Visualizing Cut and Project Tilings

Thomas Fernique

Outline

Outline

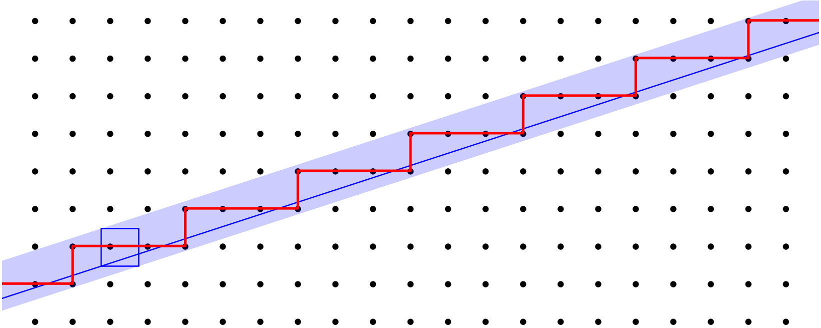
Cut and project tilings

Looking through the window

Shifting the slope

Tilting the slope

$2 \to 1$ tilings



Line $E \to$ “stripe” $E + [0,1]^2 \to$ stepped line $\to$ tiling of E.

$3 \to 2$ tilings

Plane $E \to$ "slice" $E + [0,1]^3 \to$ stepped surface $\to$ tiling of E.

$n \to d$ tilings

Definition

A $n \to d$ tiling is the (orthogonal) projection onto a d-plane $E \subset \mathbb{R}^n$, called the slope, of the d-dim. facets of $\mathbb{Z}^n$ included in $E + [0,1]^n$.

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Irrational slope (no rational line) $\to$ non-periodic tiling.

Ammann-Beenker tilings

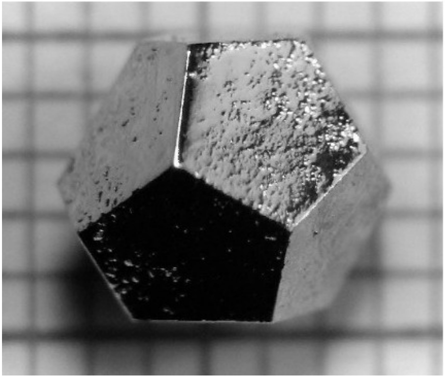
Slope generated by $\cos(k\pi/4)_{0\leq k<4}$ and $\sin(k\pi/4)_{0\leq k<4}$.

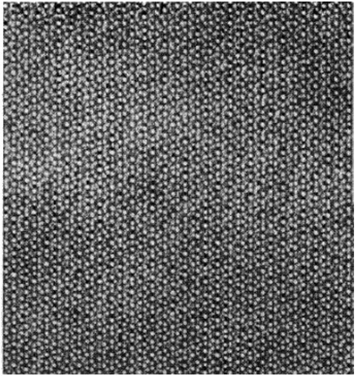
Penrose Tilings

Slope generated by $\cos(2k\pi/5)_{0\leq k<5}$ and $\sin(2k\pi/5)_{0\leq k<5}$, shifted to contain a point whose coordinates sum up to one.

Quasicrystals

Such tilings are often used to model so-called quasicrystals:





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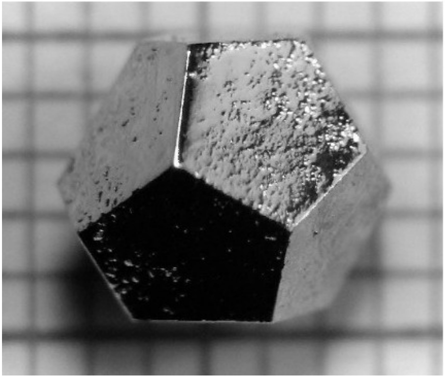
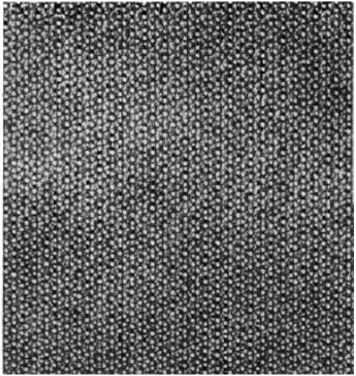


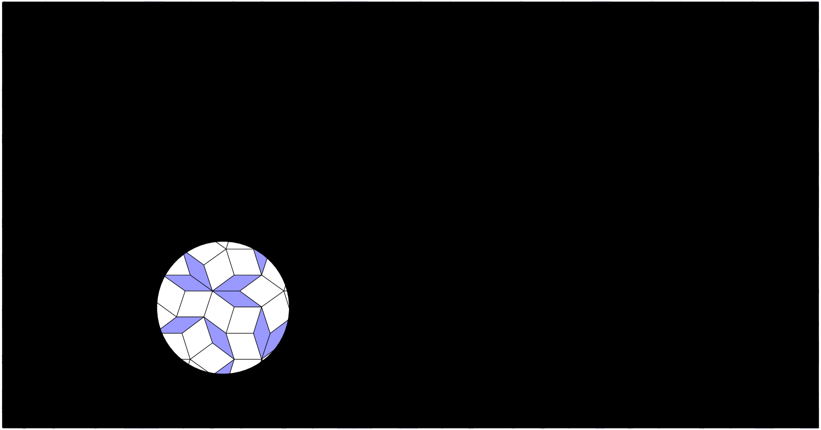


But materials are stabilized by short range energetic interactions.
They might be not aware about irrational d-dim. planes in $\mathbb{R}^n$...

Local characterization

Is there some finite set of finite patterns characterizing the tiling?

