Efficient Solutions for the λ-coloring Problem on Classes of Graphs

Daniel Posner (PESC - UFRJ)
PhD student - posner@cos.ufrj.br

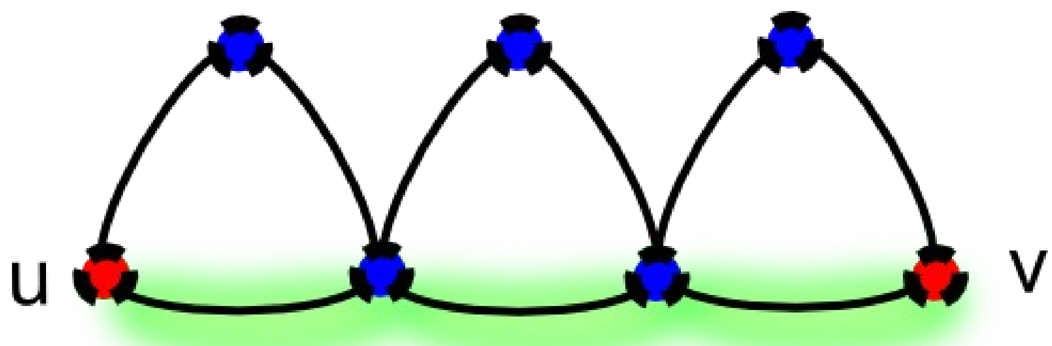
Advisor: Márcia Cerioli

LIPN – Université Paris-Nord
29th november 2011

distance

- d(u, v) = **distance** between u and v.

- **diameter** = max{d(u, v) | u, v ∈ V(G)}



- Ex: **d(u, v) = 3**; **diameter** of the graph is **3**.

coloring

coloring of a graph $G = (V, E)$

$f: V \to \mathbb{N}^*$, such that

if $uv \in E$, then $f(u) \neq f(v)$

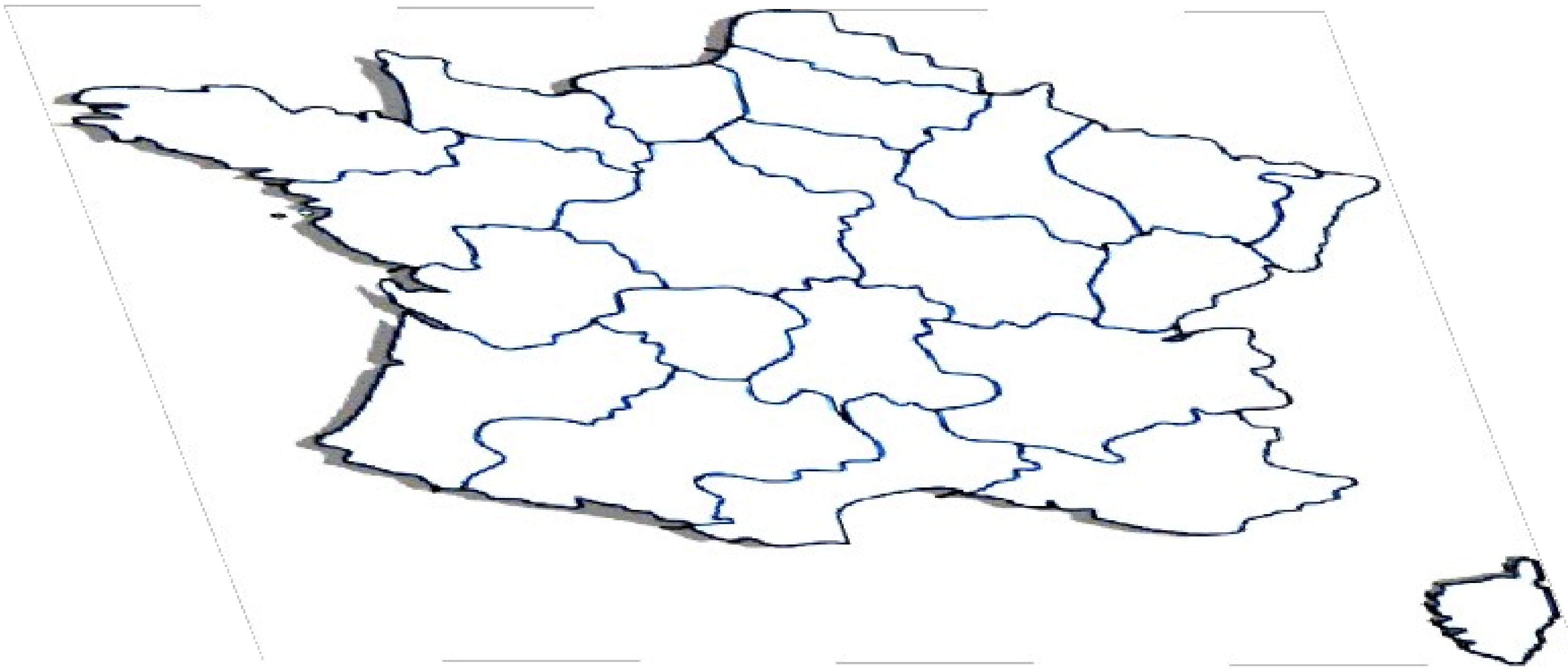
λ-coloring

λ-coloring of a graph $G = (V, E)$

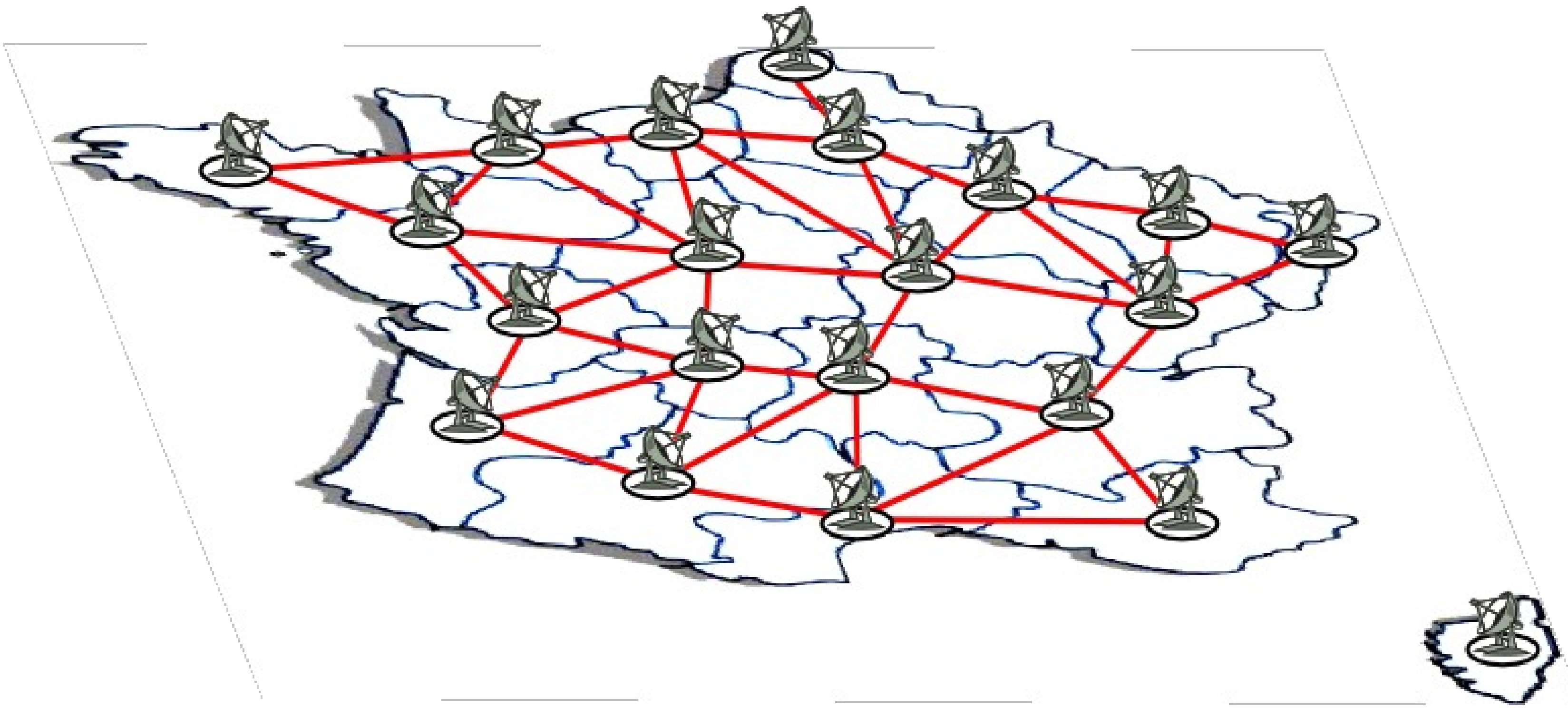
$f: V \to \mathbb{N}$, such that

- if $uv \in E$, then $|f(u) - f(v)| \geq 2$,
- if $dist(u, v) = 2$, then $f(u) \neq f(v)$

