

Telstar Balls Gone Wild

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Abstract

We describe an artistic project consisting of fabricating the 3532 different soccer balls that can be obtained by randomly assembling the 32 pieces of a classic Telstar soccer ball.

1 Soccer balls



Figure 1: The Telstar and the truncated icosahedron (WikiCommons).

Traditionally, soccer balls are made of leather pieces sewn together around an air bladder. Using multiple pieces is a necessity. Indeed, the ball has a positive (total) curvature, whereas the leather pieces have zero curvature.¹

¹Even with the famous spherical cow model, the curvature will be lower than that of a ball because it is inversely proportional to the square of the radius. Let us also emphasize that the total curvature remains zero when bending the leather piece like a cylinder, because only one of the principal curvatures is then affected, the other remaining zero – and thus the total curvature, being the product of these two curvatures, also remains zero.

Gauss's *Theorema egregium* [Gau28] shows that one cannot modify the curvature of a surface without changing its metric – which in our case would mean altering the very structure of the leather (this also explains why maps always distort the surface of the Earth in one way or another). Using multiple pieces allows the curvature to be concentrated at the edges of the pieces.

The problem then becomes designing pieces that distribute this curvature as evenly as possible over the entire surface, in order to maximize its sphericity [DT11]. The inevitable small remainder of non-sphericity is absorbed by the relative elasticity of the leather. Different strategies have been used [Wik24], the only limits to creativity seemingly being Pogorelov's theorem [Ghy14, Pog73]. One of the most famous strategies is undoubtedly that of the official ball of the 1970 World Cup, the Telstar.

The Telstar is made of hexagonal and pentagonal pieces assembled like the faces of a truncated icosahedron (Fig. 1). Curvature is thus harmoniously distributed over the 60 vertices,² forming a nicely round ball. Another advantage that explains the success of this ball is that the black and white polygons make it particularly telegenic (the rotation of the ball is especially visible).

2 Fullerenes

The truncated icosahedron structure of the Telstar is shared by a molecule discovered in 1985, the **Buckminsterfullerene C60** [KARW85]. In this structure, the 60 vertices are occupied by carbon atoms and the edges correspond to a covalent bond. A carbon atom has 4 valence electrons, and each vertex of the icosahedron has only three neighbors, leaving one free electron per carbon atom: these 60 electrons form a delocalized cloud over the entire molecule. Buckminsterfullerene is part of a larger family of chemical compounds: *fullerenes*.

A fullerene is made of hexagons or pentagons of carbon atoms, each having three covalent bonds and one delocalized electron. It thus has the structure of a planar graph whose vertices have degree 3 and whose faces are hexagons or pentagons. This structure forms a cage.³ It is of great interest in materials

²At each vertex, two hexagons and one pentagon meet, giving a total angle of $2 \times 120^\circ + 108^\circ = 348^\circ$, which is 12° of curvature relative to a flat angle of 360° . The 60 vertices yield a total curvature of $60 \times 12^\circ = 720^\circ = 4\pi$ as expected (Gauss-Bonnet theorem).

³It can also form a torus, a tube, or a sheet of graphene when the number of atoms

science for its particular properties, such as its exceptional stability (greater than that of diamond) or its conductivity due to the delocalized electrons [SHSG17, SWA15].

Two fullerenes having the same number of atoms but different structures are called **isomers**. The problem of determining all isomers of a given fullerene is classic [FM95]. Algorithms for enumerating isomers have been developed, notably **Buckygen** [BGM12].

Most of these isomers are **chiral**, meaning they are not superimposable on their mirror image and form a pair of **enantiomers** (like left and right hands). Even though their structures are very close, two enantiomers can have very different chemical properties, as illustrated by the tragic medical use of thalidomide, one enantiomer of which had the sought-after sedative anti-nausea effect while the other is teratogenic. An exact formula for the number of fullerenes (isomers and enantiomers) has recently been obtained [ESG25]. The study of isomers of a fullerene poses numerous mathematical problems [BBS25].