Local characterization



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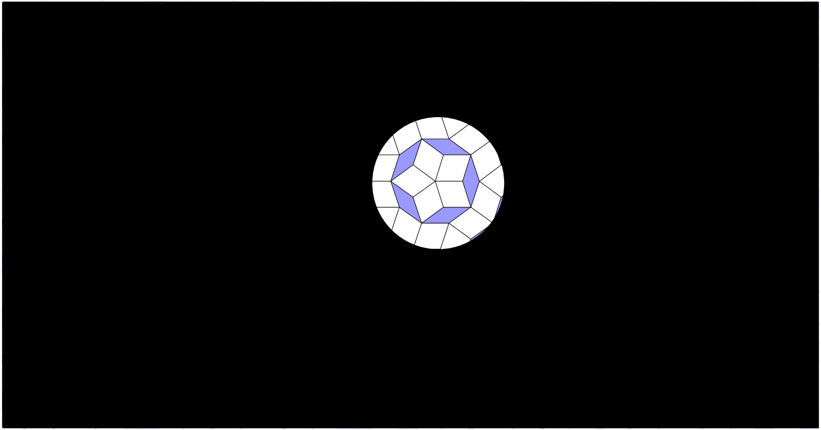

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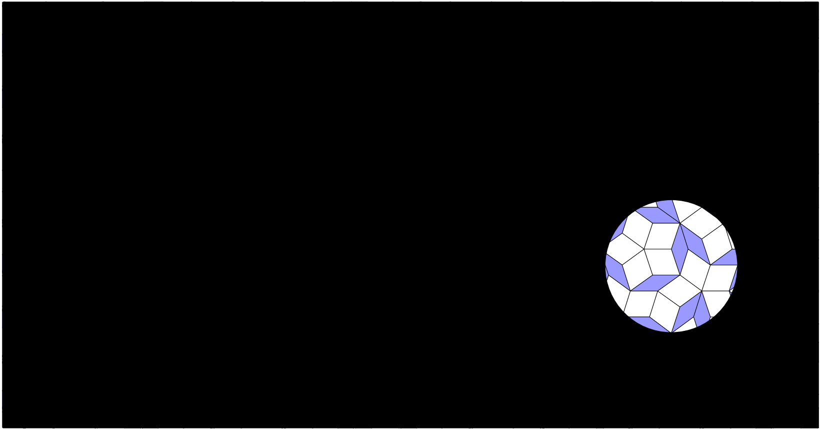

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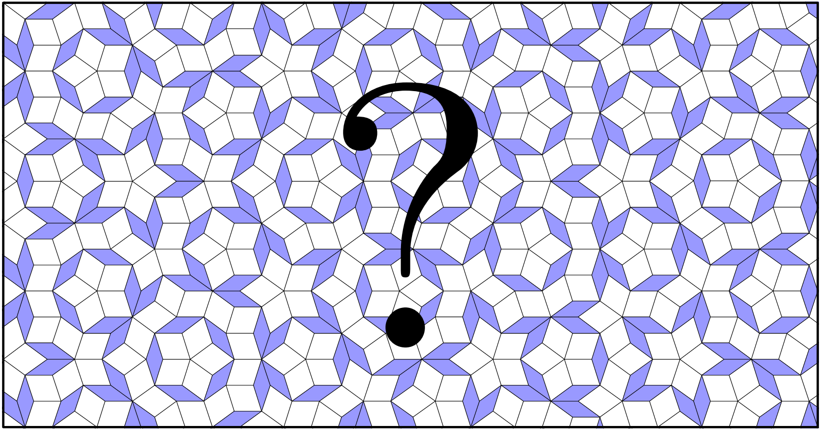

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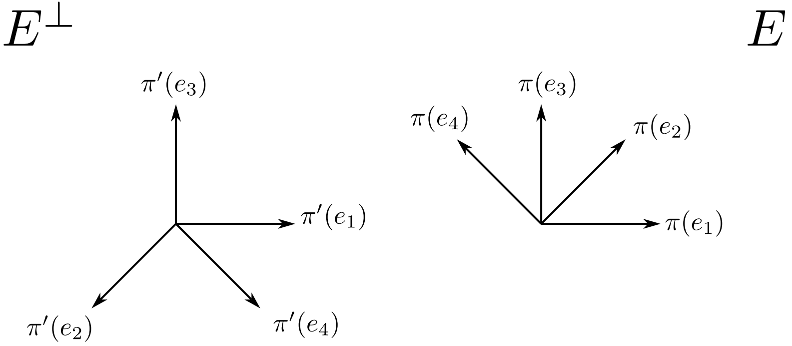
A point $x \in \mathbb{Z}^n$ that lies in $E + [0,1]^n$ thus corresponds to

- a vertex $\pi(x)$ of the tiling
- a point $\pi'(x)$ in W

Patterns in the window



Consider, e.g., a 2 plane E in $\mathbb{R}^4$.

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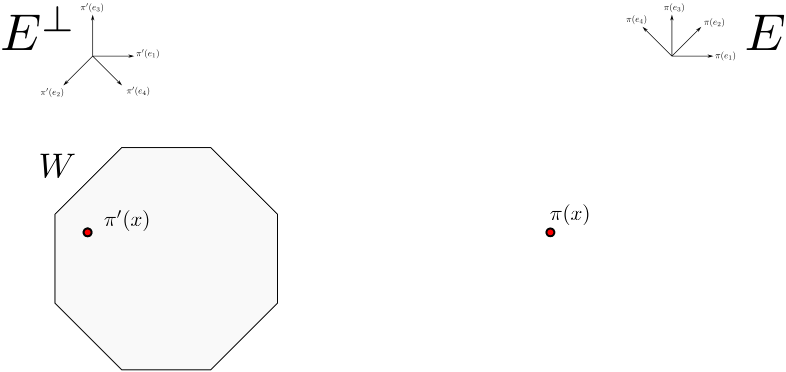

Let $x \in \mathbb{Z}^n$ be a point that lies in $E + [0,1]^n$.

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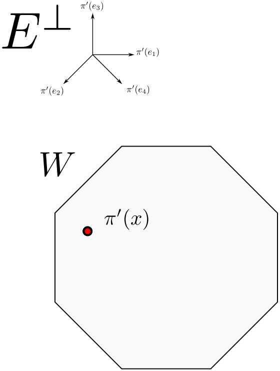

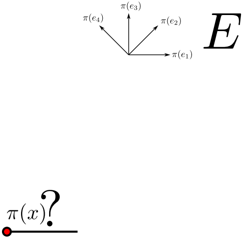

Does the tiling have an edge from $\pi(x)$ to $\pi(x) + \pi(e_1)$?