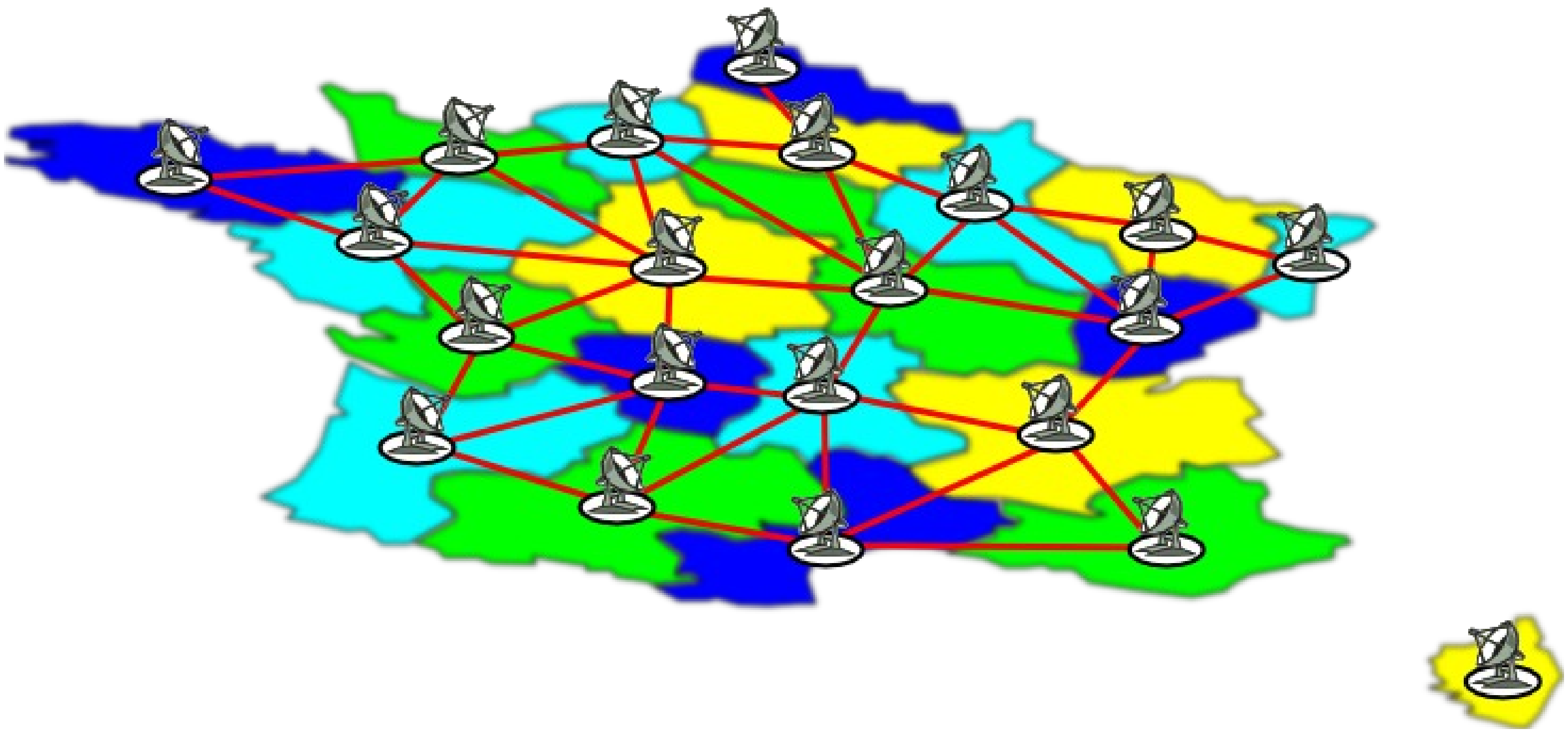
Motivation



Motivation

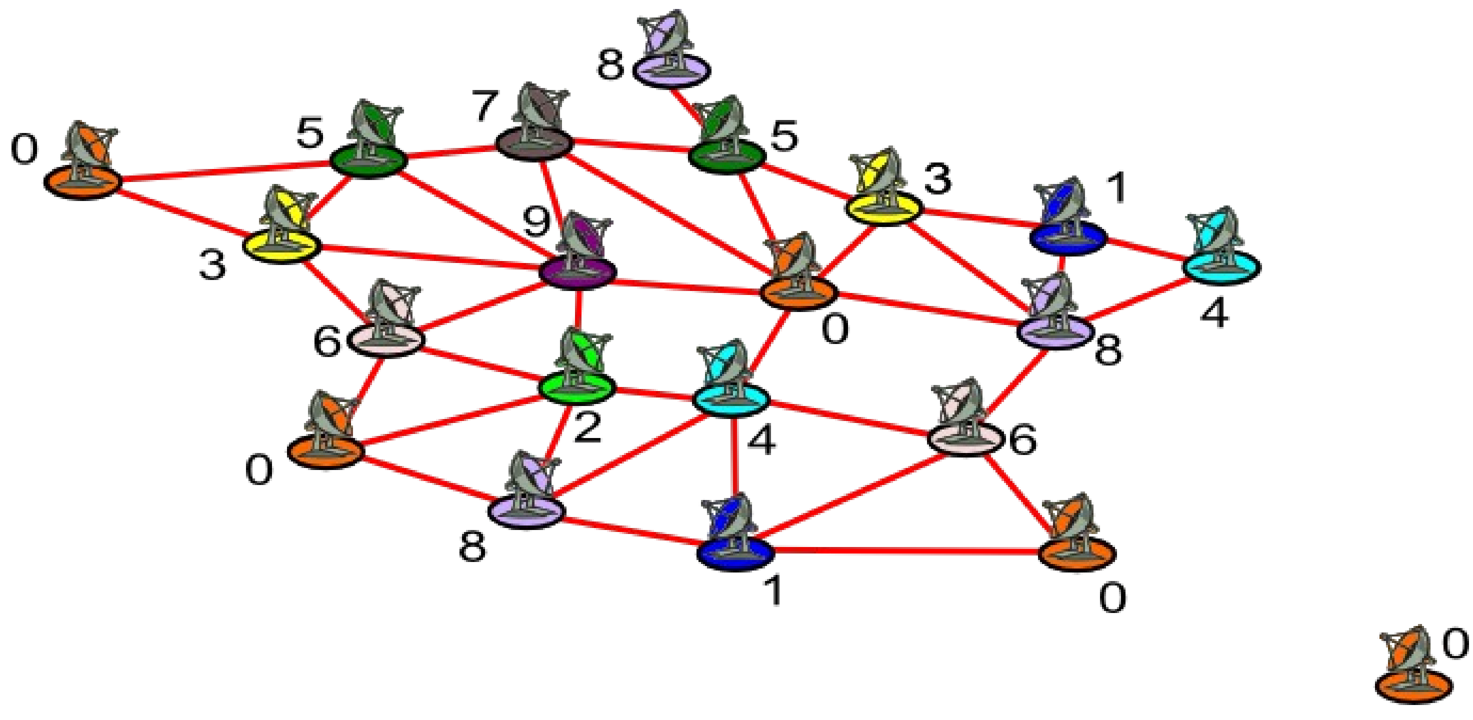


Motivation



$\chi(G) = 4$

Motivation



$\lambda(G) = 9$

Decision version

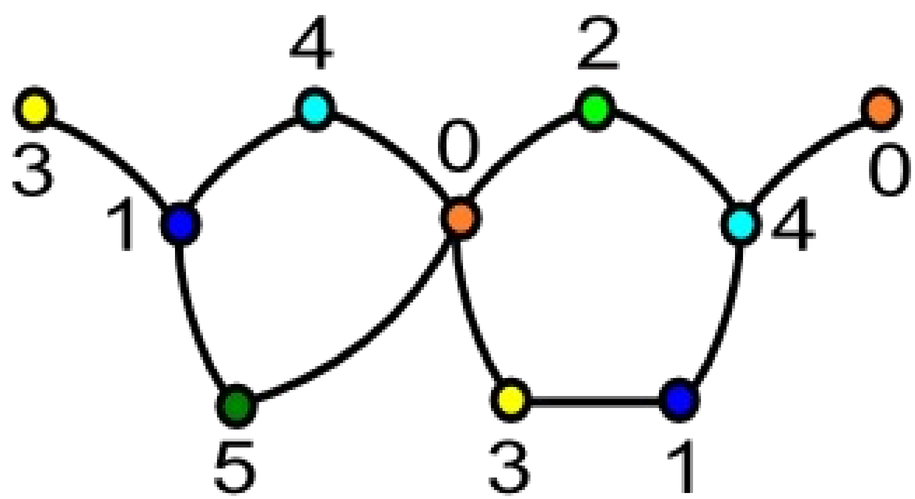
L(2,1)-coloring Problem

Instance: $G = (V, E)$, $k \in \mathbb{N}$

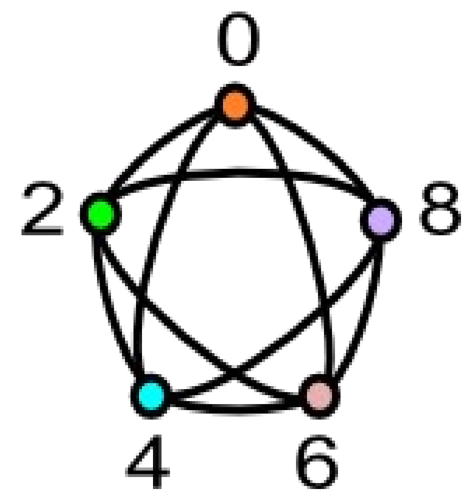
Question: Is there an λ-coloring f of G with $f : V \rightarrow \{0, 1, ..., k\}$?

