Random constraint satisfaction problems: a point of view from physics

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Outline

1. Constraint satisfaction problems
2. Random ensembles of CSP
3. Statistical physics approach and results

Constraint satisfaction problems : definitions

n variables $\quad \underline{x} = (x_1, \ldots, x_n) \in \mathcal{X}^n \quad$ discrete alphabet $\mathcal{X}$

m constraints $\quad \psi_a(\{x_i\}_{i \in \partial a}) \in \begin{cases} 1 & \text{satisfied} \\ 0 & \text{unsatisfied} \end{cases}$

solutions $\quad \mathcal{S} = \{\underline{x} : \psi_a(\underline{x}_{\partial a}) = 1 \ \forall a\}$

- decision problem, is $|\mathcal{S}| > 0$?
- counting problem, what is $|\mathcal{S}|$?
- optimization problem, what is $\max_{\underline{x}} \left[\sum_a \psi_a(\underline{x}) \right]$?

Constraint satisfaction problems : examples

- $\mathcal{X} = \{\text{True}, \text{False}\}$, ψ_a depends on k variables $x_{i_a^1}, \dots, x_{i_a^k}$
  - $\psi_a = \mathbb{1}(z_{i_a^1} \vee \cdots \vee z_{i_a^1} = \text{True})$, with $z_i \in \{x_i, \overline{x}_i\}$

    k-satisfiability problem
  - $\psi_a = \mathbb{1}(x_{i_a^1} \oplus \cdots \oplus x_{i_a^k} = y_a)$, with $y_a \in \{\text{True}, \text{False}\}$

    k-xor-satisfiability problem
- $\mathcal{X} = \{1, \dots, q\}$, $\psi_a(x_i, x_j) = \mathbb{1}(x_i \neq x_j)$
  on the edges $a = \langle i, j \rangle$ of a graph

  q-coloring problem

Worst-case complexity of the decision problem:

- *k*-xor-satisfiability easy for all *k*
- *k*-satisfiability, *q*-coloring difficult for $k, q \geq 3$

Random constraint satisfaction problems

What about their "typical case" complexities ?

"typical"= with high probability in some random ensemble of instances

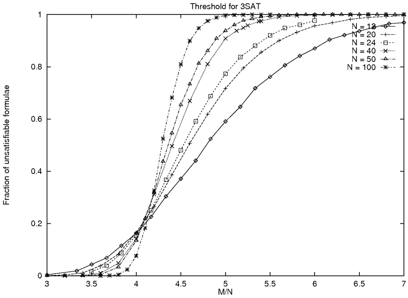
Examples :

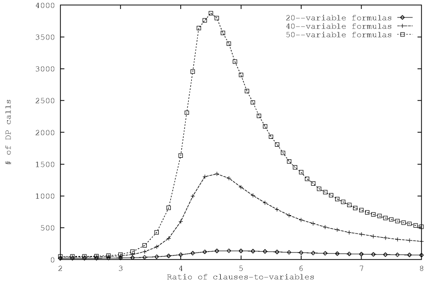
- coloring Erdös-Rényi random graphs $G(n, m)$
  choose m edges uniformly at random in the $\binom{n}{2}$ possible ones
- random (xor)satisfiability ensembles
  choose m hyperedges (k-uplets of variables), among $\binom{n}{k}$

Most interesting regime : $n, m \to \infty$ with $\alpha = m/n$ fixed

Random constraint satisfaction problems

Phase transition for the unsatisfiability probability :





associated to a peak in the hardness of solving

Random constraint satisfaction problems

A few rigorous results for random *k*-satisfiability and *q*-coloring :

- existence of a sharp threshold $\alpha_s(k)$ [in fact $\alpha_s(k,n)$] [Friedgut]
- upper and lower bounds on $\alpha_s(k)$
  [Chao and Franco, Frieze and Suen, Achlioptas, Dubois et al]
- asymptotics of $\alpha_s(k)$ at large *k* [Achlioptas, Moore, Naor, Peres]

But :

- no precise value of $\alpha_s(k)$ for small *k*
- unsatisfactory understanding of algorithmic difficulty at $\alpha < \alpha_s(k)$

Why physics ?

Statistical mechanics :

- configuration space $\underline{x} = (x_1, \ldots, x_n)$
- energy function $E(\underline{x})$
- temperature T
- Gibbs-Boltzmann distribution $\mu(\underline{x}) = \exp[-E(\underline{x})/T]/Z$

Low-temperature statistical physics $\approx$ combinatorial optimization

randomness in the distribution of instances $\approx$ disordered systems

Outcomes of the physics approach

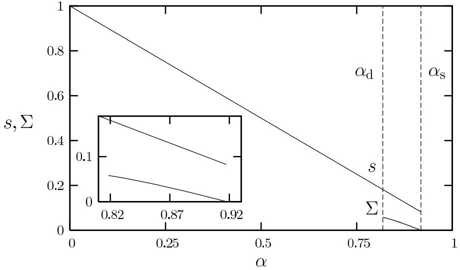
- quantitative estimation of $\alpha_s(k)$
- refined picture of the satisfiable phase
- analysis of known algorithms
- suggestion of new ones