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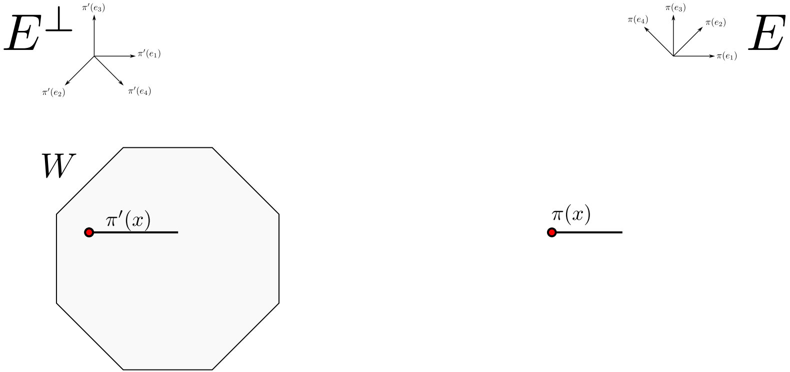
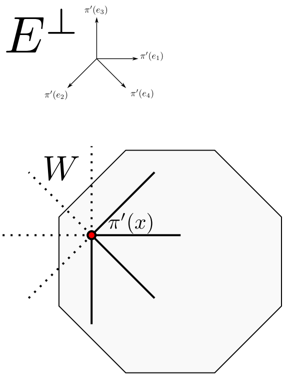
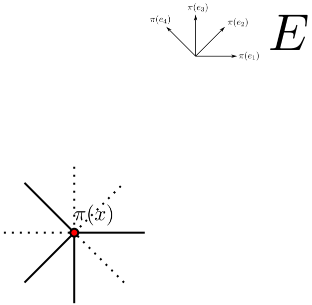


By definition: yes iff moving from $\pi'(x)$ by $\pi'(e_1)$ leads into W.

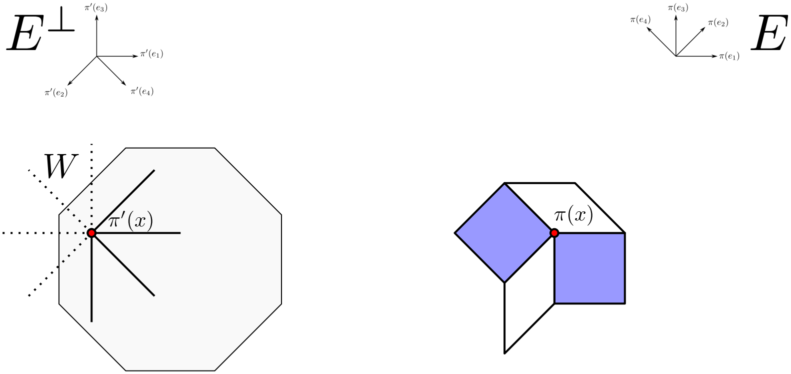
Patterns in the window





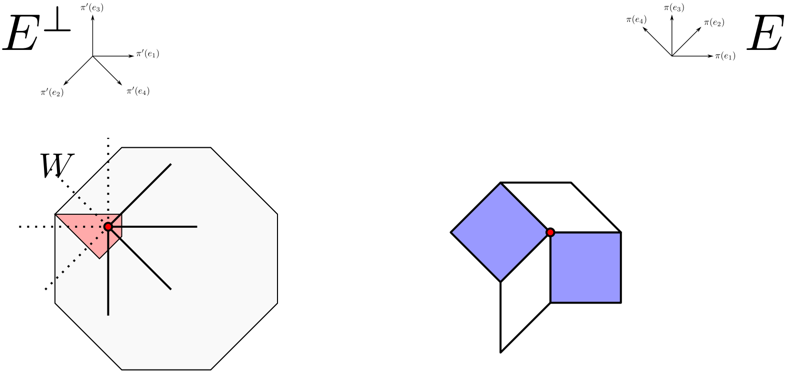
The same holds in every direction.

Patterns in the window



This yields the pattern around $\pi(x)$ in the tiling!

Patterns in the window



The same holds for every $\vec{y}$ s.t. the exact same moves from $\pi'(y)$ lead into W. This associates a polyhedral cell with the pattern.

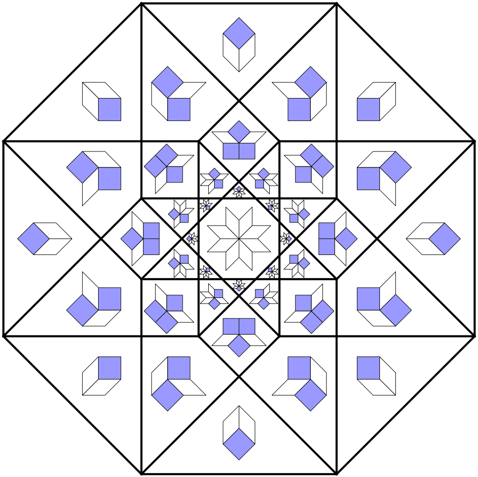
Maps and atlas

Definition

Given a vertex x of a tiling and $r \in \mathbb{N}$, the r-map of center x is the pattern formed by the tiles that contains a vertex within r edges from x.

The r-atlas of a tiling is the set of its r-maps.

Previous slide $\rightsquigarrow$ partition of W in cells associated with some r-atlas.



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Points are even *uniformly distributed*: the proportion of points $\pi'(\{0, \ldots, N\}^n)$ inside $A \subset W$ tends to $\mu(A)/\mu(W)$ for $N \to \infty$. Cells areas thus yield pattern *frequencies*.