the minimum span is denoted λ

Examples



$\lambda = 5$



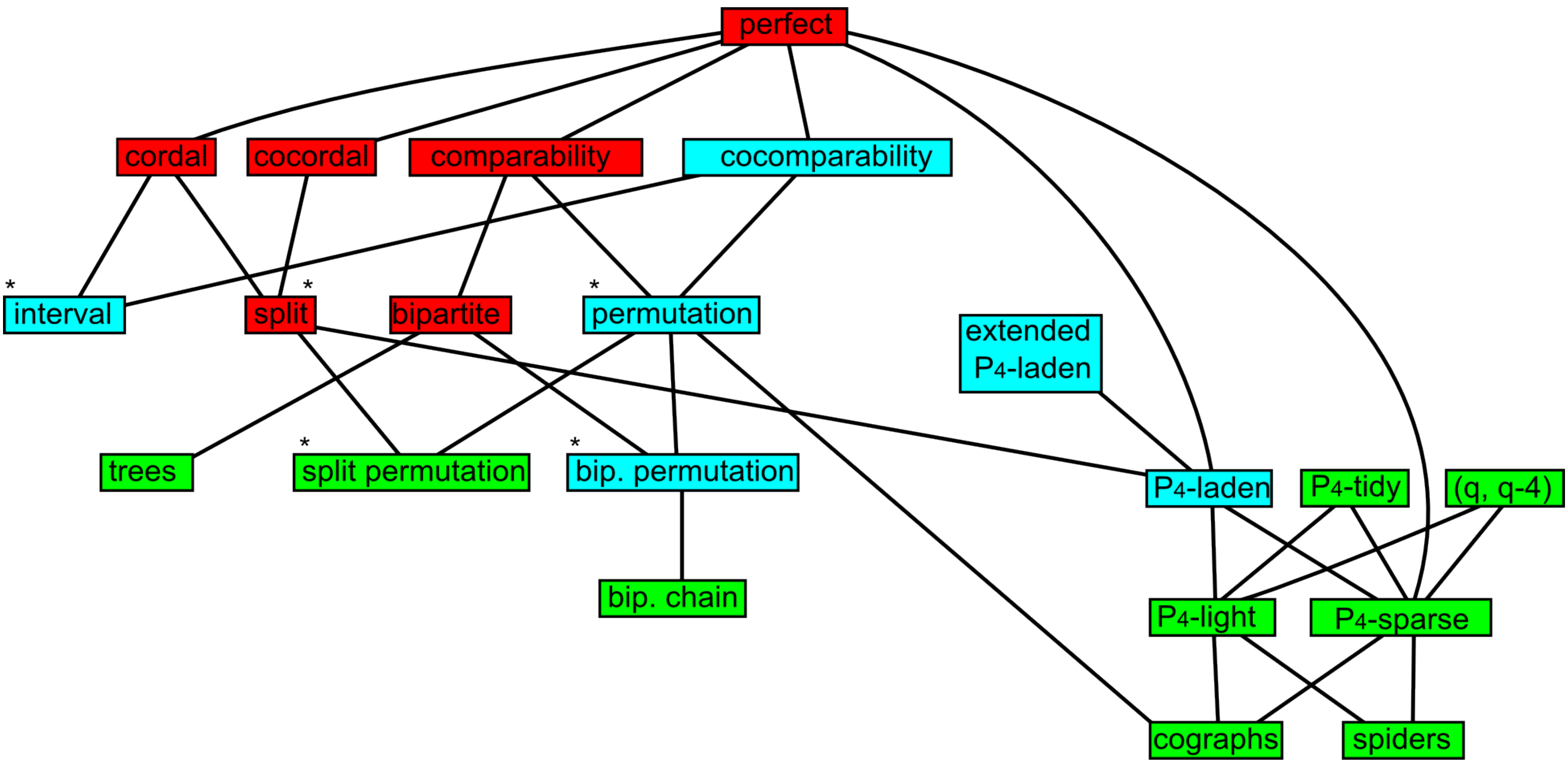
$\lambda(Kn) = 2n-2$

$\lambda \geqslant \Delta + 1$

Known results

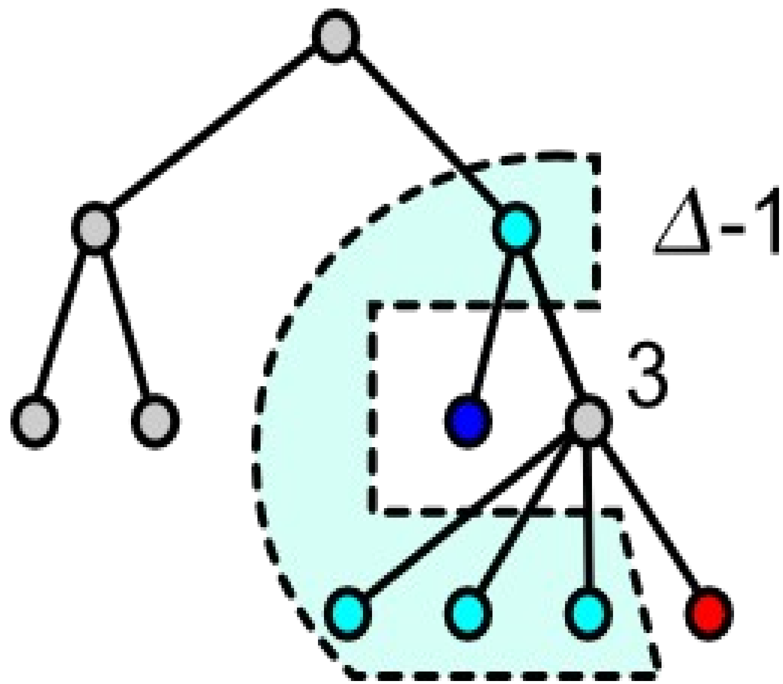
Class	Comp.	Class	Comp.
trees	P	diameter 2	NP-c
p-quasi trees	P	k fixed	NP-c
		proper interval	Open
bipartite chain	P	permutation	Open
bipartite planar	NP-c	regular grids	P
bipartite permutation	Open		
		cographs	P
split	NP-c	P4-tidy	P
split permutation	P	graphs (q, q-4), q fixed	P

Known results



Trees

- $\lambda = \Delta + 1$ or $\Delta + 2$

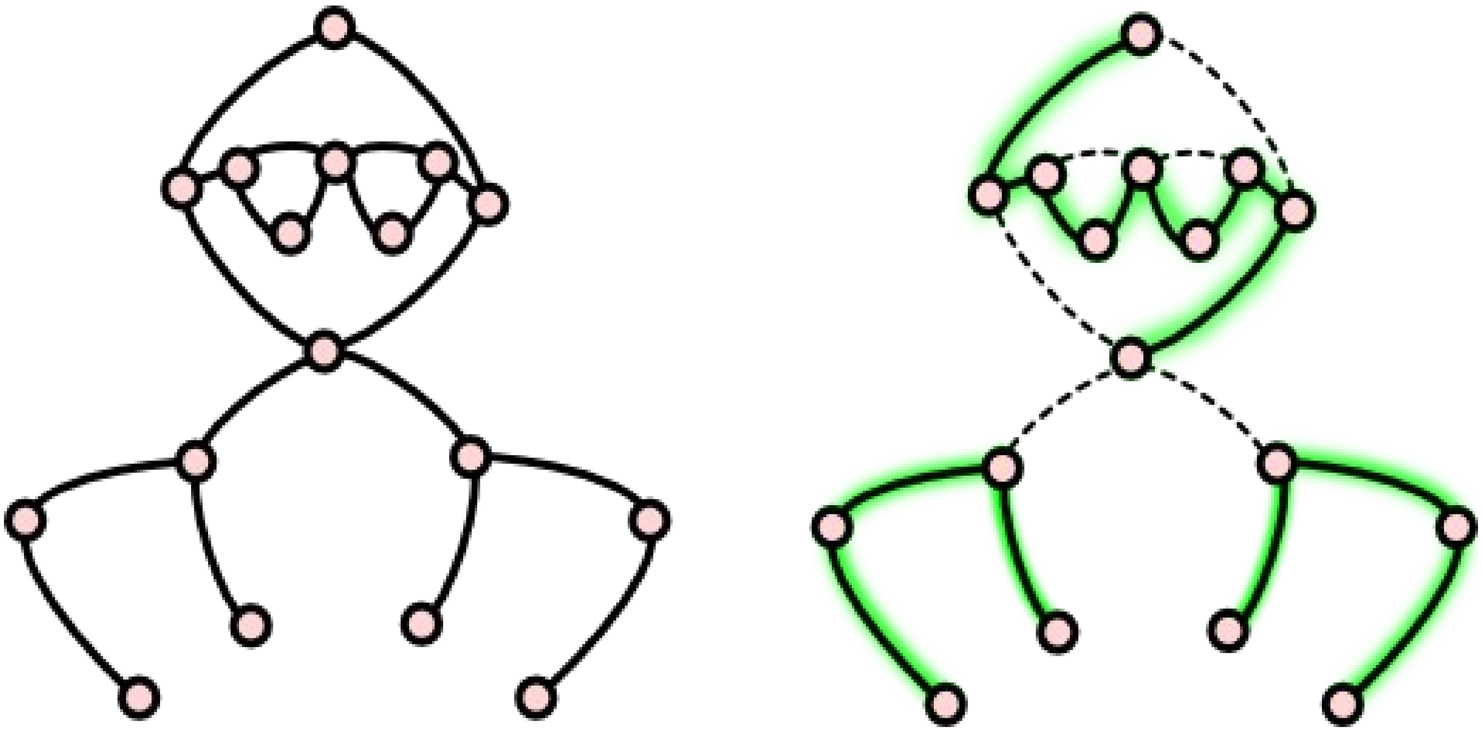


Trees

- [Griggs e Yeh 92] **conjectured λ-col. of trees** was **NP-complete.**
- [Chang e Kuo 96] showed an **$O(n\Delta^{4.5})$** algorithm.
- [Hasunuma et al. 09] gave a **linear time** algorithm.
- It is still open a **structural characterization** of trees.

pv(G)

- pv(G) = minimum number of disjoint paths



pv(G) = 3

$\lambda(G)$ and pv(G)

[Griggs e Yeh 92]

- $\lambda(G \wedge K_1) = n \Leftrightarrow G^c$ is hamiltonian

[Georges et al. 94]

- $pv(G^c) \geq 2 \Leftrightarrow \lambda = n + pv(G^c) - 2$
- $pv(G^c) = 1 \Leftrightarrow \lambda \leq n - 1$

(q, q-4) graphs

A graph **G** is **(q, q-4)** if **each set** of q vertices induces at most q - 4 P_4's. [Babel e Olariu]

Ex.: q = 4 a.k.a. cografos (G is cograph ⇔ P_4-free)

q = 5 a.k.a. P_4-sparse

q = 7 superclass of P_4-lite (P_4-tidy and perfect)

p-componente separável

Teo (Jamison e Olariu): If G is $(q, q-4)$, then:

(i) union of two $(q, q-4)$ graphs or;

