS when adding/removing a tile...

Lot of open questions!

Problem

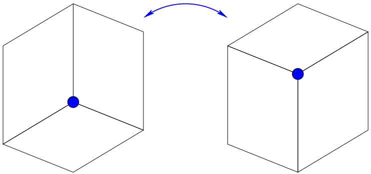


How to draw uniformly at random such rhombus tilings?

Flip (or phason-flip)

Definition (Flip)

A flip is a 180° rotation of three rhombi which form a hexagon.

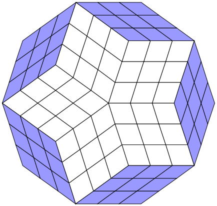


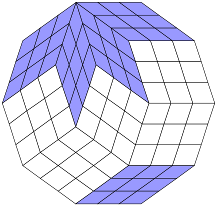
Theorem (Kenyon 1993)

Rhombus tilings of a simply connected domain are flip-connected.

Markov chain Monte Carlo method

Discrete-time Markov chain on rhombus tilings:
Choose uniformly at random a vertex, a direction, and try to flip.





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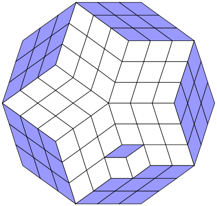

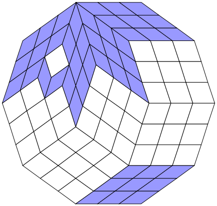

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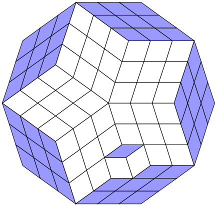

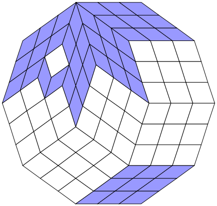

Flip-connectedness $\rightsquigarrow$ ergodic chain $\rightsquigarrow$ convergence in probability.
Symmetric chain $\rightsquigarrow$ uniform stationary (equilibrium) distribution.

Stationary distribution: at the limit. In practice: how many steps?

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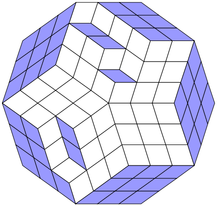

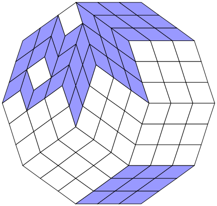

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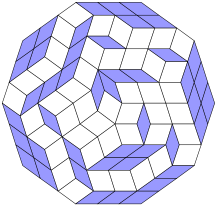

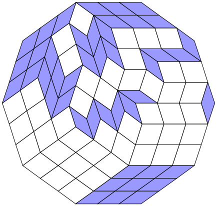

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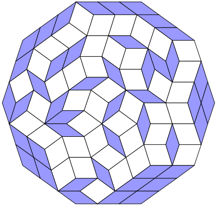

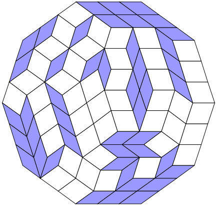

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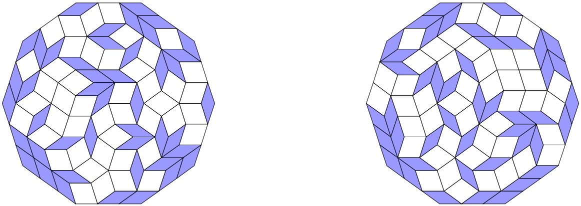

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Stationary distribution: at the limit. In practice: how many steps?

Mixing time

Total variation between two distributions:

$$||\mu - \nu|| := \frac{1}{2} \sum_{x \in \Omega} |\mu(x) - \nu(x)|.$$

Measure of the convergence towards the stationary distribution:

$$d(t) = \max_{x \in \Omega} ||P^t(x, \cdot) - \pi||.$$

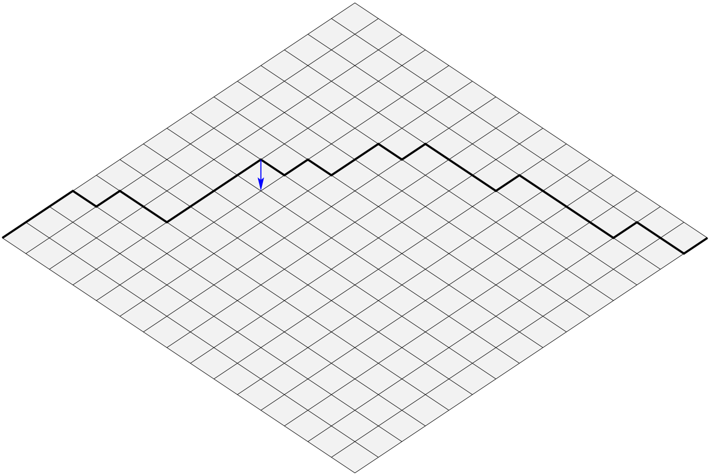
Mixing time:

$$\tau := \min\{t \mid d(t) \leq 1/(2e)\}.$$

Theorem (Half-life)