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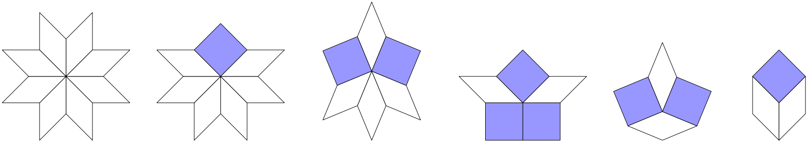
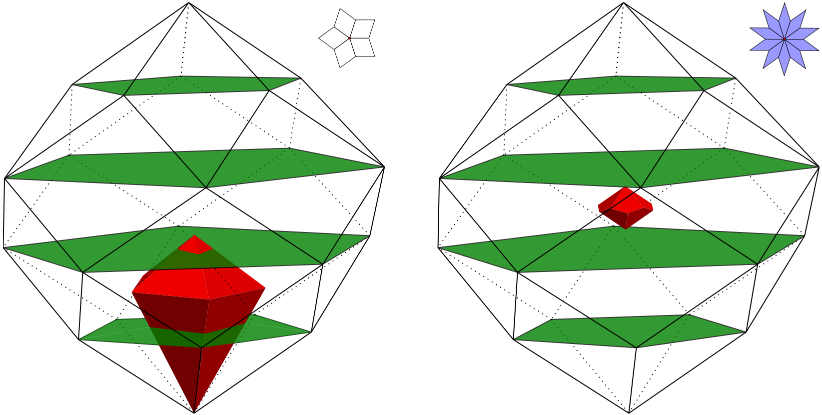
For Ammann-Beenker tilings:



$17-12\sqrt{2}$	$29\sqrt{2}-41$	$34-24\sqrt{2}$	$10\sqrt{2}-14$	$6-4\sqrt{2}$	$\sqrt{2}-1$
2.94%	1.22%	5.89%	14.21%	34.31%	41.42%

Penrose tilings

For Penrose tilings, $\pi'(\mathbb{Z}^n)$ densely fills only *sections* of W:



To $r \in \mathbb{N}$ corresponds a partition of W where cells which intersect the sections $\pi'(\mathbb{Z}^n)$ yields the patterns of the r-atlas.

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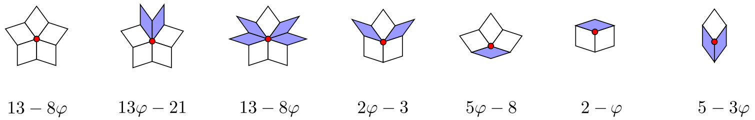

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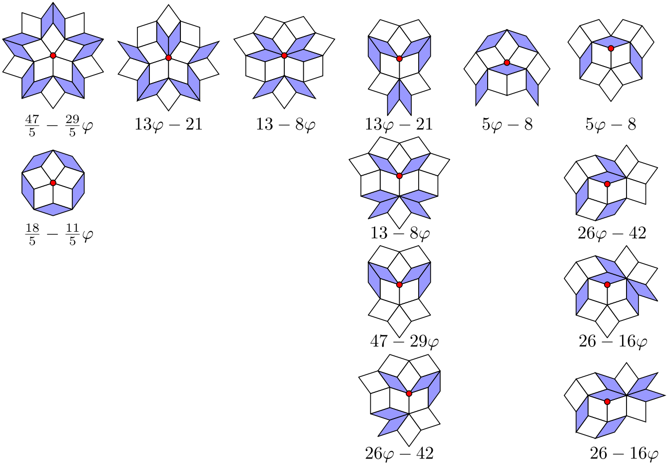

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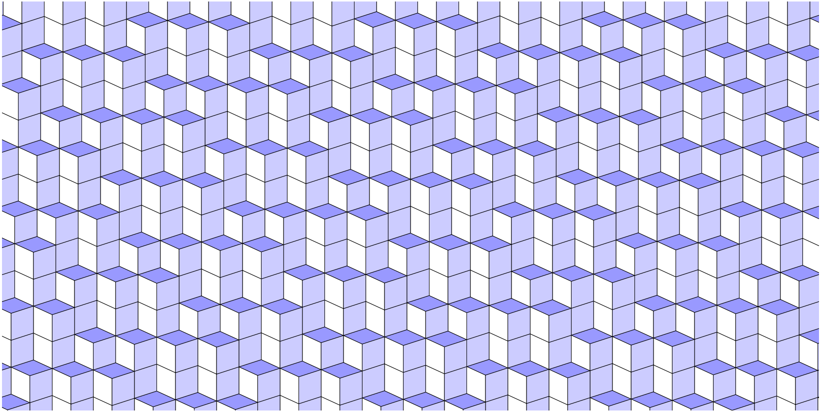

Shift from E to $E + t$, for some $t \in \mathbb{R}^n$ yields sparse "flips".
Same finite patterns/frequencies since $\pi'\mathbb{Z}$ is unif. distrib. in $E^\perp$.

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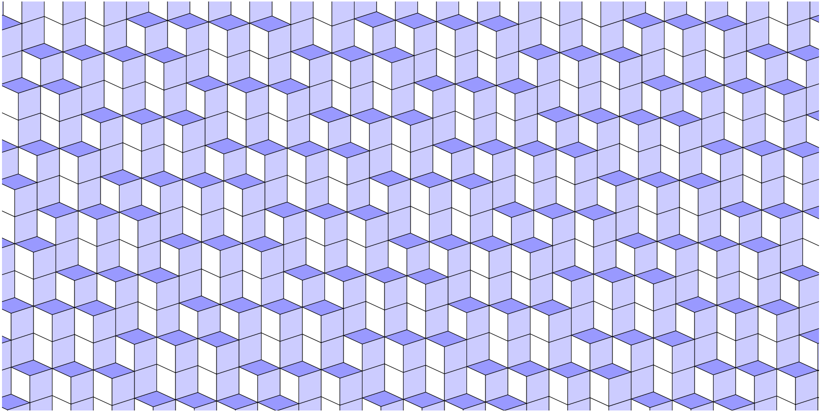

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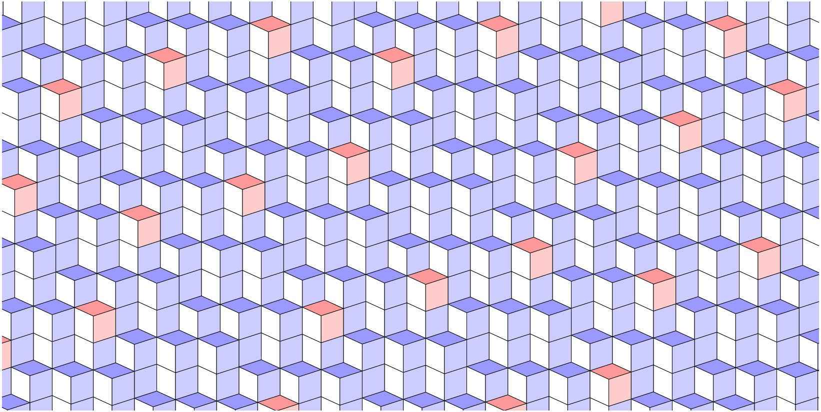