(ii) join of two $(q, q-4)$ graphs or;

(iii) spider where the head is a $(q, q-4)$ graph or;

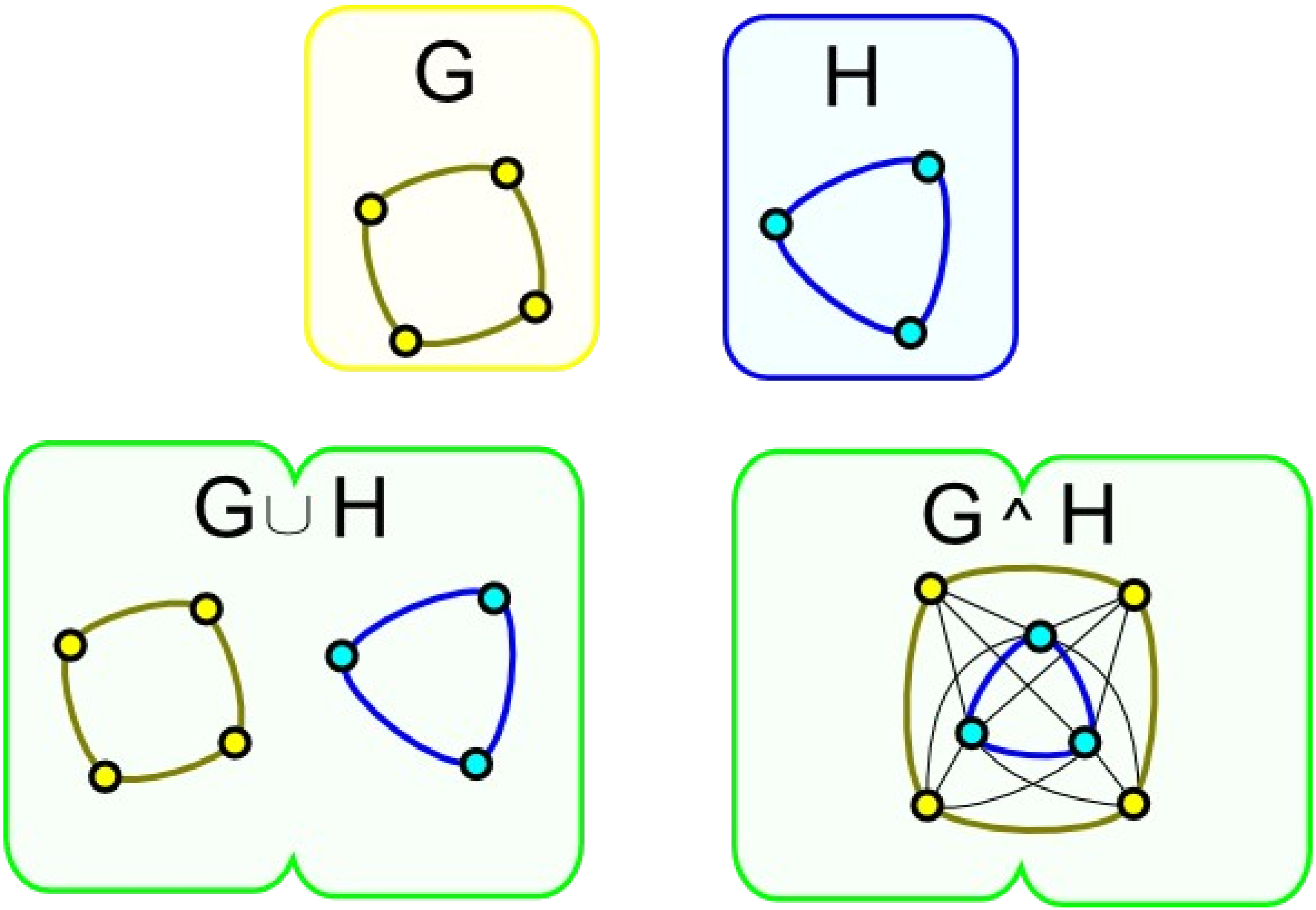
(iv) it has a separaple p-component $H = (H_1, H_2)$,

$|H| \leq q$,

$G[V \setminus H_2] = G[V \setminus H] \wedge G[H_1]$,

$G[V \setminus H_1] = G[V \setminus H] \cup G[H_2]$.

Union and Join



Spider graph

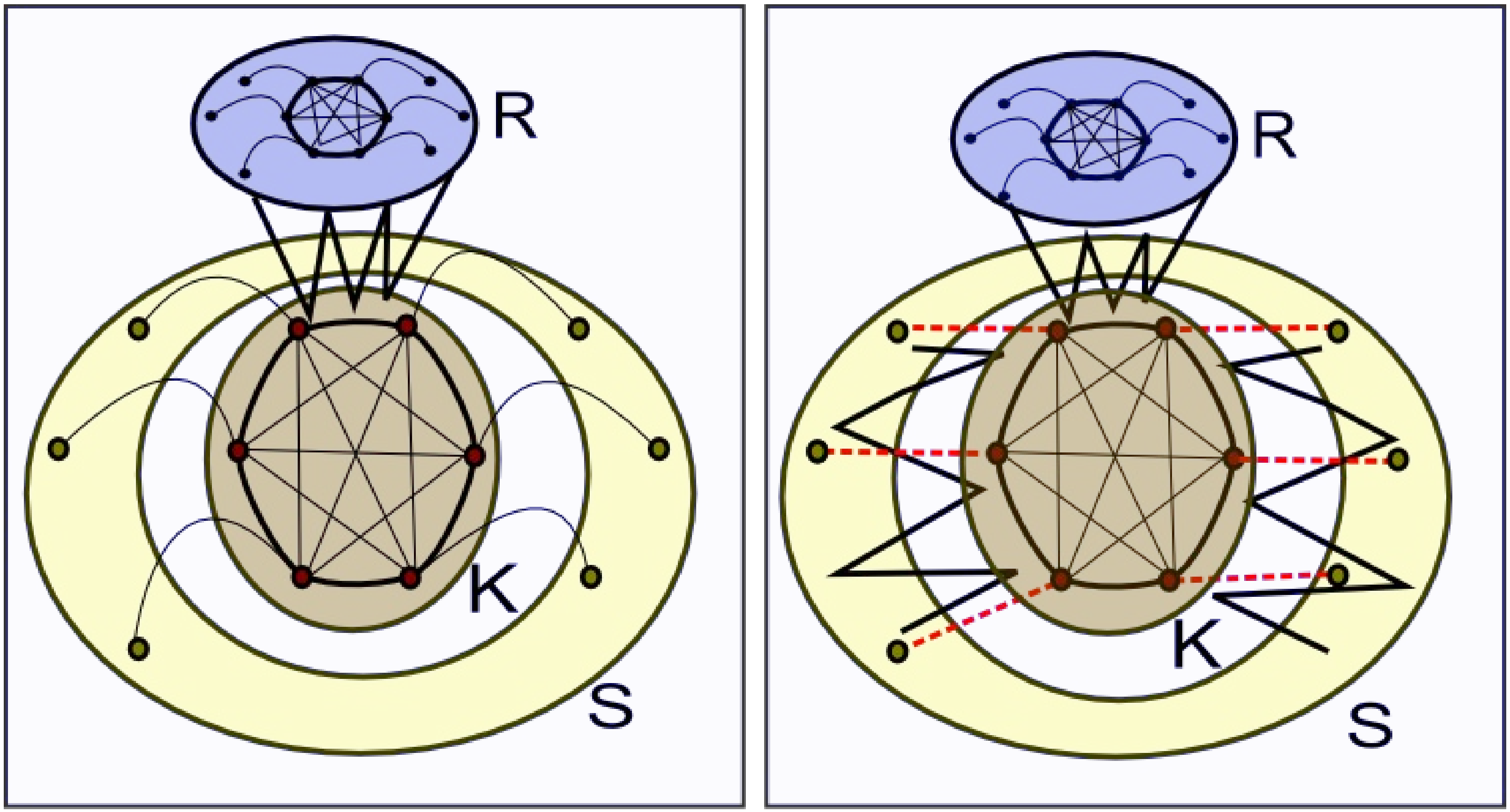
If $G = (V, E)$ is a **spider**, then

- $V = S \cup K \cup R$.
  S is a stable set.
  K is a clique,
  $|S| = |K|$
- $G[R \cup K] = G[R] \wedge G[K]$
- bijective function $f : S \rightarrow K$