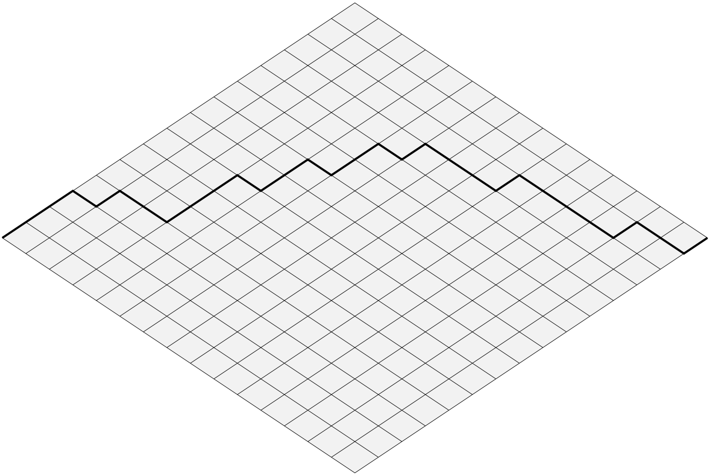
For an ergodic Markov chain, one has: $d(t) \leq \exp(-\lfloor t/\tau \rfloor)$.

The two-letter case



On a length n word: choose a vertex, a direction, and try to flip.

The two-letter case



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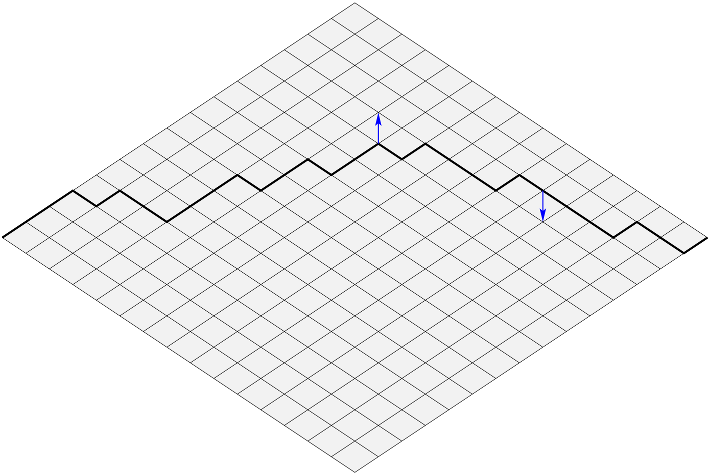

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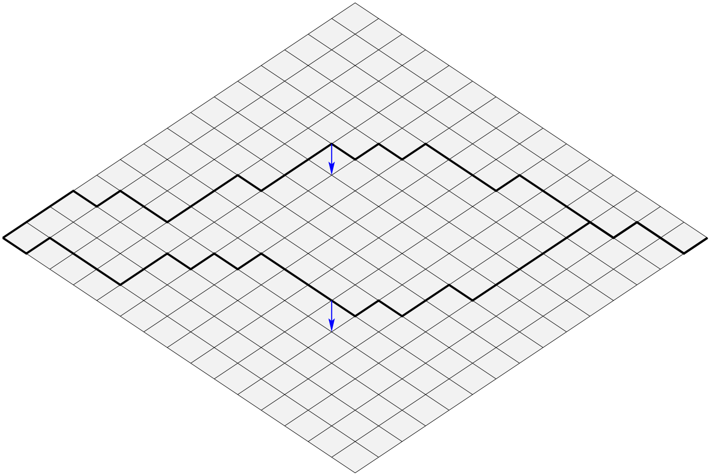
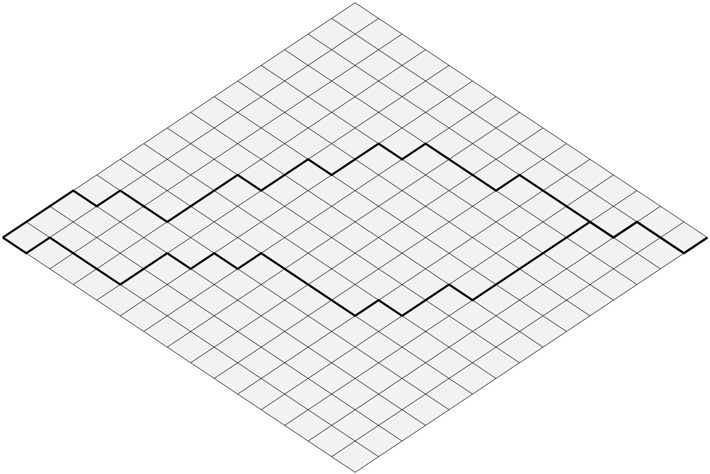