3 Balls gone wild

Let us now turn to our actual project. The Telstar ball has the same structure as Buckminsterfullerene. Thus, it also has isomers: these are the balls that can be obtained by assembling the 20 hexagons and 12 pentagons of the Telstar in all possible ways.

Like Buckminsterfullerene, the Telstar has 3532 isomers: 92 that are superimposable on their mirror image and 1720 pairs of enantiomers (i.e., $92 + 1720 = 1812$ different structures up to isometry). Each of these balls is unique, characterized by how its pieces are assembled, and therefore also by its shape, since the way the pieces are assembled governs the distribution of curvature over the surface. The project consists of fabricating a unique copy of each of these 3532 isomers, assembled from the standard pieces of a soccer ball.

This raises a first technical question, addressed in paragraph 4: how to describe each ball in a way that facilitates its assembly?

Furthermore, we have chosen to order all these balls from roundest to least round. More precisely, we used the classic notion of the **isoperimetric**

is large enough to ensure greater stability than the cage – this is not the case for the 60 atoms of Buckminsterfullerene.

quotient to measure this roundness. The isoperimetric quotient measures how spherical a solid is (it is maximal for a sphere). Formally, the isoperimetric quotient q of a solid with volume V and surface area S is defined by

$$q := 36\pi \frac{V^2}{S^3}.$$

Here, since all balls have the same surface area S (that of 12 pentagons and 20 hexagons) and the function $V \rightarrow V^2$ is monotonic, ordering by the isoperimetric quotient simply amounts to ordering by volume. Since a ball and its mirror image obviously have the same volume, we assigned a single number to each pair of enantiomers.

This raises a second technical question, addressed in paragraph 5: how to measure the volume of a ball?

4 Fabrication

A Telstar is made by assembling the pieces one by one with a saddle stitch (not forgetting to insert an air bladder before closing). The assembly rule is simple: just never sew two pentagons together (the *Isolated Pentagon Rule* (IPR)). Indeed, the Telstar is the only fullerene with 20 hexagons that has no two adjacent pentagons. What about the isomers?

Rather than seeking a more or less complex “rule” for each isomer, we aim to cut the polyhedron so that it can be unfolded onto a flat surface. The resulting figure is called a *net* or *unfolding*: simply folding it back yields the original polyhedron. More precisely, since we want to sew pentagons and hexagons along their edges, we look for an *edge unfolding*, meaning we only cut along edges. Figure 2 shows an edge unfolding of the truncated icosahedron (the Telstar) found in 1525 by Albrecht Dürer [Dü25].

Obtaining an edge unfolding of a Telstar isomer can be broken down into three steps:

1. determine the *dual* of the isomer, i.e., the graph whose vertices correspond to the faces of the isomer, two vertices being connected by an edge if and only if their corresponding faces share an edge;
2. choose a *spanning tree* of this graph, which is done by removing the maximum number of edges, in an arbitrary order, while preserving the graph’s *connectivity*;

L In anders das mach auß zweinzig sechsecker flachen feldern/ gleichseitig vnd wincklich/ so man darzu thut zwelf fünfecker flacher felder/ so die gleichseitig gegen den sechsecken seiden sind/ vnd in inen selbs auch gleich wincklich vnd ordentlich an cynander gesetzt werden/ wie ich das offen im plano hernach hab aufgerissen/ So man dann das alles zusamen schleust/ so würt ein corpus daraus/ das gewinnet zwey vnd sechsig ecke/ vnd neunzig scharpfer seiten/ dis Corpus rüret in einer hohlen kugel mit allen seinen ecken an.