(a) edges: thin spider

(b) no edges: thick spider

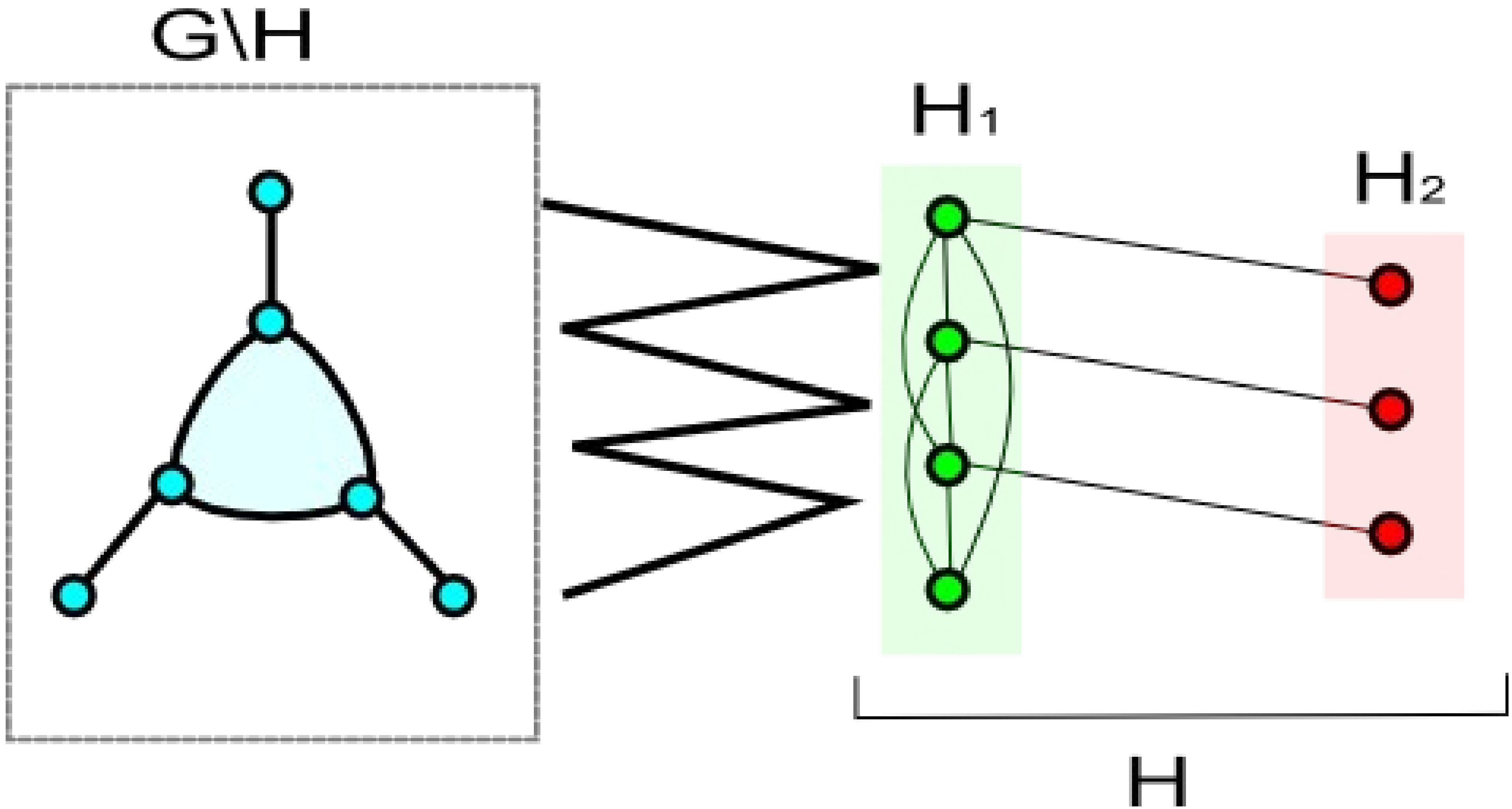
Thin spider and Thick spider



(a)

(b)

Separaple p-component



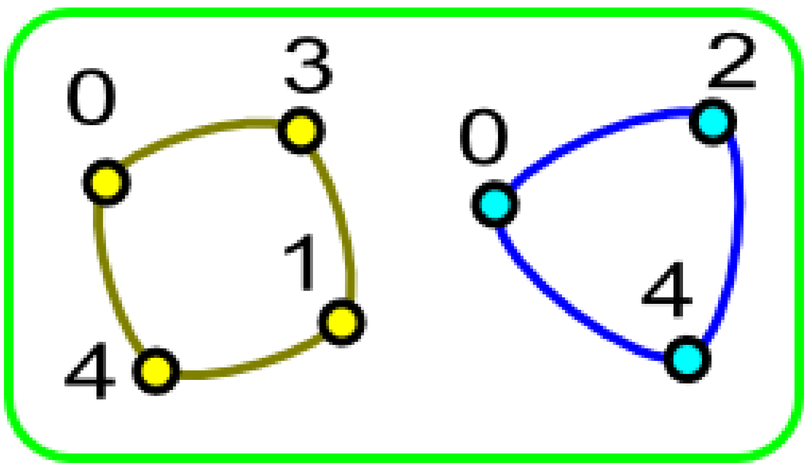
λ-coloring of union and join

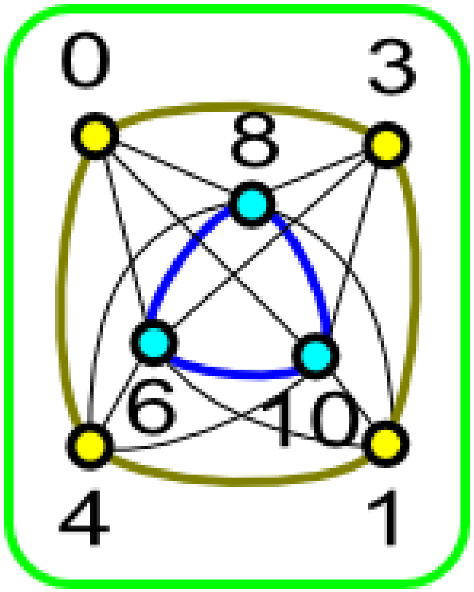
- λ-coloring of G ∪ H

$$\lambda = \max\{ \lambda(G), \lambda(H) \}$$

- λ-coloring G ^ H

$$\lambda = \lambda'(G) + \lambda'(H) + 2$$

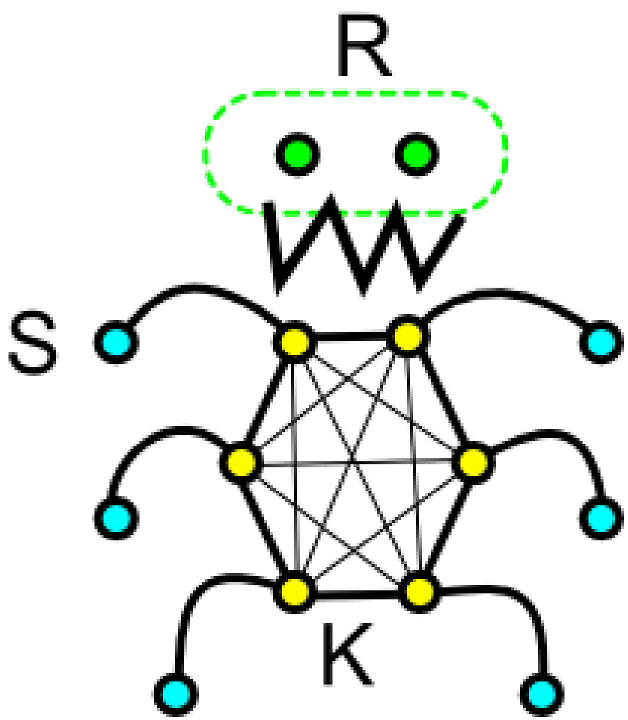


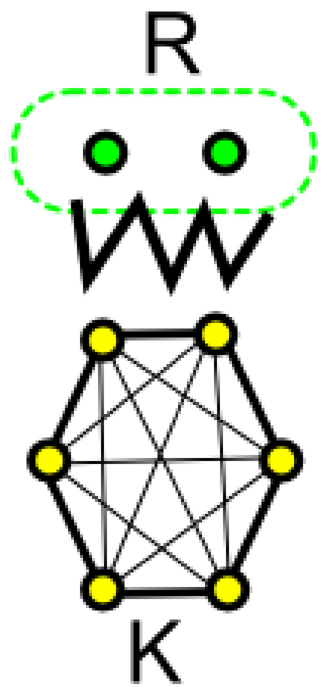


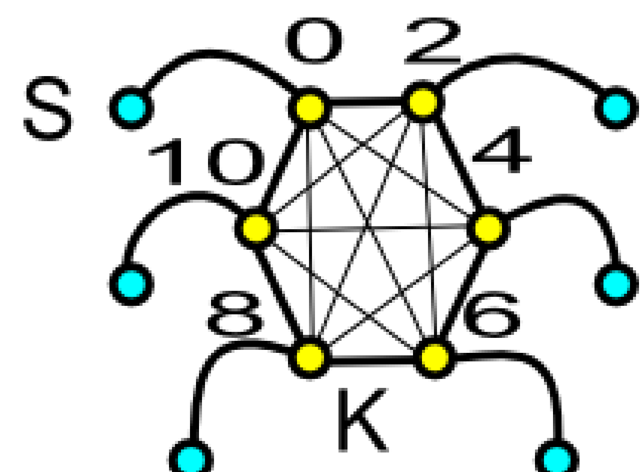
λ-coloring of thin spider

<u>Teo</u> If G is thin spider with |K| > 3.

then $\lambda = \max\{\, |R| - 1, \lambda(G[R]) \,\} + 2\,|K|$







|K|-1 em {0, …, 2|K| - 2}

λ-coloring of thick spider

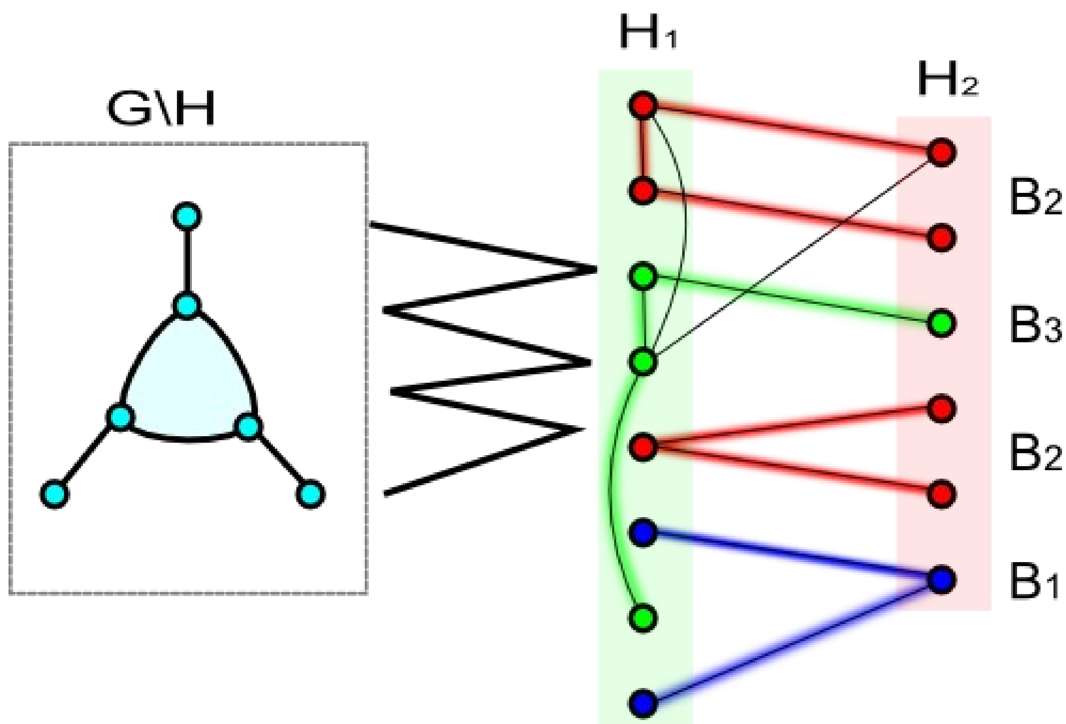
Teo If G is thick spider with $|K| \geq 3$