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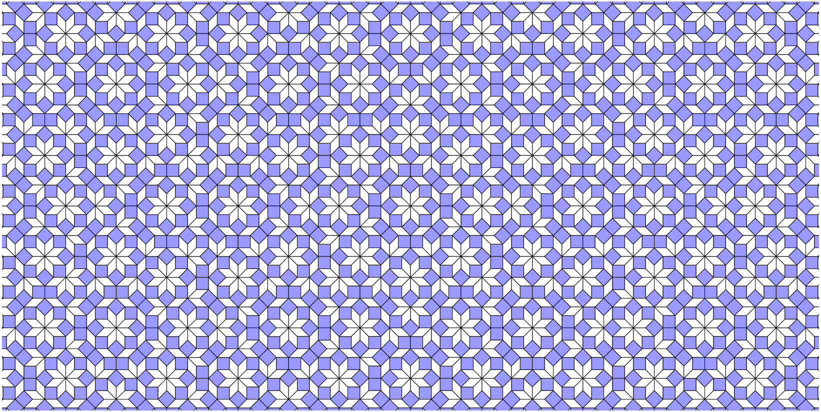

Still same finite patterns/frequencies, but <u>lines</u> of flips!?

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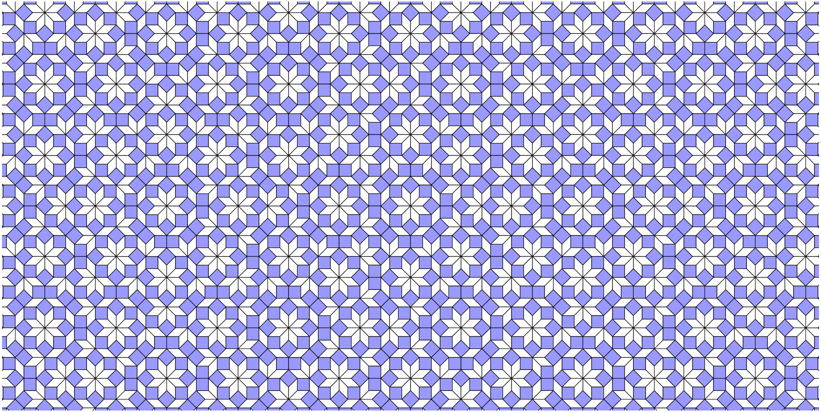

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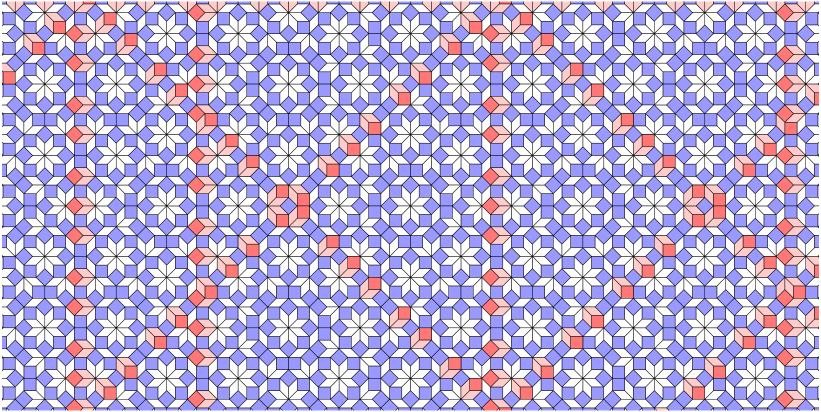

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$$(\sqrt{2}, 1, 0, -1), \quad (1, \sqrt{2}, 1, 0), \quad (0, 1, \sqrt{2}, 1), \quad (1, 0, -1, -\sqrt{2}).$$

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A subperiod $\vec{p} \in E$ can be written $\vec{p} = \vec{q} + \vec{x}$ with

- $\vec{q} \in \mathbb{Z}^n$ (the "floor part"),
- $\vec{x}$ in a $n-d-1$ dim. unit face of $\mathbb{Z}^n$ (the "fractional part").

In particular, $\pi'\vec{x}$ is in a facet of the window.

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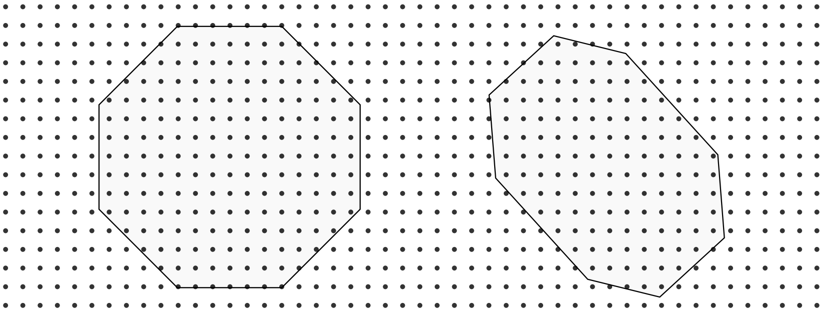
In particular, $\pi'\vec{x}$ is in a facet of the window. Hence $\pi'\vec{q}$ as well:

$$\pi'\vec{p}=0=\pi'\vec{q}+\pi'\vec{x}.$$

This "yields" lines of vertices that project in a facet of the window.

In the window



Shift: case with a subperiod (left) *vs* generic case (right).

