Conformal Field Theory (CFT) with central charge $c = 1$ coupled to gravity

Vincent Vargas[1] [2] [3]

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[1] DMA, ENS Paris
[2] Joint work with Rhodes
[3] Other collaborators: Duplantier, Lacoin, Sheffield

Outline

1. CFT coupled to Liouville Quantum Gravity
   - Framework
   - CFT with central charge $c < 1$
   - CFT with central charge $c = 1$ or $2d$-string theory

2. The building blocks of $2d$-string theory
   - The Liouville measure
   - Other Tachyon fields
   - Liouville Brownian motion

Framework

On $(\Omega, \mathcal{F}, P)$, we will work in dimension 2 with two independent GFFs X, Y on a domain D and associated "smooth" cut-off approximations $X_\varepsilon, Y_\varepsilon$. We suppose that the family of centered Gaussian processes $(X_\varepsilon(x))_{x \in \mathbb{R}^d}$ is such that:

- Variance: $\mathbb{E}[X_\varepsilon(x)^2] = \ln \frac{1}{\varepsilon} + \ln C(x, D) + o(1)$ where $C(x, D)$ conformal radius.
- Covariance: $\mathbb{E}[X_\varepsilon(x) X_\varepsilon(y)] \sim \ln \frac{1}{|x-y|+\varepsilon}$
- Filtration $\mathcal{F}_\varepsilon^X = \{X_l(x);\ x \in \mathbb{R}^d,\ \varepsilon \leq l\}$
- For all $\varepsilon < \varepsilon'$, $(X_\varepsilon(x) - X_{\varepsilon'}(x))_{x \in \mathbb{R}^d}$ independent from $\mathcal{F}_{\varepsilon'}^X$

We define $(Y_\varepsilon(x))_{x \in \mathbb{R}^d}$ similarly.

Framework

Notations:

- Filtration $\mathcal{F}_\varepsilon = \mathcal{F}_\varepsilon^X \cup \mathcal{F}_\varepsilon^Y$
- $M_\varepsilon^{\gamma,\beta}(dx) = e^{\gamma X_\varepsilon(x) + i\beta Y_\varepsilon(x)} dx$, $\gamma, \beta \geq 0$
- $(\varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} M_\varepsilon^{\gamma,\beta}(dx))_{\varepsilon > 0}$ is a $\mathcal{F}_\varepsilon$-martingale.

Other Frameworks: universality

In fact, working with any log-correlated field in any dimension and with other "smooth" cut-offs leads to similar results:

- Kahane, (1985): theory based on σ-positivity of the logarithmic kernel
- Robert, V.(2006, 2008): theory based on convolutions of any log-correlated field in any dimension.
- Duplantier, Sheffield (2008): theory based on the H^1 decomposition of the GFF in dimension 2

For these questions, see our review with R. Rhodes (2013).

2-dimensional quantum gravity

In this talk, we consider the continuous model first considered by Polyakov in 1981: Quantum geometry of bosonic strings, *Phys. Lett B*. The continuous model is parametrized by γ and μ (cosmological constant).

The KPZ relation (Knizhnik, Polyakov, Zamolodchikov, 1988) was derived within this framework by David (1988) and Distler, Kawai (1989).

2-dimensional quantum gravity

It is conjectured to be the limit of random planar maps weighted by a statistical physics system (CFT with central charge $c \leq 1$) and conformaly mapped to a domain D:

- Ambjorn, Durhuus, Jonsson (2005): Quantum geometry: A Statistical Field Theory Approach
- Duplantier, Sheffield (2008): Liouville Quantum gravity and KPZ
- Sheffield (2010): Conformal weldings of random surfaces: SLE and the quantum gravity zipper

2-dimensional quantum gravity

Within this framework:

- Background metric g and curvature R
- Liouville action:

$$S_L(X) = \frac{1}{4\pi} \int_D (g^{ab}\partial_a X(x)\partial_b X(x) + QRX(x) + \mu e^{\gamma X(x)})\sqrt{|g|}d^2x$$

.

- μ cosmological constant (set to 0 here)
- $\gamma = \frac{\sqrt{25-c}-\sqrt{1-c}}{\sqrt{6}}$ (KPZ relation) with $c \leq 1$.
- Random metric: $e^{\gamma X} g$
- Liouville measure: $e^{\gamma X}\sqrt{|g|}d^2x$
- $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$

In this talk, the background metric will be flat, i.e. g will be the standard Euclidean metric.

2-dimensional quantum gravity

Can one define

- Random metric: $e^{\gamma X(x)}$ for $\gamma \leq 2$?
- Liouville measure: $e^{\gamma X} d^2x$ for $\gamma \leq 2$?