$$\lambda = \begin{cases} \lambda(G[R]) + 2\,|K| & \text{if } \lambda(G[R]) \geq |R| + \left\lceil \frac{|K|}{2} \right\rceil - 2 \\ n + \left\lceil \frac{|K|}{2} \right\rceil - 2 & \text{otherwise} \end{cases}$$

pv(G) in separable p-component

Teo If G has a separable p-component H = (H1, H2), then

$$pv(G) = \min_{\psi \in C_H}\left\{ \max\left\{ pv(G \setminus H) - |B1(\psi)|,\ \left\lceil \frac{|B3(\psi)|}{2} \right\rceil,\ 1\right\} + |B2(\psi)| \right\}$$



FPT

FPT(fixed parameter tractable) in q(G)

q(G) = smallest q for which G is (q, q-4) graph

Algorithm FPT in q(G)

Linear algorithms for (q, q-4) graphs with q fixed

Ex: $O(2^q n)$ or $O(q^q n)$

λ-coloring of separable p-component

Teo If G is (q, q-4) graph, q fixed, with a separable p-component then λ can be obtained in linear time.

proof.

G^c is (q, q-4) graph and H^c is a separable p-component.

If G has less than 2q vertices, one can obtain λ in $O(2q^{4q})$.

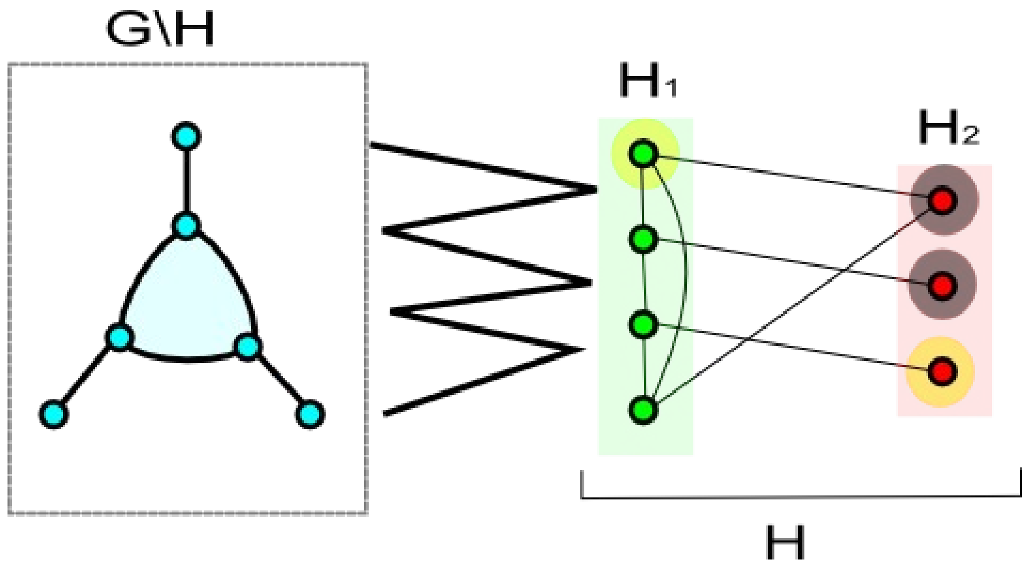
Otherwise, $pv(G^c)$ can be obtained in O (n), as $|C_H| \leq q^q$

λ-coloring of separable p-component

Teo If G is (q, q-4) graph, q fixed, with a separable p-component then λ can be obtained in linear time.

proof.

If **d(u,v)** ≥ **3**, then **u**, **v** ∈ **H_1 U H_2.**



λ-coloring of separable p-component

Teo If G is (q, q-4) graph, q fixed, with a separable p-component then λ can be obtained in linear time.

proof.

Let **G'** be obtained from G merging vertices in the same class.

If pv(**G'^{c}**) > 1, then λ(G') = n' + pv(G'^{C}) – 2 (Georges et al.)

If pv(**G'^{c}**) = 1, then λ(G') = n' -1 (use hamiltonian path.)

λ-coloring of separable p-component

Teo If G is (q, q-4) graph, q fixed, with a separable p-component then λ can be obtained in linear time.

proof.

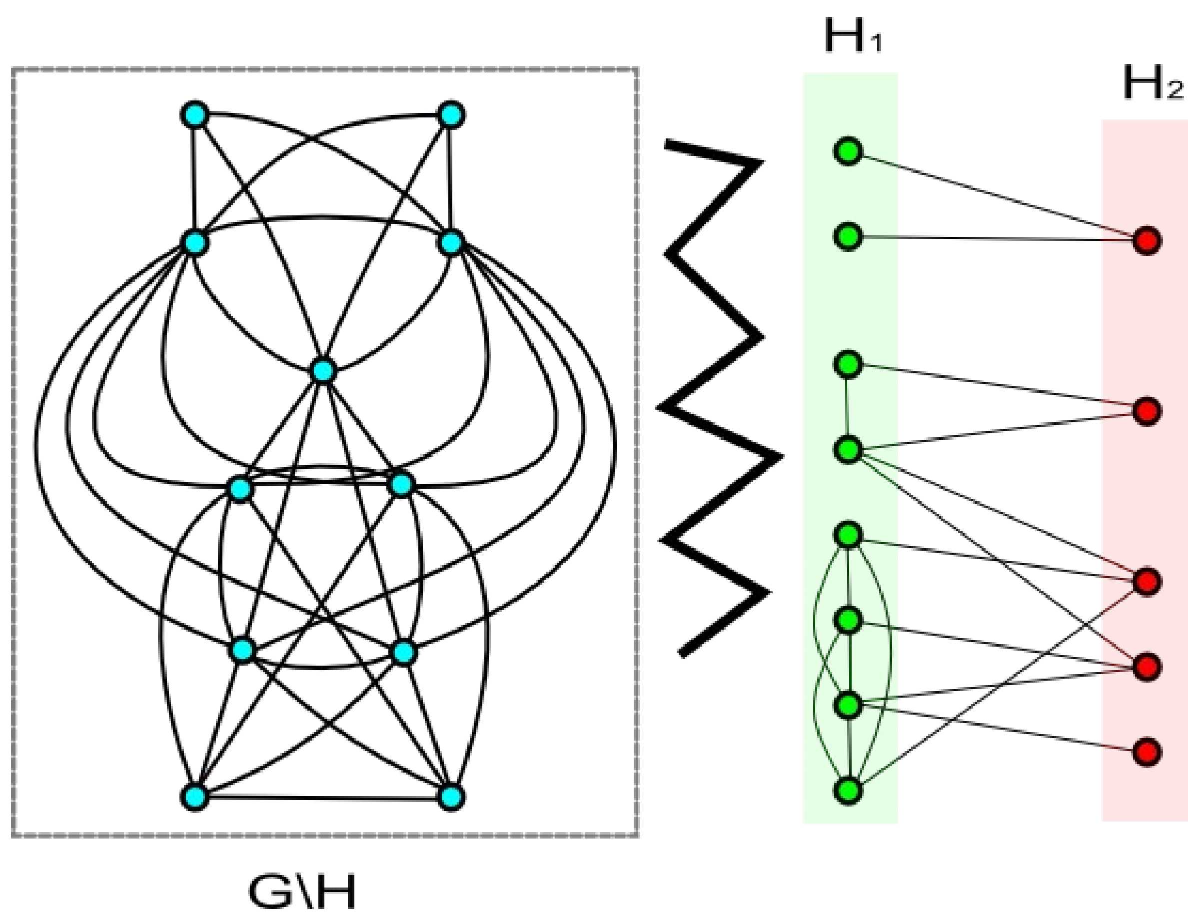
Assign the **same color** to the **merged vertices**

For each **O(q ^{q})** possible **G'** one can obtain **λ'**.