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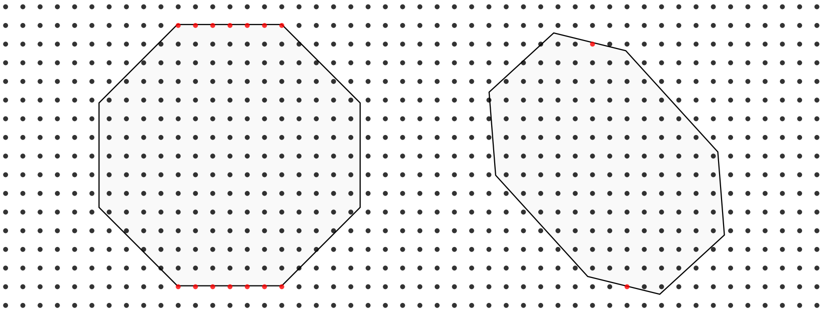

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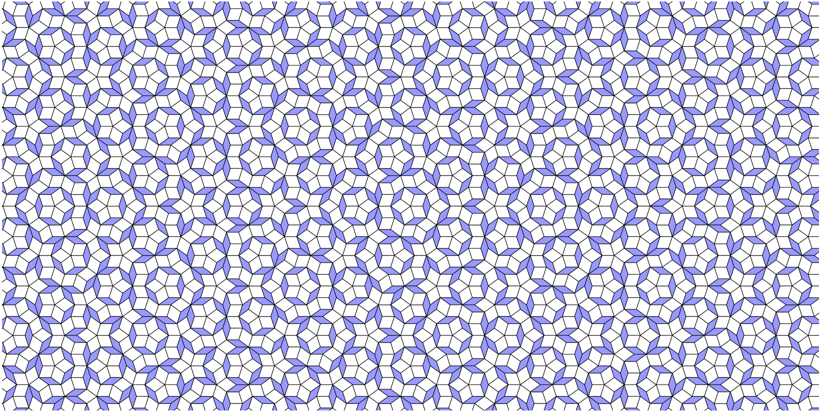
Penrose tilings



A similar phenomenon occurs for Penrose tilings.

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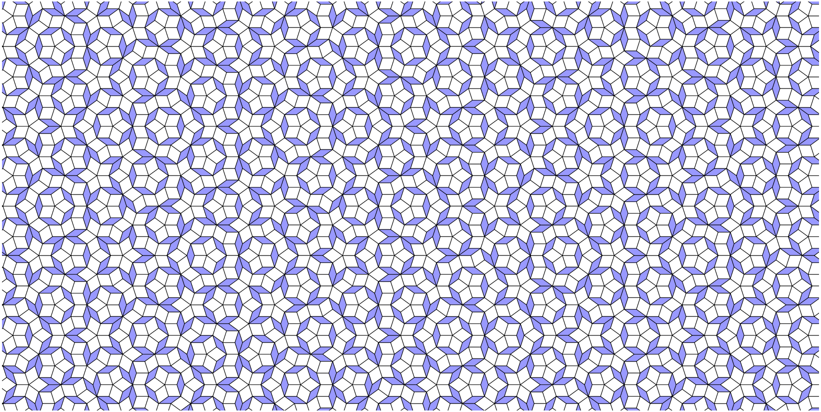

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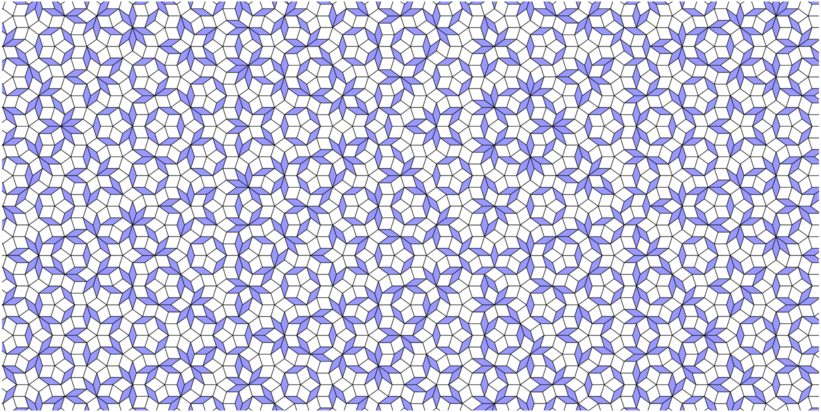

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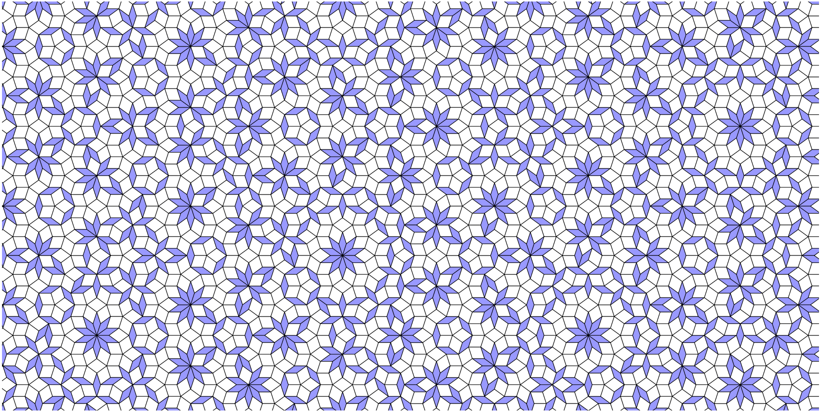

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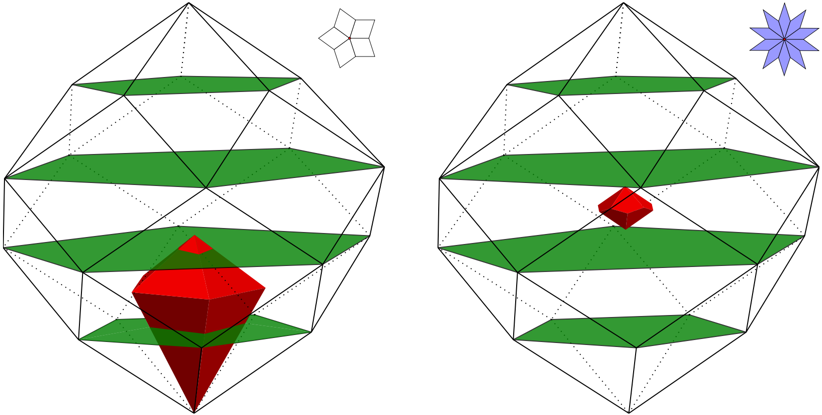

Penrose: E contains a point whose coordinates sum up to one.
"Transversal" shifts yield generalized Penrose tilings.

Outline

Coincidences

When the slope E is modified into a (non-parallel) E':

- the window and cells associated with patterns are deformed;
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How cells are create/deleted?