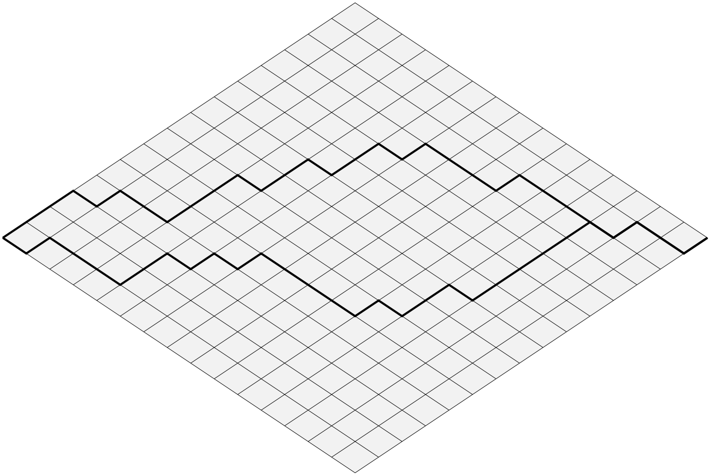
Coupling of two words x and y $\rightsquigarrow$ *coalescence time* T_{xy}.

The two-letter case



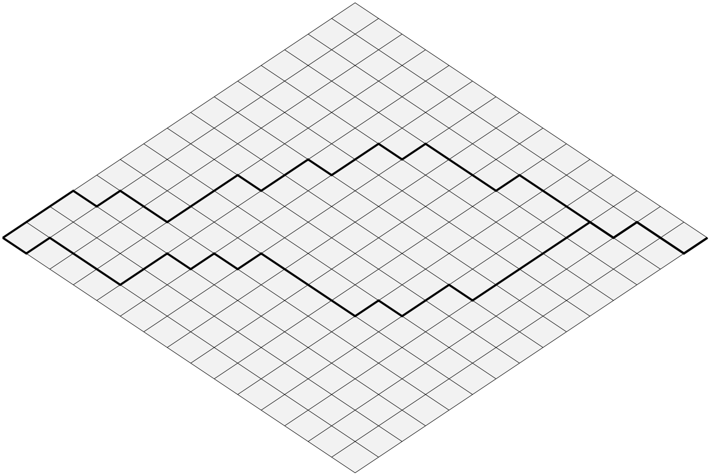
Coupling of two words x and $y \rightsquigarrow$ *coalescence time* T_{xy}.

The two-letter case



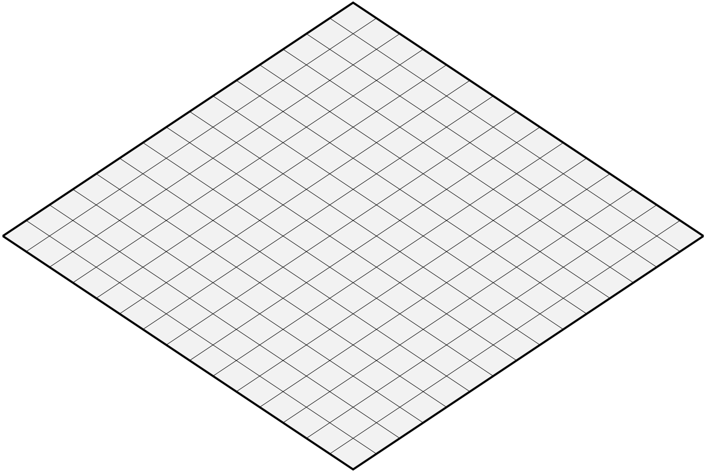
Theorem: $\tau = O(\mathbb{E}T)$, where $T = \max_{x,y} T_{xy}$.