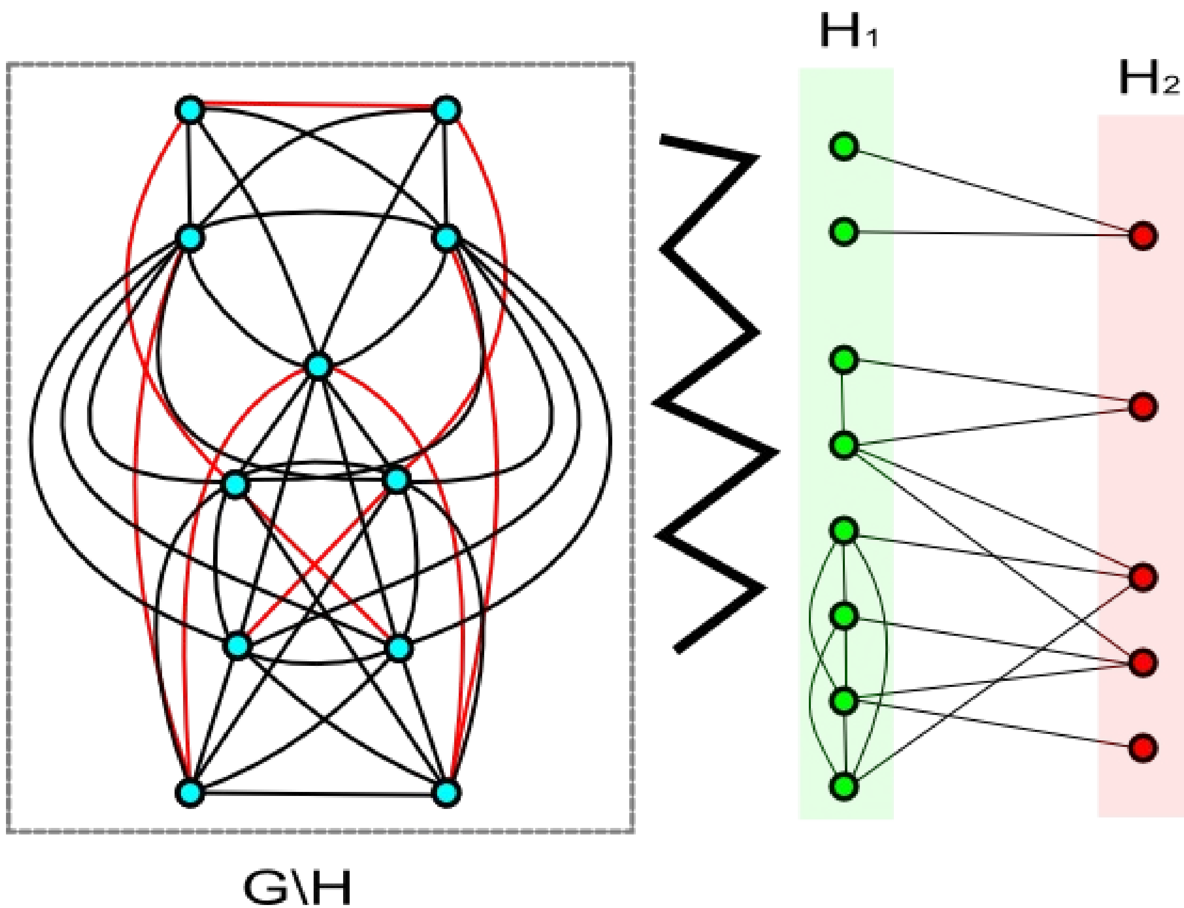
λ will be the **minimum** among all these **λ'**