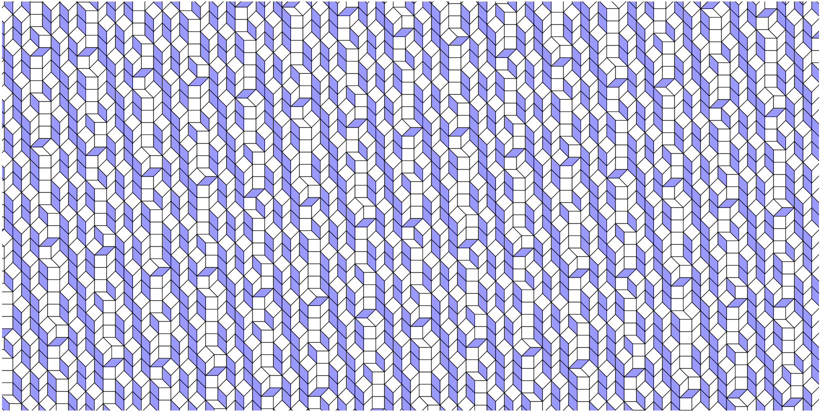
The limit case is a cell reduced to a point:

Definition

A coincidence of a $n \to d$ tiling is a set of $n-d+1$ unit faces of $\mathbb{Z}^n$ of dim. $n-d-1$ with a common intersection in W.

E.g. 3 lines in $\mathbb{R}^2$ for Ammann-Beenker; 4 planes in $\mathbb{R}^3$ for Penrose.

Generic case



Generic case: no coincidence.
The r-atlas is unchanged by a slight enough tilt.

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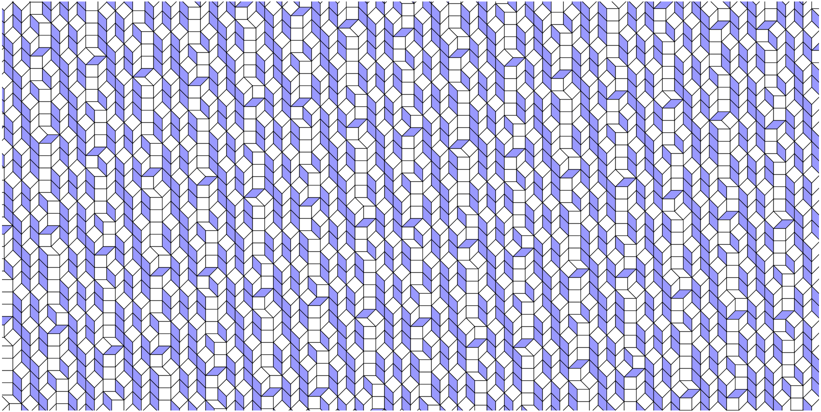

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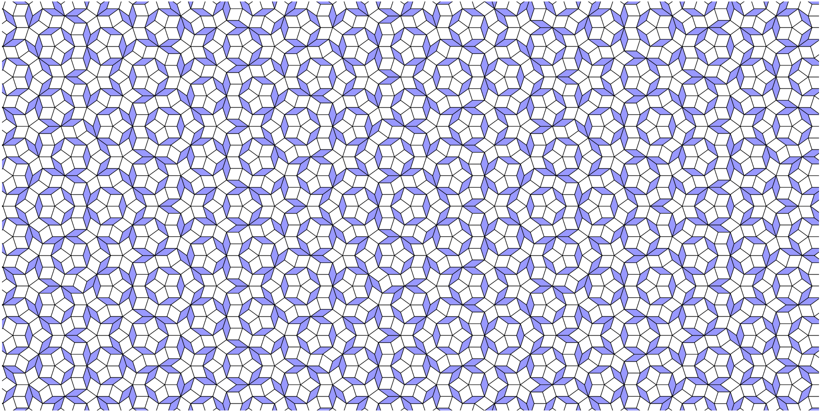

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Finitely many of them even suffice to characterize the slope.
Every tilt hence destroys some of them and creates new patterns.

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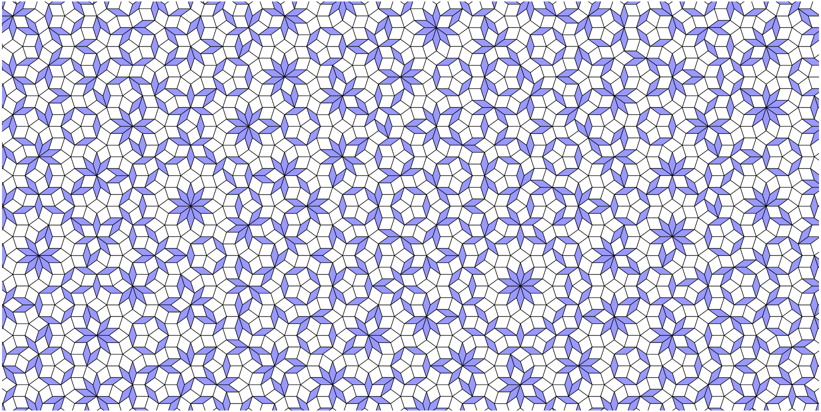

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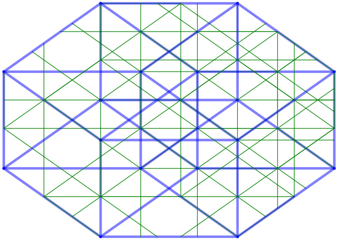
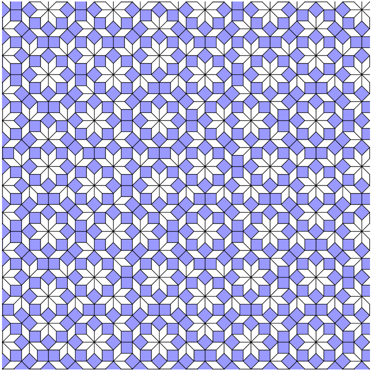
Ammann-Beenker tilings





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However, they characterize a one-parameter family of slopes.
The r-atlas is unchanged by a slight enough tilt along this curve.