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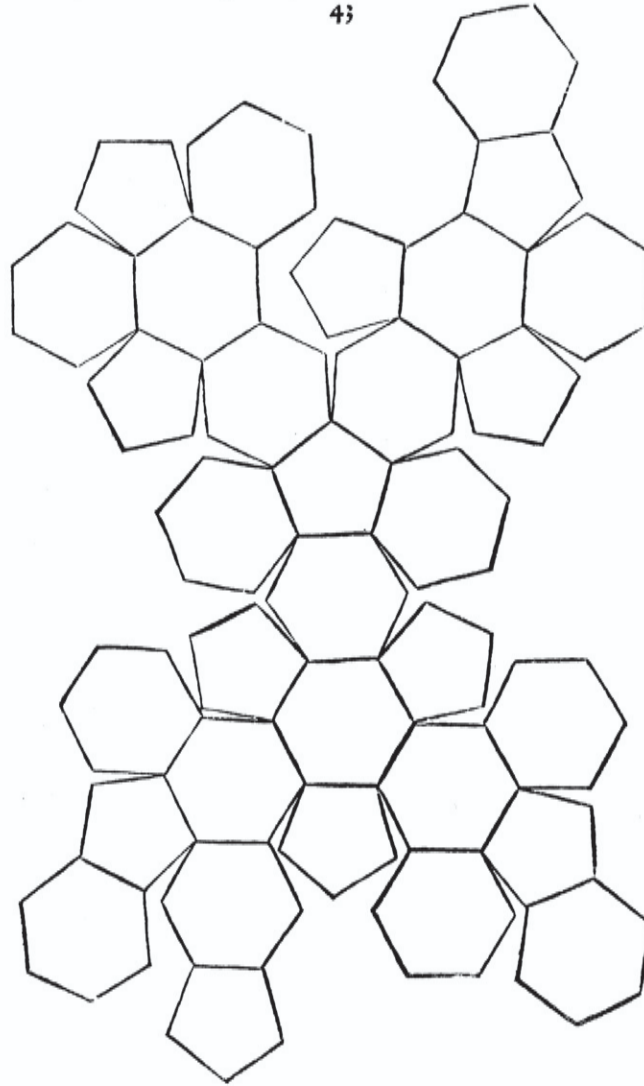


Figure 2: Edge unfolding of the truncated icosahedron (Dürer, 1525).

- cut the edges between two faces of the Telstar that correspond in the dual to vertices not connected by an edge of the spanning tree.

It then suffices to unfold to lay everything flat. The resulting edge unfolding depends on the spanning tree constructed in the second step. Different unfoldings of the Telstar are given in Fig. 3.