The heuristic picture of the satisfiable phase

Exponential number of solutions for $\alpha < \alpha_s$, $\sim \exp[ns(\alpha)]$

Sudden disappearance at α_s : $s(\alpha_s^-) > 0$

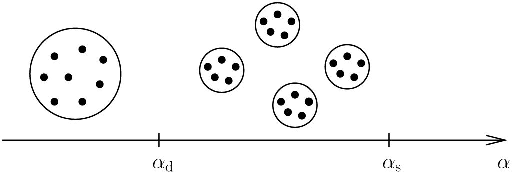
Clustering transition at another threshold $\alpha_d < \alpha_s$



[here for 3-xor-sat]

$s(\alpha) = \Sigma(\alpha) + s_{\rm int}(\alpha)$

$\Sigma(\alpha_s) = 0$



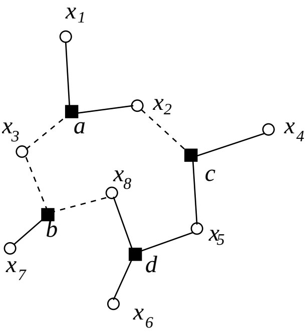
[more complicated picture for sat and col]

Methods

If the formula F has solutions,

define $\mu(\underline{x}) = \frac{1}{Z} \prod \psi_a(\underline{x}_{\partial a})$ uniform measure on $\mathcal{S}$

Factor graph representation of a formula :



Crucial property : in the $n, m \to \infty$ limit with $\alpha = m/n$ fixed
local convergence of the factor graph to a random Galton-Watson tree

Methods

For a tree factor graph $\mu(\underline{x})$ is an easily parameterized object

(Belief Propagation is exact)

For a locally tree like factor graph it is almost the same

(within a cluster)

- for $\alpha < \alpha_{\rm d}$, $\mu(\underline{x}) \approx \mu_{\rm tree}(\underline{x})$
- for $\alpha_{\rm d} < \alpha < \alpha_{\rm s}$, decomposition over the clusters,

$$\mu(\underline{x}) \approx \sum_c w_c \, \mu^c_{\rm tree}(\underline{x})$$

Properties of w_c yields the value of $\Sigma(\alpha)$, hence $\alpha_{\rm s}$

Message passing algorithms

Sequential generation from μ : for $t = 1, \ldots, n$

- choose $i(t)$ u.a.r. in $\{1, \ldots, n\} \setminus \{i(1), \ldots, i(t-1)\}$
- draw $\sigma_{i(t)}$ according to $\mu(\sigma_{i(t)} | \sigma_{i(1)}, \ldots, \sigma_{i(t-1)})$

At the end, $\underline{\sigma}$ is distributed according to μ

$\Rightarrow$ solves the uniform generation problem $\Rightarrow$ construction problem

BUT : needs an oracle to compute the marginals

Practical approximate implementation : replace the oracle with

- belief propagation (in absence of clustering)
- survey propagation : to find a solution (non-uniformly), we only need to detect $\mu(\sigma_{i(t)} | \sigma_{i(1)}, \ldots, \sigma_{i(t-1)}) = 0$ this can be done in the case of clustering with SP

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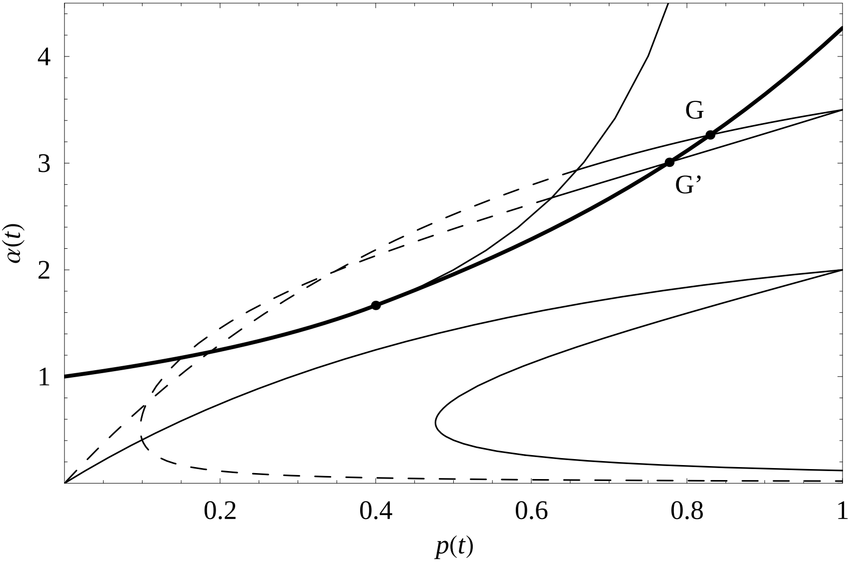
Analysis of DPLL



Random Walk algorithms