We will discuss in this talk the construction of the Liouville measure (and other Tachyon fields). One must distinguish 2 cases:

- $\gamma < 2$: $e^{\gamma X} d^2x$
- $\gamma = 2$: $-Xe^{2X} d^2x$ and $e^{2X} d^2x$. Are these measures the same?

The Liouville measure for $\gamma < 2$

Theorem (Kahane, 1985)

There exists a random measure $M^{\gamma,0}$ such that following limit exists almost surely in the space of Radon measures:

$$\varepsilon^{\frac{\gamma^2}{2}} M_{\varepsilon}^{\gamma,0}(dx) \underset{\varepsilon \to 0}{\to} M^{\gamma,0}(dx).$$

$M^{\gamma,0}$ is called Gaussian multiplicative chaos associated to the Green kernel in D.

The Liouville measure for $\gamma < 2$

Theorem (Kahane, 1985)

The measure $M^{\gamma,0}$ is different from 0 if and only if $\gamma^2 < 4$.

Theorem (Kahane, 1985)

For $\gamma^2 < 4$, the measure $M^{\gamma,0}$ "lives" almost surely on a set of Hausdorff dimension $2 - \frac{\gamma^2}{2}$.

CFT with central charge $c = 1$ coupled to Gravity

- Polyakov action on a domain D

$$S(X, Y) = \frac{1}{4\pi}\int_D |\nabla Y(x)| d^2x + \frac{1}{4\pi}\int_D |\nabla X(x)|^2 + QR(x)X(x) d^2x,$$

  R is the curvature and $Q = 2$

- Equivalence class of random surfaces:

$$(X, Y) \to (X \circ \psi + 2\ln|\psi'|, Y \circ \psi),$$

  where $\psi : \tilde{D} \to D$ is a conformal map. See Ginsparg, Moore (1993), Lectures on 2D gravity and 2D string theory or Duplantier, Sheffield (2008).

Critical Gaussian multiplicative chaos: Liouville measure for $\gamma = 2$

Theorem (Duplantier, Rhodes, Sheffield, V., 2012)

There exists a random measure M such that following limit exists almost surely in the space of Radon measures:

$$\varepsilon^2 (2 \ln \frac{1}{\varepsilon} - X_\varepsilon(x)) M_\varepsilon^{2,0}(dx) \underset{\varepsilon \to 0}{\to} M'(dx).$$

The measure M' has no atoms. M' is called critical Gaussian multiplicative chaos associated to the Green kernel.

Critical Gaussian multiplicative chaos: Liouville measure for $\gamma = 2$

Theorem (Duplantier, Rhodes, Sheffield, V., 2012)

The following limit exists almost surely (along suitable subsequences) in the space of Radon measures:

$$\sqrt{\ln \frac{1}{\varepsilon}}\,\varepsilon^2 M_\varepsilon^{2,0}(dx) \underset{\varepsilon \to 0}{\to} \sqrt{\frac{2}{\pi}}\, M'(dx).$$

Theorem (Barral, Kupiainen, Nikula, Saksman, Webb, 2013)

The measure M' lives on a set of Hausdorff dimension 0.

Looking for other Tachyon fields

We want to study the limit of $(\varepsilon^{\frac{\gamma^2}{2}-\frac{\beta^2}{2}} M_\varepsilon^{\gamma,\beta}(dx))_{\varepsilon>0}$.

We introduce the following phase

$$\mathcal{P} := \left\{\gamma + \beta \leq 2, \gamma \in \left]1, 2\right[\right\} \cup \{\gamma^2 + \beta^2 < 2\}.$$

Complex Gaussian multiplicative chaos: Phase diagram

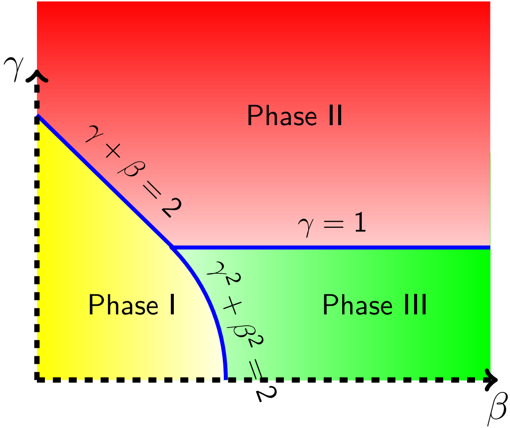

Figure: Phase diagram

Looking for other Tachyon fields

Theorem (Lacoin, Rhodes, V., 2013)

There exists $p > 1$ such that for γ, β in phase $\mathcal{P}$:

1. *For all compactly supported bounded measurable function f, the martingale*

$$(\varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} \int_D f(x)\, M_\varepsilon^{\gamma,\beta}(dx))_\varepsilon$$

   is uniformly bounded in L_p.