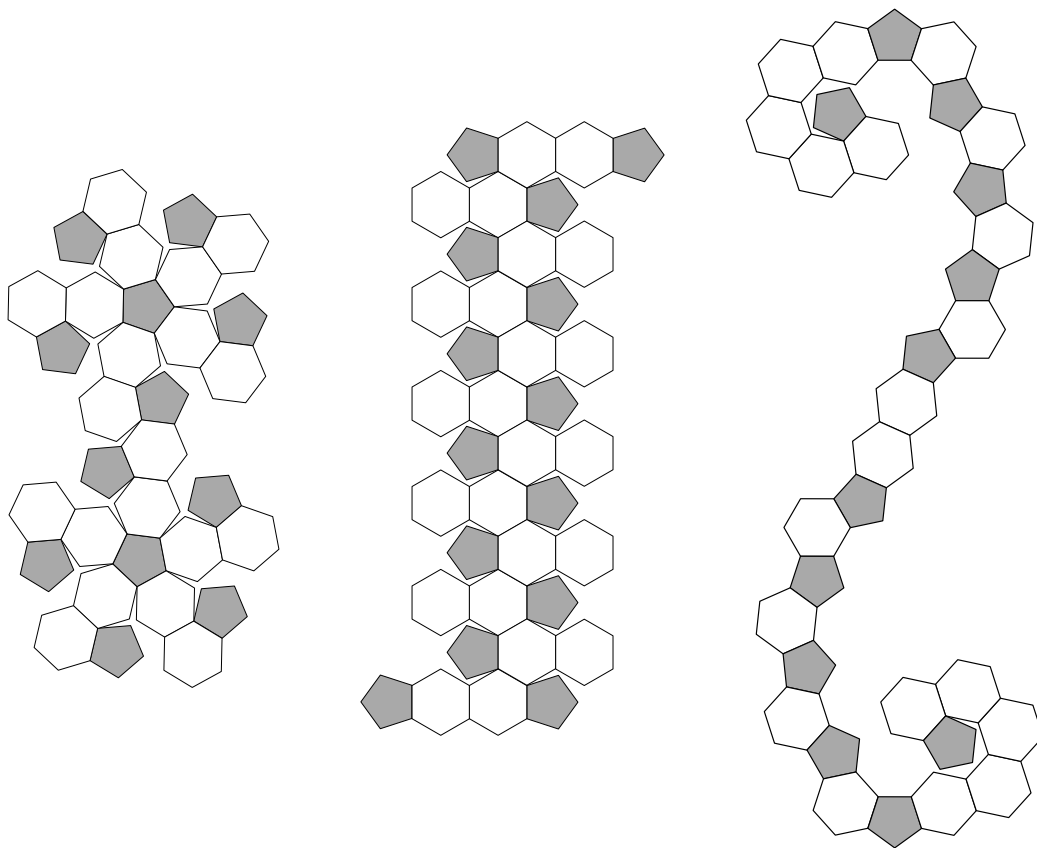


Figure 3: Three edge unfoldings of the Telstar. The first two are classic. The third is obtained by “peeling” the ball like an orange (more precisely, it is the unfolding of the spiral defined in [FM95]).

However, there is no guarantee that a given spanning tree yields a *valid* unfolding, i.e., one where there is no overlap of faces when unfolding into the plane! The question of the existence of a valid edge unfolding is indeed an open mathematical problem known as “Dürer’s Problem,” formalized in [She75] and widely studied (see, for example, [O’R11], Chap. 7, or [DO07],

Chap. 22). While counterexamples are known for non-convex polyhedra, the case of convex polyhedra – and in particular fullerenes – remains an open question! In the case of Telstar isomers, however, we were easily able to find a valid edge unfolding for each isomer. We even found many, and the problem that arose was rather: which one to choose? Which is the most practical for the seamster who is supposed to use it to assemble the ball? We hypothesized that a good unfolding is a “compact” unfolding. For example, the most compact of the four unfoldings of the Telstar shown in Fig. 3 is the last one, while the least compact is the third (very nice but probably not very practical for the seamster). Formally, we followed the following heuristic:

1. choose an arbitrary initial face;
2. add a face that minimizes the distance to the initial face;
3. repeat the previous step until all faces have been added.

A priori, we could have encountered an isomer that leads to a deadlock situation where no further face can be added without creating an overlap, but this never happened (regardless of the choice of the initial face). Furthermore, on the unfolding intended for the seamster, we also numbered the faces and added markers to indicate the seams between faces that end up far apart in the unfolding, see Fig. 4.

5 Modeling

Since the balls are numbered by decreasing volume, we need to determine this volume. A naive solution is, of course, to make each ball and then immerse it in a container of water and measure the displaced volume. . . But besides the impracticality of the process, it requires fabricating *all* the balls before being able to number them. Therefore, we adopted another approach: model each ball computationally and thus estimate its volume.

An algorithm like buckygen ([BGM12]) allows generating all balls as *abstract graphs*: vertices are faces and edges connect two adjacent faces, but nothing specifies how the faces are embedded in space. To define such an embedding, we proceed in three steps.