The two-letter case



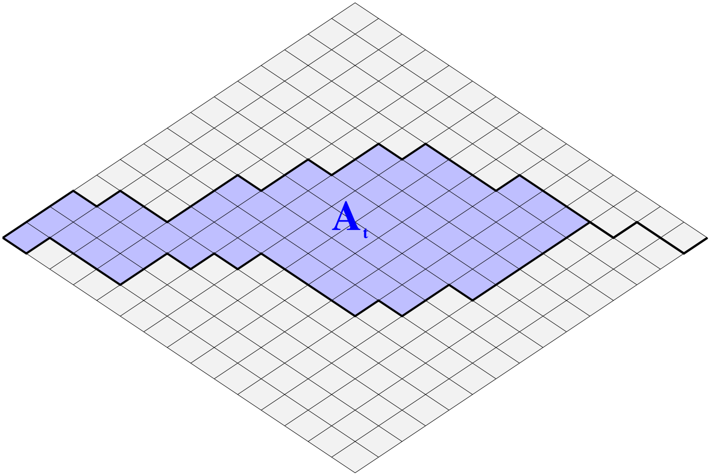
If x is completely below y at time t, then this holds for $t' > t$.

The two-letter case



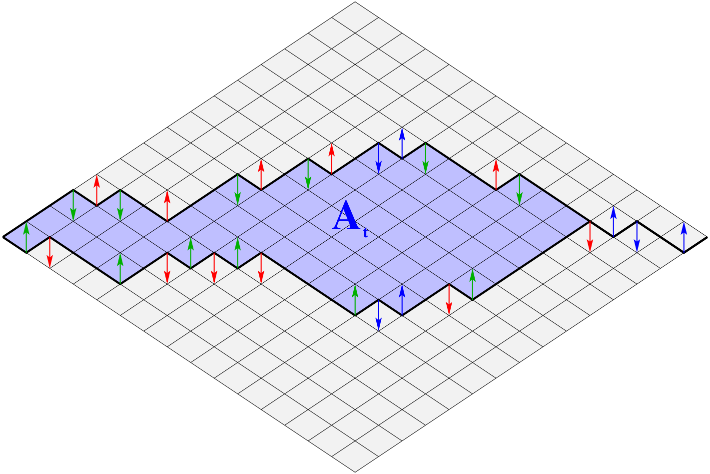
Top and bottom words thus sandwich all the other words $\rightsquigarrow T$.

The two-letter case



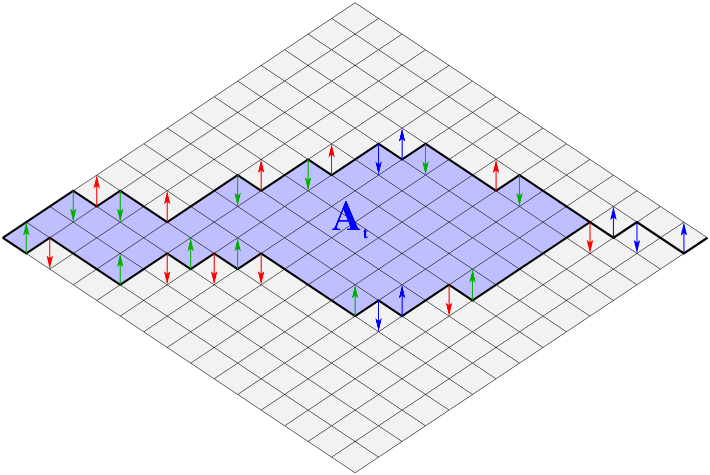
Let A_t be the sandwiched area at time t. $A_0 = n^2$. $A_T = 0$.

The two-letter case



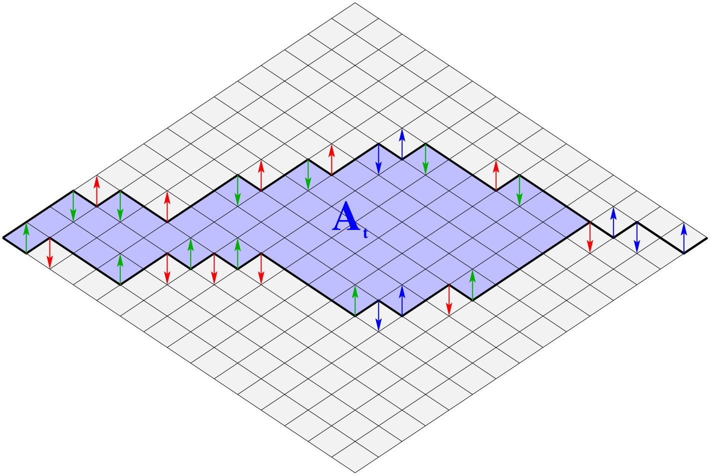
Let $\Delta A_t := A_{t+1} - A_t$. Claim: $\mathbb{E}(\Delta A_t | A_t) \leq 0$ (count flips by type).