2. *The $\mathcal{D}'(D)$-valued martingale:*

$$M_\varepsilon^{\gamma,\beta} : \varphi \to \varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} \int_{\mathbb{R}^d} \varphi(x) M_\varepsilon^{\gamma,\beta}(dx)$$

   converges almost surely in the space $\mathcal{D}_2'(D)$ of distributions of order 2 towards a non trivial limit $M^{\gamma,\beta}$.

Looking for other Tachyon fields

We denote

$$M_{X,Y}^{\gamma,\beta}(dx) = e^{\gamma X(x) + i\beta Y(x) - \frac{\gamma^2}{2}\mathbb{E}[X(x)^2] + \frac{\beta^2}{2}\mathbb{E}[Y(x)^2]} C(x,D)^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} dx,$$

where $C(x,D)$ is the conformal radius. This is because we do not renormalize by the mean!

Looking for other Tachyon fields

Under the above equivalence class ($\psi : \tilde{D} \to D$)

$$M^{\gamma,\beta}_{X\circ\psi+2\ln|\psi'|,Y\circ\psi}(\varphi) = |\psi' \circ \psi^{-1}|^{2\gamma-\frac{\gamma^2}{2}+\frac{\beta^2}{2}-2} M^{\gamma,\beta}_{X,Y}(\varphi \circ \psi^{-1}),$$

for every function $\varphi \in C^2_c(\tilde{D})$

Tachyon Fields are conformally invariant. One must solve

$$2\gamma - \frac{\gamma^2}{2} + \frac{\beta^2}{2} - 2 = 0 \leftrightarrow \gamma \pm \beta = 2, \quad \gamma \in]1,2[.$$

Liouville Brownian motion for $\gamma = 2$

The Liouville Brownian motion for $\gamma = 2$ can be thought of as the solution of the following formal SDE

$$\begin{cases} \mathcal{B}^x_{t=0} = x \\ d\mathcal{B}^x_t = e^{-X(\mathcal{B}^x_t)}\, d\bar{B}_t. \end{cases} \tag{1}$$

where $\bar{B}$ is a Brownian motion. By the Dambis-Schwarz theorem (or rather the Knight theorem in dimension 2) we can rewrite (1) as

$$\mathcal{B}^x_t \overset{law}{=} x + B_{\langle \mathcal{B}^x \rangle_t},$$

where $(B_r)_{r\geq 0}$ is another two-dimensional Brownian motion and the quadratic variation $\langle \mathcal{B}^x \rangle$ of $\mathcal{B}^x$ is given by:

$$\langle \mathcal{B}^x \rangle_t := \inf\{s \geq 0 \,:\, \int_0^s e^{2X(x+B_u)}\, du \geq t\}\,.$$

Liouville Brownian motion for $\gamma = 2$

Of course, the above considerations are formal but it is natural to consider the regularized field X_ε and to take the limit as $\varepsilon \to 0$ of $\mathcal{B}^{\varepsilon,x}$ where $\mathcal{B}^{\varepsilon,x}$ is given by:

$$\mathcal{B}_t^{\epsilon,x} \stackrel{law}{=} x + B_{\langle \mathcal{B}^{\epsilon,x} \rangle_t}, \qquad (2)$$

where $(B_r)_{r \geq 0}$ is another two-dimensional Brownian motion and the quadratic variation $\langle \mathcal{B}^{\varepsilon,x} \rangle$ of $\mathcal{B}^{\epsilon,x}$ is given by:

$$\langle \mathcal{B}^{\epsilon,x} \rangle_t := \inf\{ s \geq 0 \, : \, \sqrt{|\ln \varepsilon|}\varepsilon^2 \int_0^s e^{2X_\varepsilon(x+B_u)} \, du \geq t \}$$

Finally, we introduce the following notation

$$F^\varepsilon(x,t) = \sqrt{|\ln \varepsilon|}\varepsilon^2 \int_0^t e^{2X_\varepsilon(x+B_u)} \, du.$$

Liouville Brownian motion for $\gamma = 2$

Theorem (Rhodes, V., 2013)

Almost surely in X, for M' all y (and all $y \in \mathbb{Q}^2 \cap D$), the family $(F^\epsilon(y,\cdot))_\epsilon$ converges in law under $\mathbb{P}^B$ in $C(\mathbb{R}_+)$ equipped with the sup-norm topology towards a continuous increasing mapping $F(y,\cdot)$. Let us define the process $t \mapsto \langle \mathcal{B}^y \rangle_t$ by:

$$\forall t \geq 0, \quad F(y, \langle \mathcal{B}^y \rangle_t) = t.$$

The law of the Liouville Brownian motion $\mathcal{B}^y$ starting from y is then given by

$$\mathcal{B}^y_t = y + B_{\langle \mathcal{B}^y \rangle_t}.$$

The process $\mathcal{B}^y$ is reversible with respect to M'.