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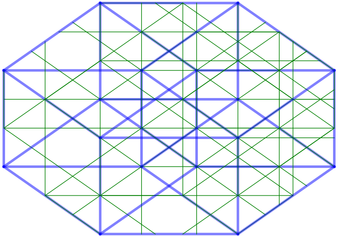

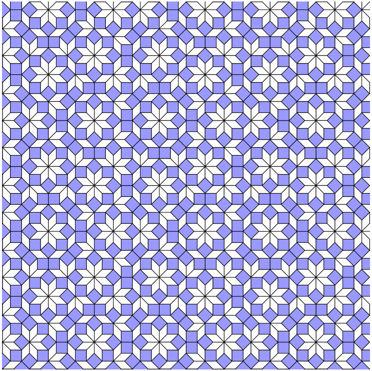

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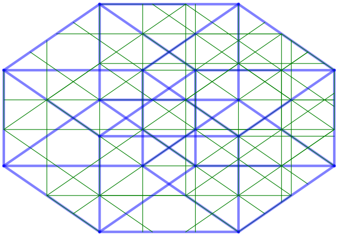

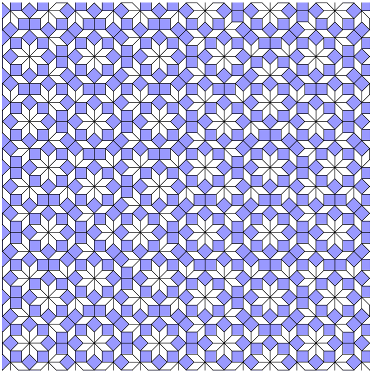

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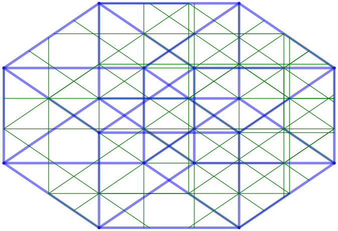

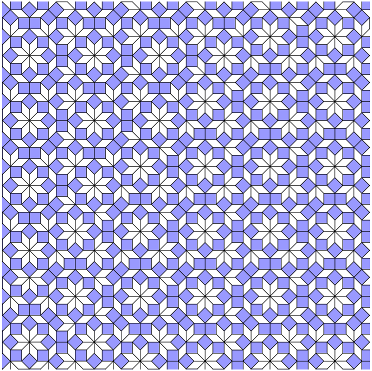

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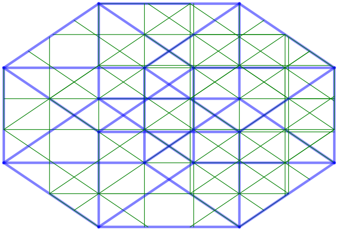

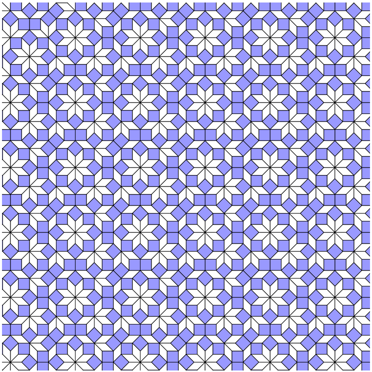

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Et voilà.

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What about local characterization?

Where are the theorems!?

Triangulated packing with three sizes of disks

Thomas Fernique

The problem

Definition

A packing of disks is a set of interior disjoint disks (in $\mathbb{R}^2$).
It is is triangulated if its contact graph is a triangulation.

Example: the hexagonal compact packing (equal disks).

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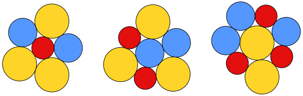
Triangulated packings are good candidates to maximize the density and predict structures for new materials made of, e.g., nanodisks.

Proof sketch

First, prove there is finitely many <u>local</u> possibilities:



For that: polynomial systems, interval arithmetic, various tricks.

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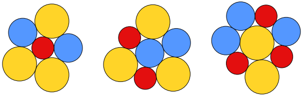

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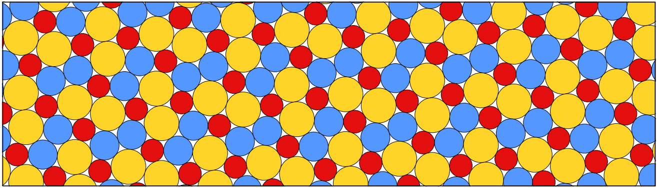
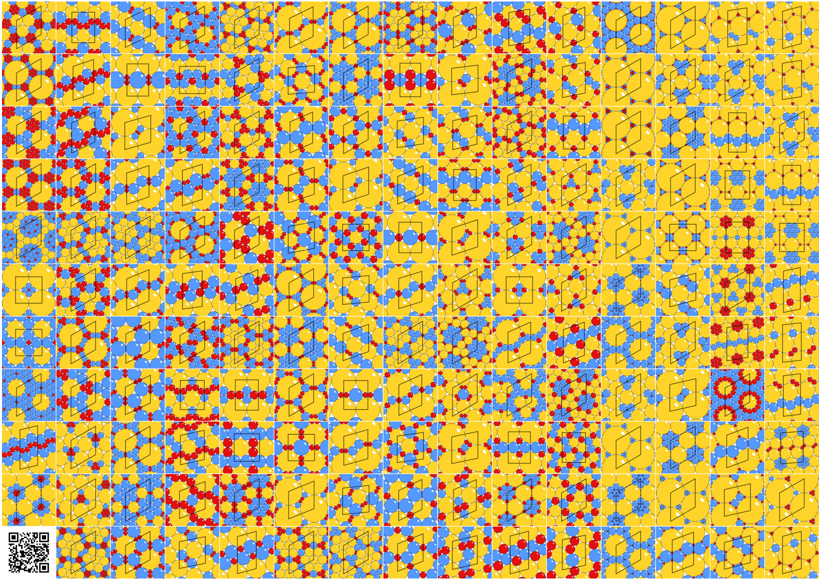
Then, proves that this can be extended over the whole plane:



For that: draw the packings! (11 page appendix in the paper).

In a single picture



An applet to investigate all the 164 cases

Telstar Balls Gone Wild

Thomas Fernique

Telstar

This is a soccer ball with a shape of truncated icosahedron.
Equivalently: sew 20 hexagons and 12 isolated pentagons!

Let's forget isolation rule!

Many new "balls" becomes possible, actually 1812 up to isometry. Hard to play soccer with. Known as fullerenes in chemistry.

An applet to visualize them all

The concept: to realize a unique version of each variant.
First three pieces: acquired by the Palais de la Découverte.
Order yours at fernique@lipn.fr