The two-letter case



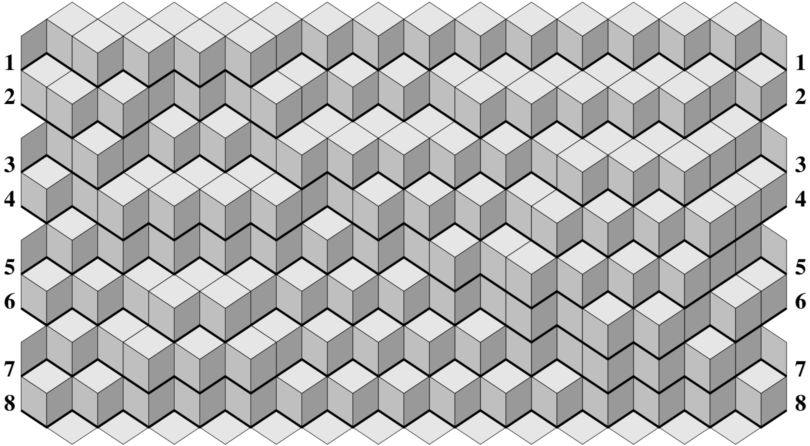
Theorem: $(\mathbb{E}(\Delta A_t) \leq 0,\ \Pr(|\Delta A_t| \geq 1) > \alpha) \Rightarrow \mathbb{E}T \leq \frac{\max A_t^2}{\alpha}$.

The two-letter case



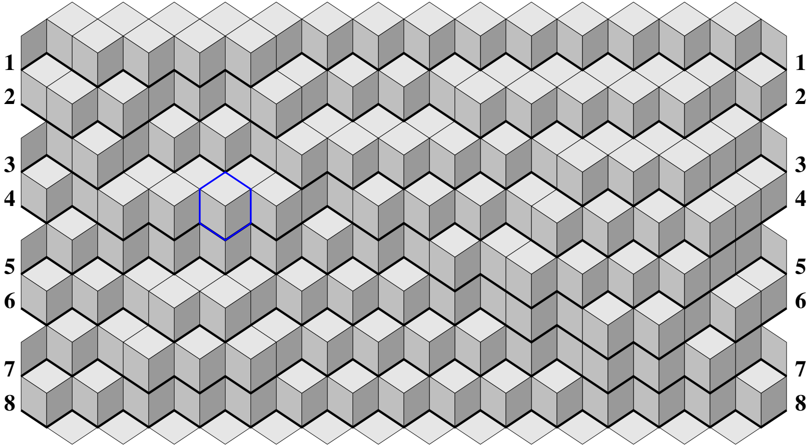
Here: $\max A_t^2 = A_0^2 = n^4$ and $\alpha \geq \frac{1}{2n} \rightsquigarrow \mathbb{E}\,T \leq n^5 \rightsquigarrow \tau = O(n^5)$.

The dimer case (Luby-Randall-Sinclair 1995)



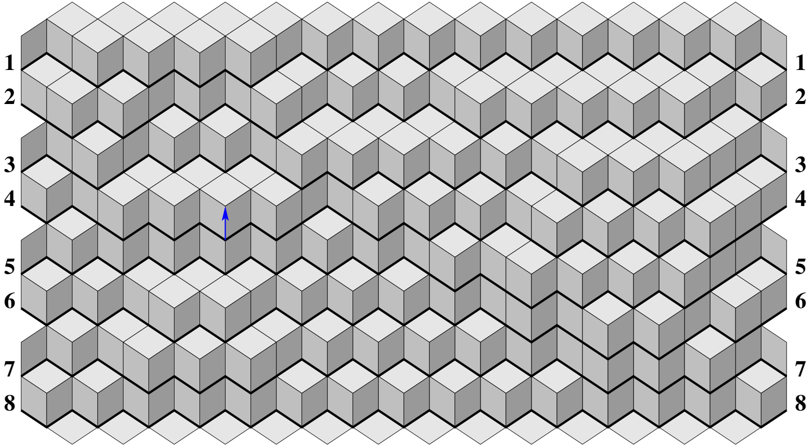
Dimer $\sim$ family of non-intersecting paths $\rightsquigarrow$ n two-letter words.