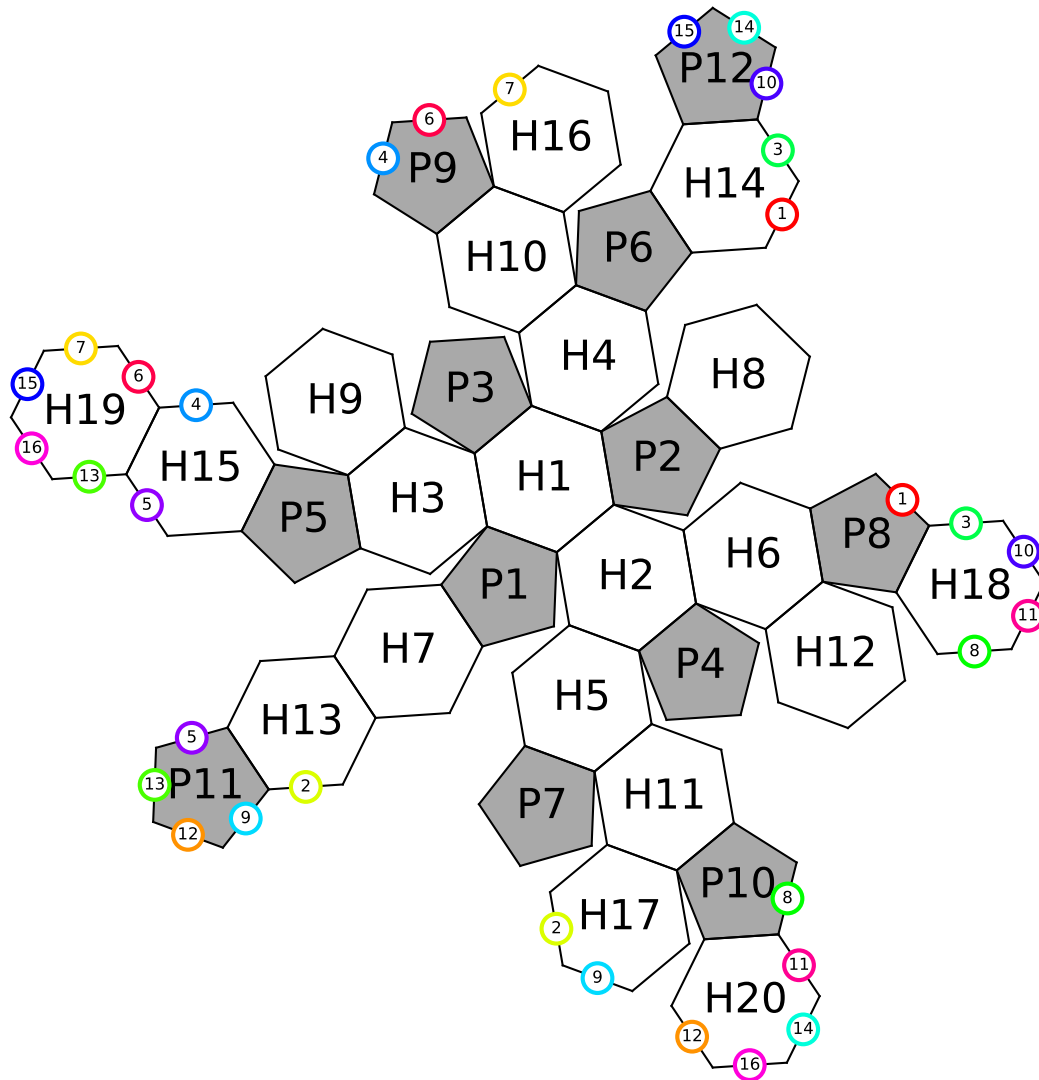


Figure 4: A “compact” net of the Telstar, with numbered faces and markers for certain seams.

First step. First, the unfolding of the ball is “pressed” onto a sphere. More precisely, each vertex of the unfolding is mapped onto the unit sphere tangent to the plane of the unfolding via the inverse of the stereographic projection (a vertex of the ball generally appears multiple times in the unfolding: we randomly choose one of these occurrences). Although very crude (Fig. 5, left), this embedding proves to be a good basis for being transformed into a more realistic embedding, quickly and without running into self-intersecting embeddings and other inextricable configurations.