The dimer case (Luby-Randall-Sinclair 1995)



Flip on the dimer $\rightsquigarrow$ flip on one of these words.

The dimer case (Luby-Randall-Sinclair 1995)



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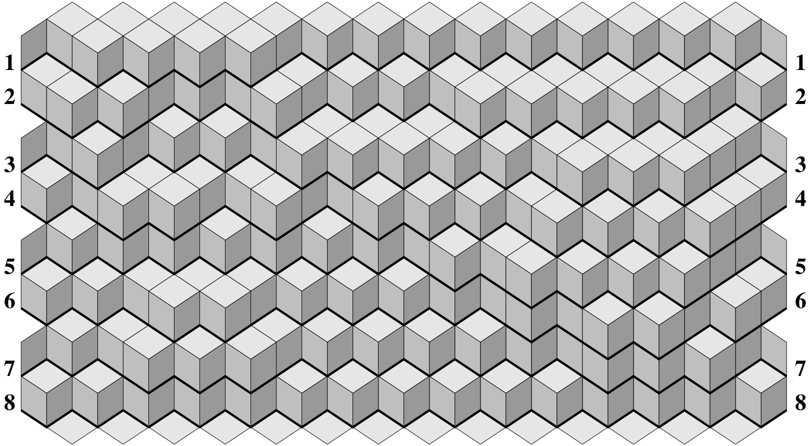

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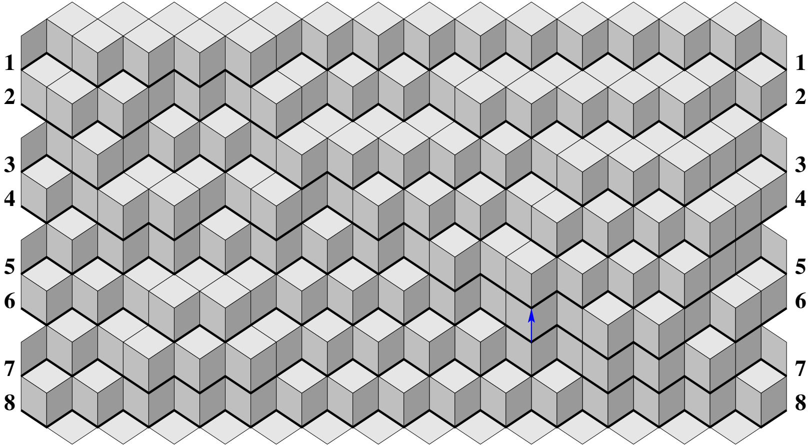
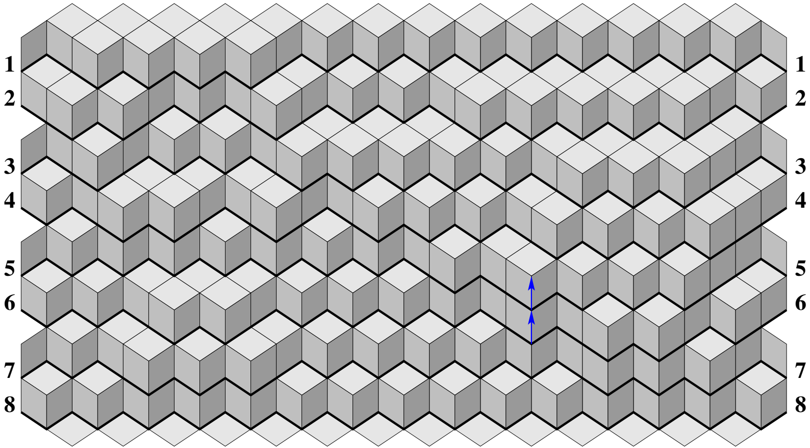


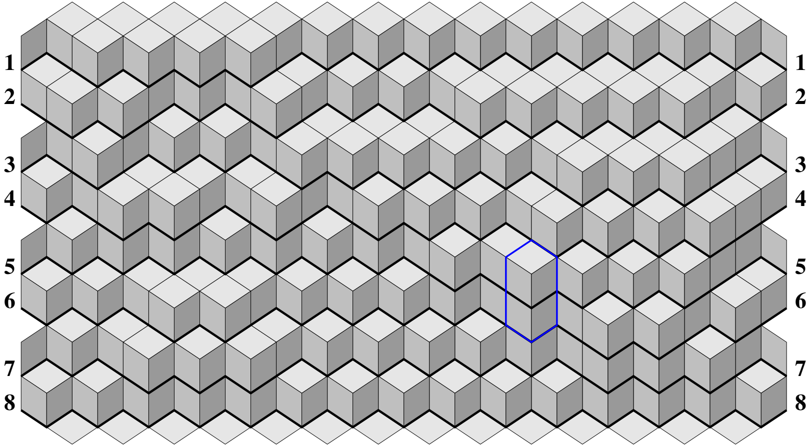
The converse does not always hold $\rightsquigarrow$ Markov chain modification.

The dimer case (Luby-Randall-Sinclair 1995)



Choose a vertex, a direction, and try to flip a *tower*.

The dimer case (Luby-Randall-Sinclair 1995)



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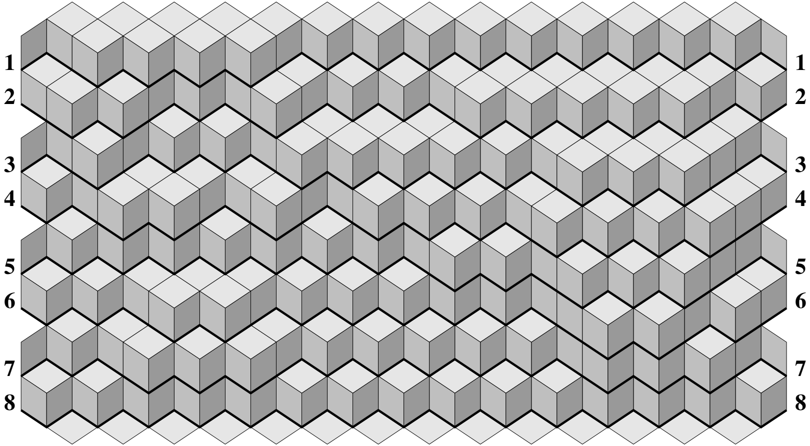

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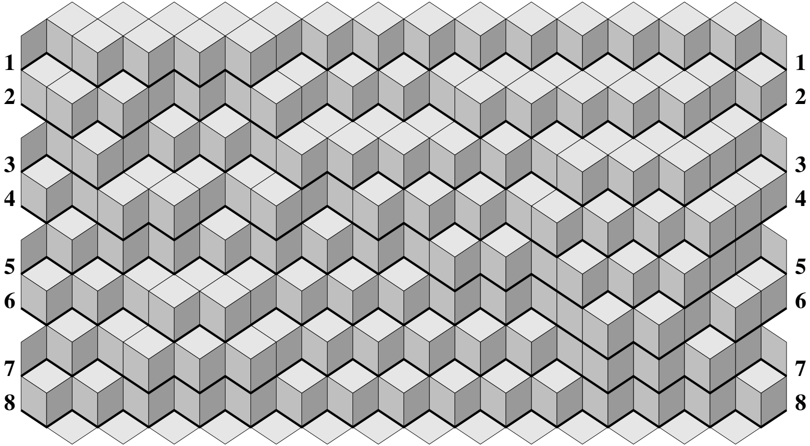
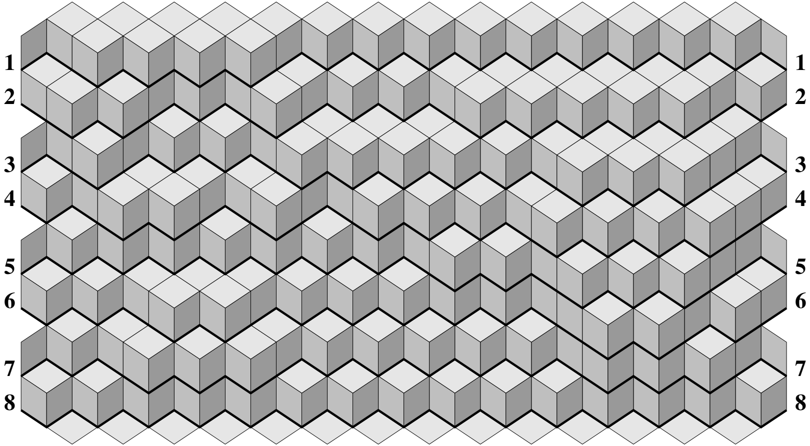