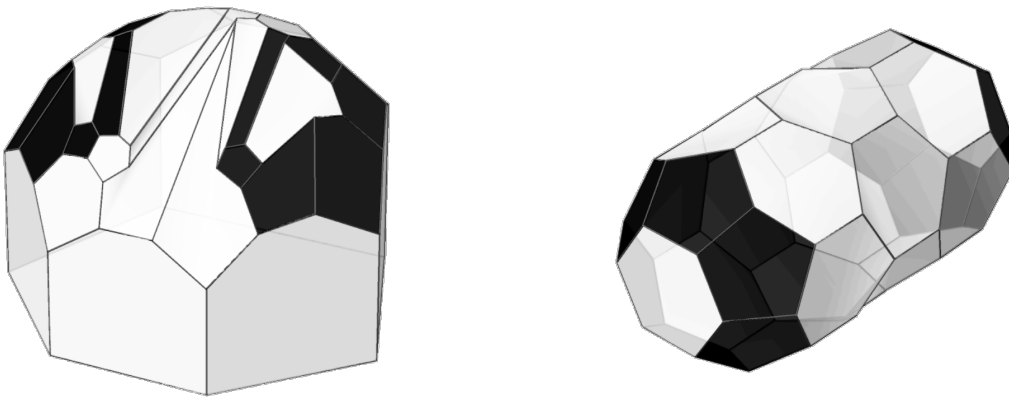


Figure 5: On the left, the embedding obtained by stereographic lifting of an unfolding (all vertices lie on a single sphere). The faces are not regular at all, but at least it is a topological sphere. On the right, the same embedding after iterations of forces between vertices. The ball is still very angular but is beginning to take shape.

Second step. An iterative algorithm modifying the embedding by “forces” is then applied. The idea of this type of algorithm is to imagine that the edges of the embedded graph are springs that try to return to their equilibrium length by stretching or contracting. This is modeled by forces exerted by the spring-edges on each vertex, forces that cause the vertices to move. A repulsive force between distant vertices is also generally added. Here, we followed the variant proposed in [Ead84], putting springs not only on the edges of the graph, but also, within each face, between pairs of non-adjacent vertices. This latter point is intended to model the fact that the faces are not flexible polygons, but leather pieces with limited elasticity.

Third and final step. The embedding begins to take shape, but the vertices of a given face are generally not coplanar, i.e., the faces cannot be flat (Fig. 5, right). This is perfectly normal, because one can demonstrate that only the truncated icosahedron (the Telstar) can be realized with perfectly flat faces ⁴ But they can certainly be “curved” like rolling a sheet of paper into a cylinder ⁵. How to model this? We implemented the Catmull-Clark algorithm [CC78], which consists of iteratively subdividing a surface to round it. Three subdivisions seem a good compromise between ball roundness and model complexity (Fig. 6).

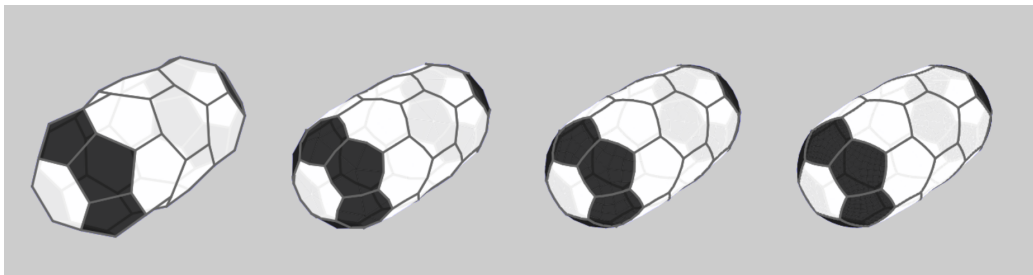


Figure 6: From left to right: successive Catmull-Clark subdivisions.

The actual volume of the ball is then estimated from this embedding: the faces (after Catmull-Clark subdivision they are quadrilaterals) are divided into triangles from their center, then for each triangle we consider the tetrahedron obtained by adding a point inside the ball (the center of mass, for example), and the volume of the ball is defined as the sum of the volumes of all these tetrahedra.

We have begun hand-stitching a unique, numbered piece of each ball in natural leather. So far, only about a dozen balls have been hand-stitched (see, e.g., Fig. 7), and the process is ongoing. The model of every ball can be viewed here:

<https://lipn.univ-paris13.fr/~fernique/gallery/animball.html>

⁴Indeed, the dihedral angle between two adjacent faces is then determined by the type of the three faces meeting at one of its vertices – one then shows by “propagation” that all vertices necessarily see the same three types of faces, which only allows the dodecahedron of pentagons, the infinite plane of hexagons, and the truncated icosahedron. Only the last is a Telstar isomer.