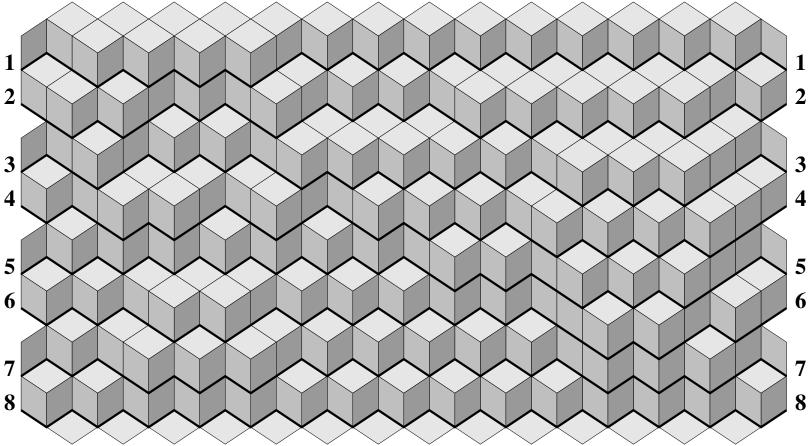
More precisely, flip a tower of height h with probability $\frac{1}{h}$.

The dimer case (Luby-Randall-Sinclair 1995)



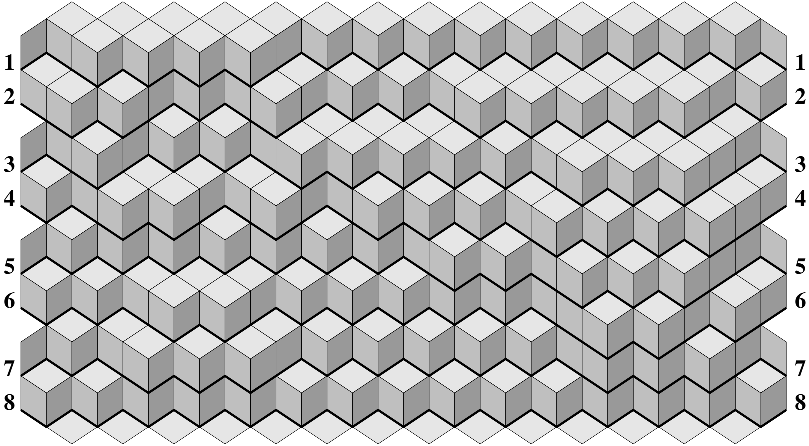
Still ergodicity and symmetric $\rightsquigarrow$ uniform stationary distribution.

The dimer case (Luby-Randall-Sinclair 1995)



Coupling $\rightsquigarrow V_t := \sum_i A_t^i \rightsquigarrow \mathbb{E}(\Delta V_t | V_t) \leq 0$ thanks to the factor $\frac{1}{h}$.

The dimer case (Luby-Randall-Sinclair 1995)



Here: $\max V_t \leq V_0 = n^3$ and $\Pr(|\Delta V_t| \geq 1) \geq \frac{1}{2n} \rightsquigarrow \tau = O(n^7)$.

Further cases? Other methods?

Tight bounds (Wilson 2001):

- two-letter case: $\tau = \Theta(n^3 \log(n))$;
- dimer case: $\tau = \Theta(n^4 \log(n))$.

Open question: Mixing times for general rhombus tilings?

Random sampling without flip?

- two-letter words: k updates $1 \rightarrow 2$ uniformly at random on 1^n;
- random boxed dimer tiling in $O(n^5)$ (Borodin-Gorin 2008);
- for more general rhombus tilings?

Some references for this lecture:

- Henry Cohn, Michael Larsen, James Propp, *The Shape of a typical boxed plane partition*, New York J. Math. **4** (1997).
- Henry Cohn, Richard Kenyon, James Propp, *A Variational principle for domino tilings*, J. Amer. Math. Soc. **14** (2001).
- Dieter Joseph, Veit Elser, *A Model of quasicrystal growth*, Phys. Rev. B **79** (1997).
- Michael Luby, Dana Randall, Alistair Sinclair, *Markov chain algorithms for planar lattice structures*, SIAM J. Comput. **31** (2001).

These slides and the above references can be found there:

`http://www.lif.univ-mrs.fr/~fernique/qc/`