⁵The already cited theorem of Pogorelov [Ghy14, Pog73] then assures us that any fullerene can be realized, even with pieces that are not at all elastic, for example, paper.

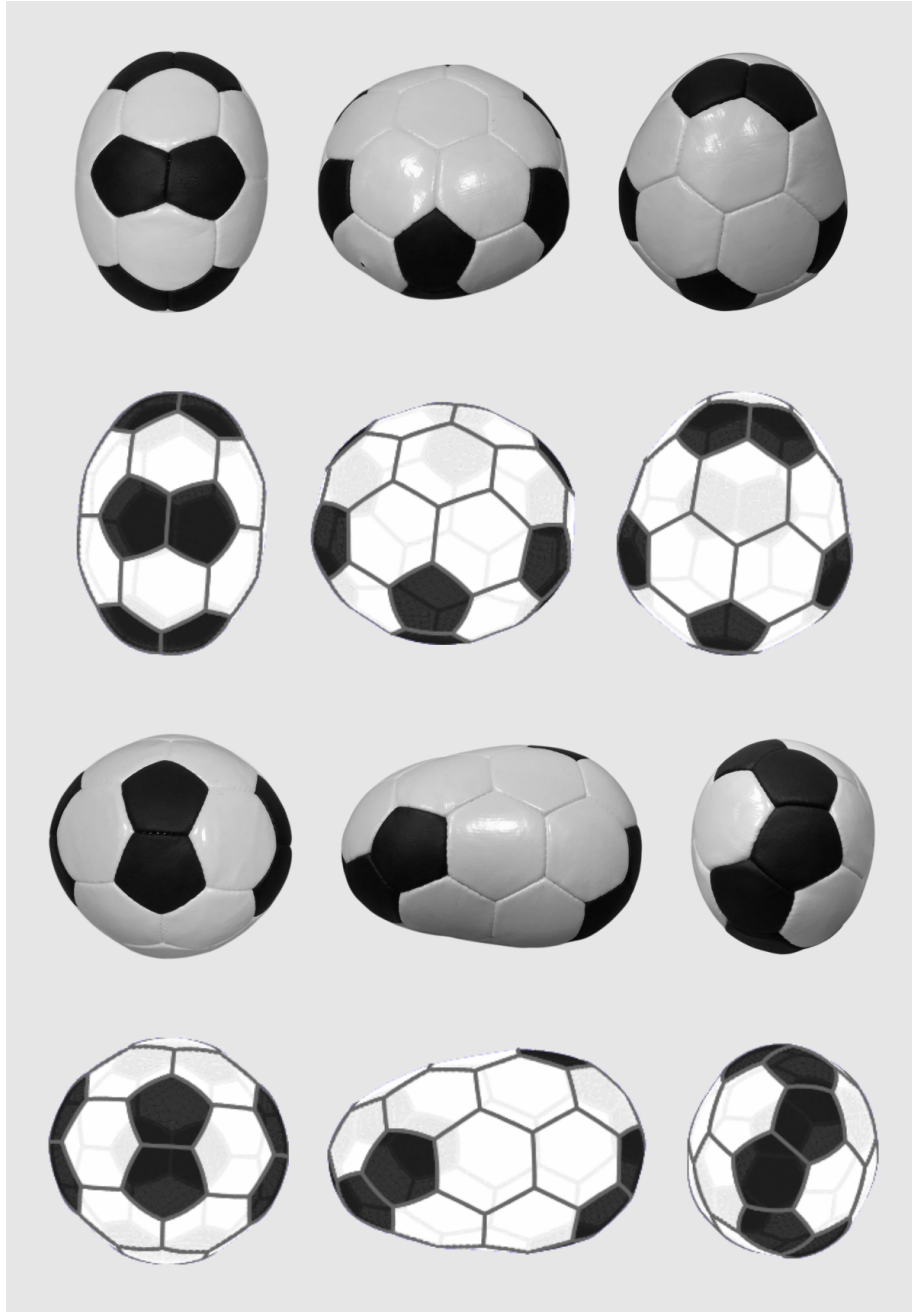


Figure 7: Some actual balls and their modeling.

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