Complexity: $O(n\ 2q^{5q})$

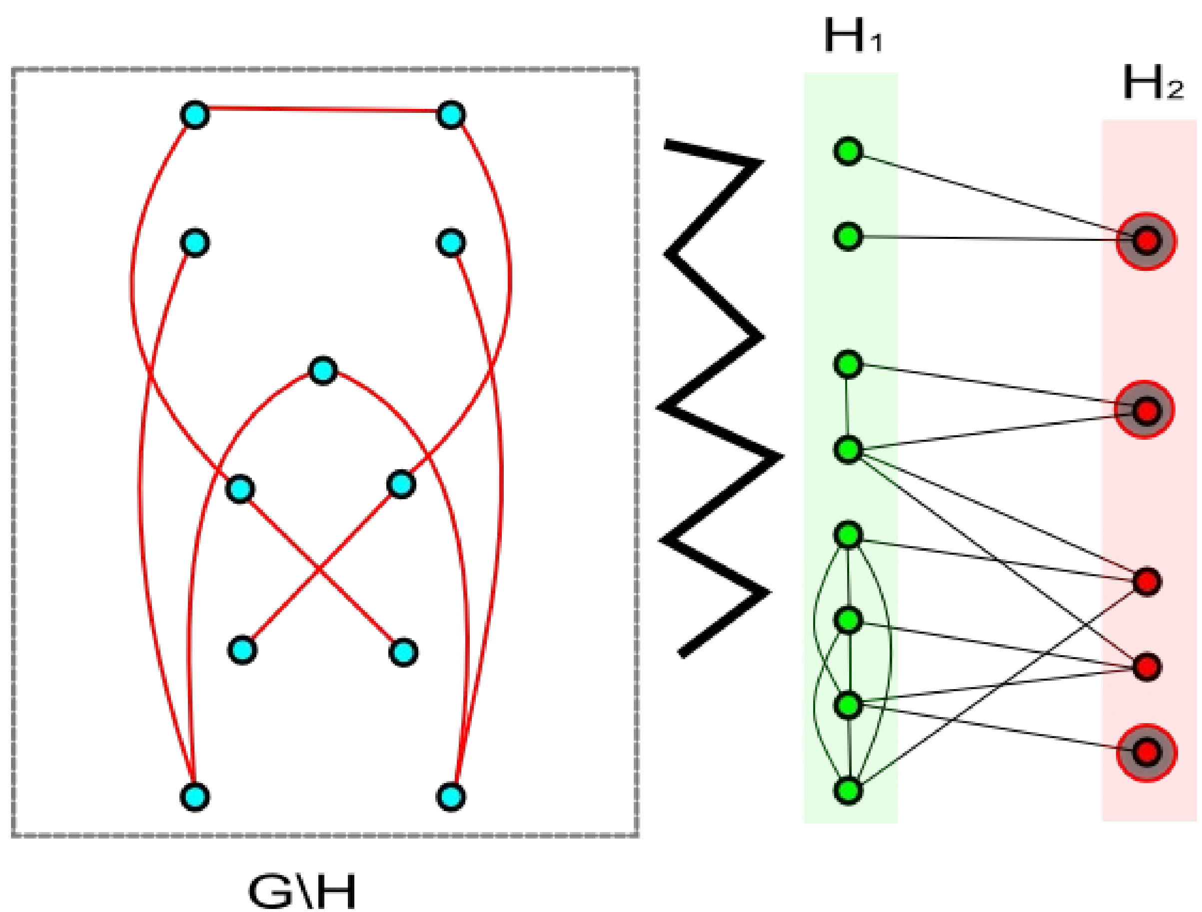
Example



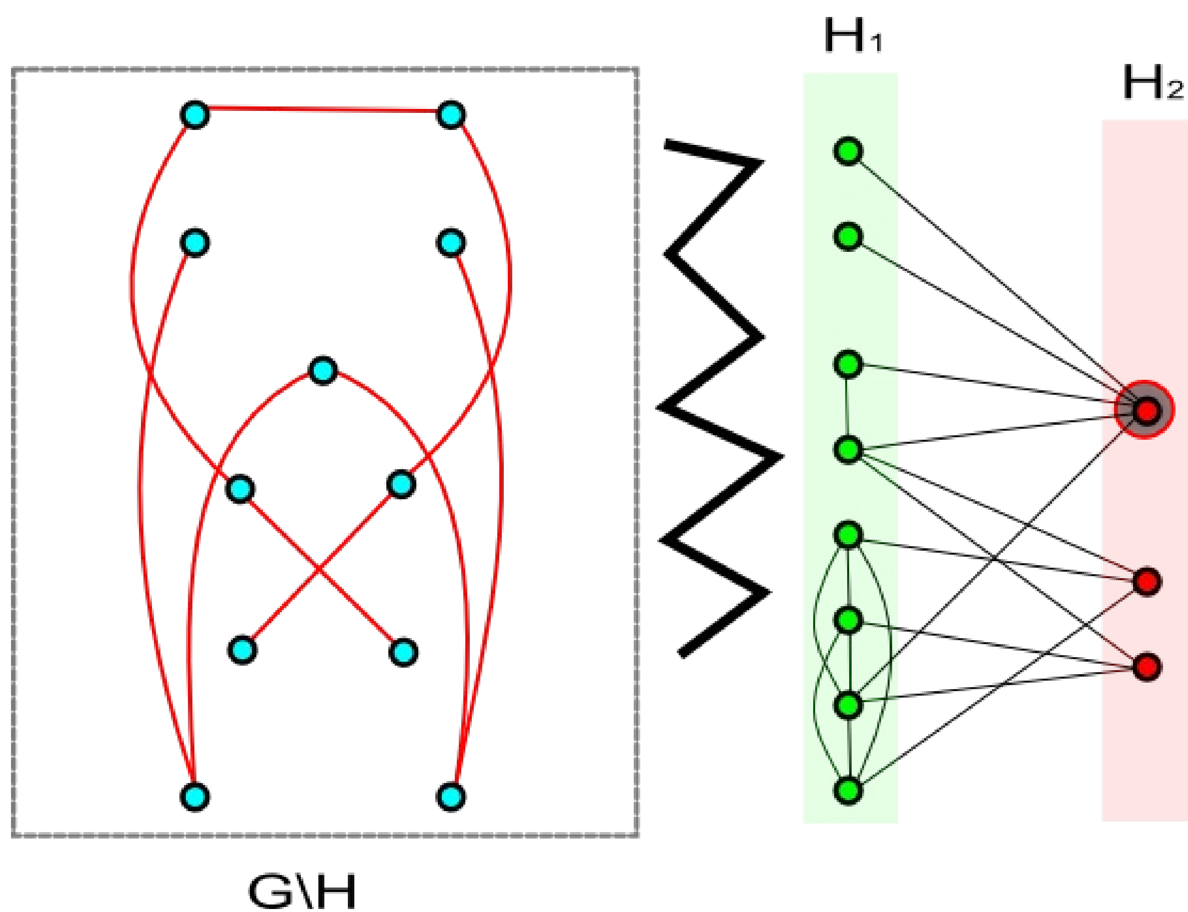
Example



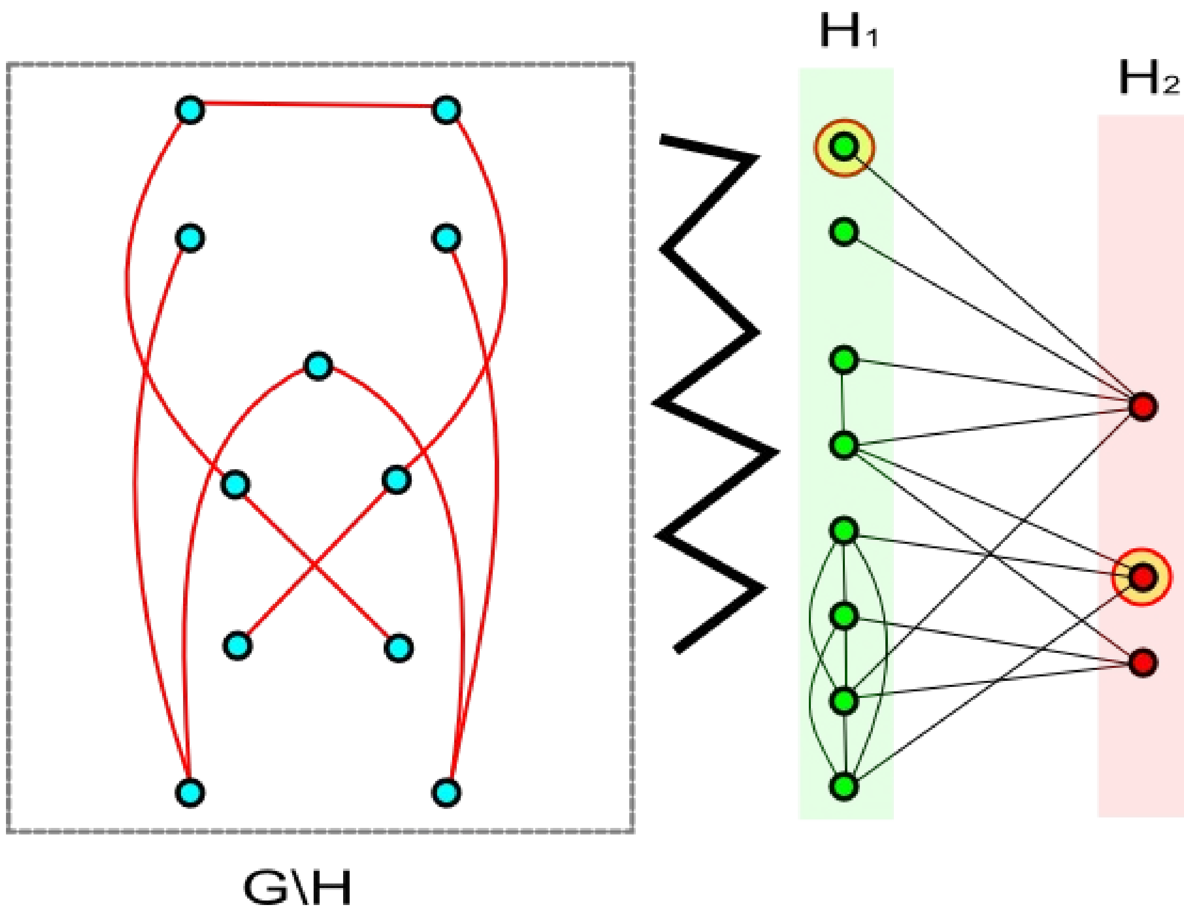
Example



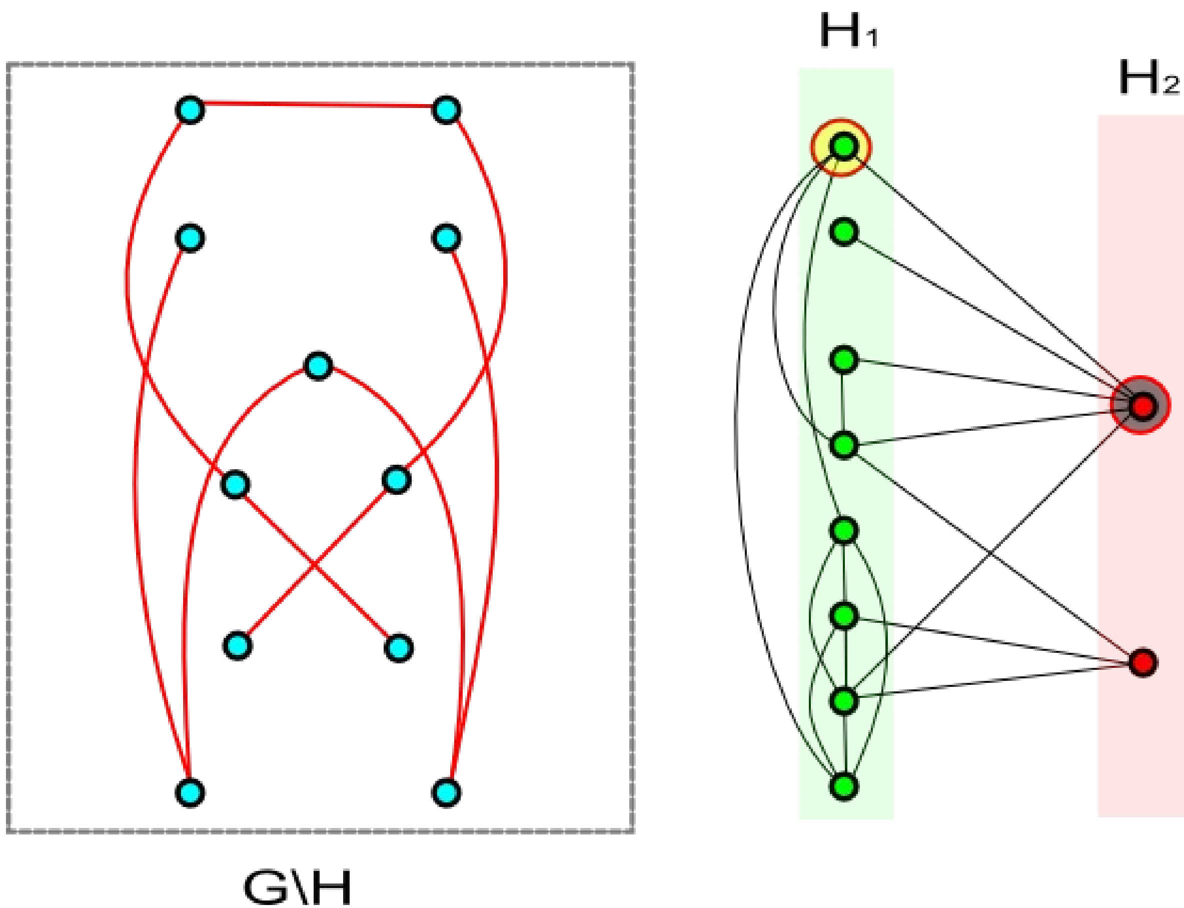
Example



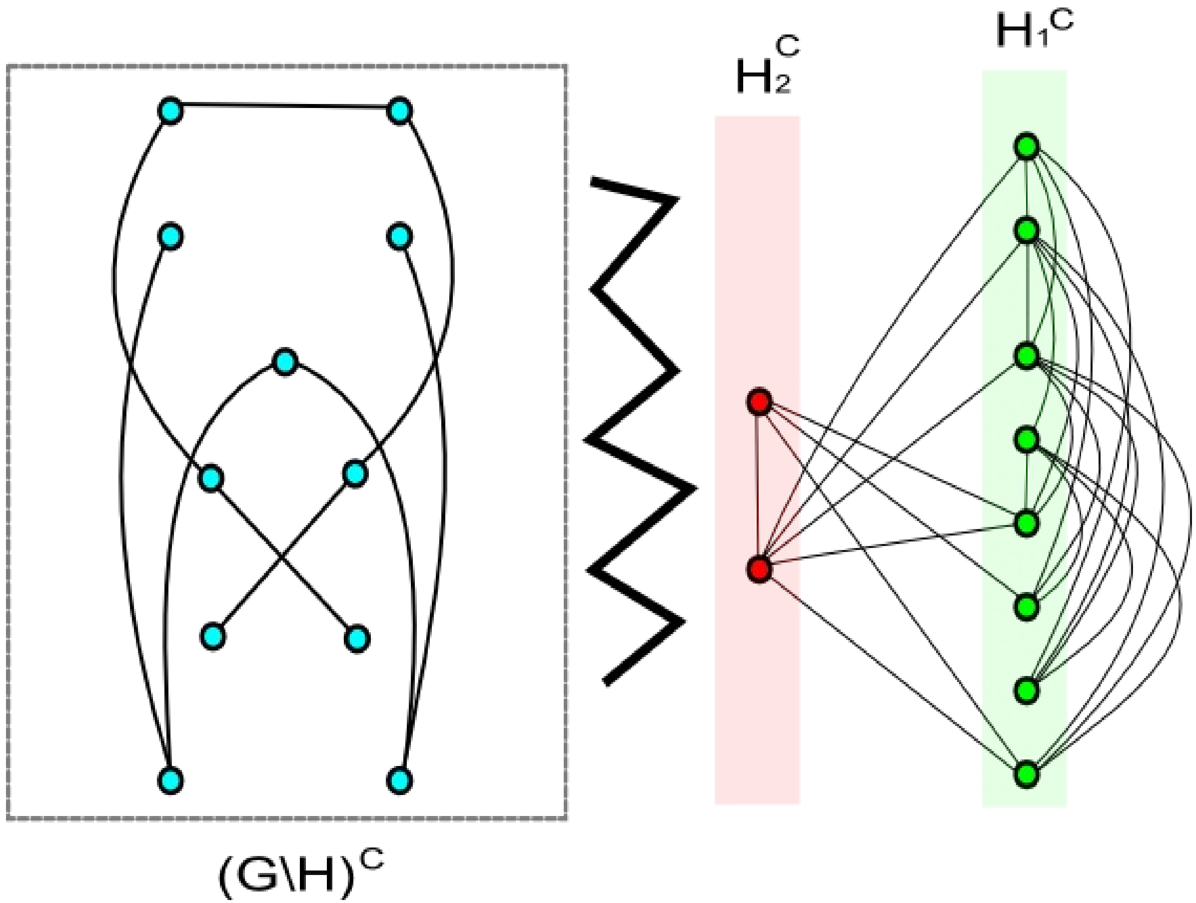
Example



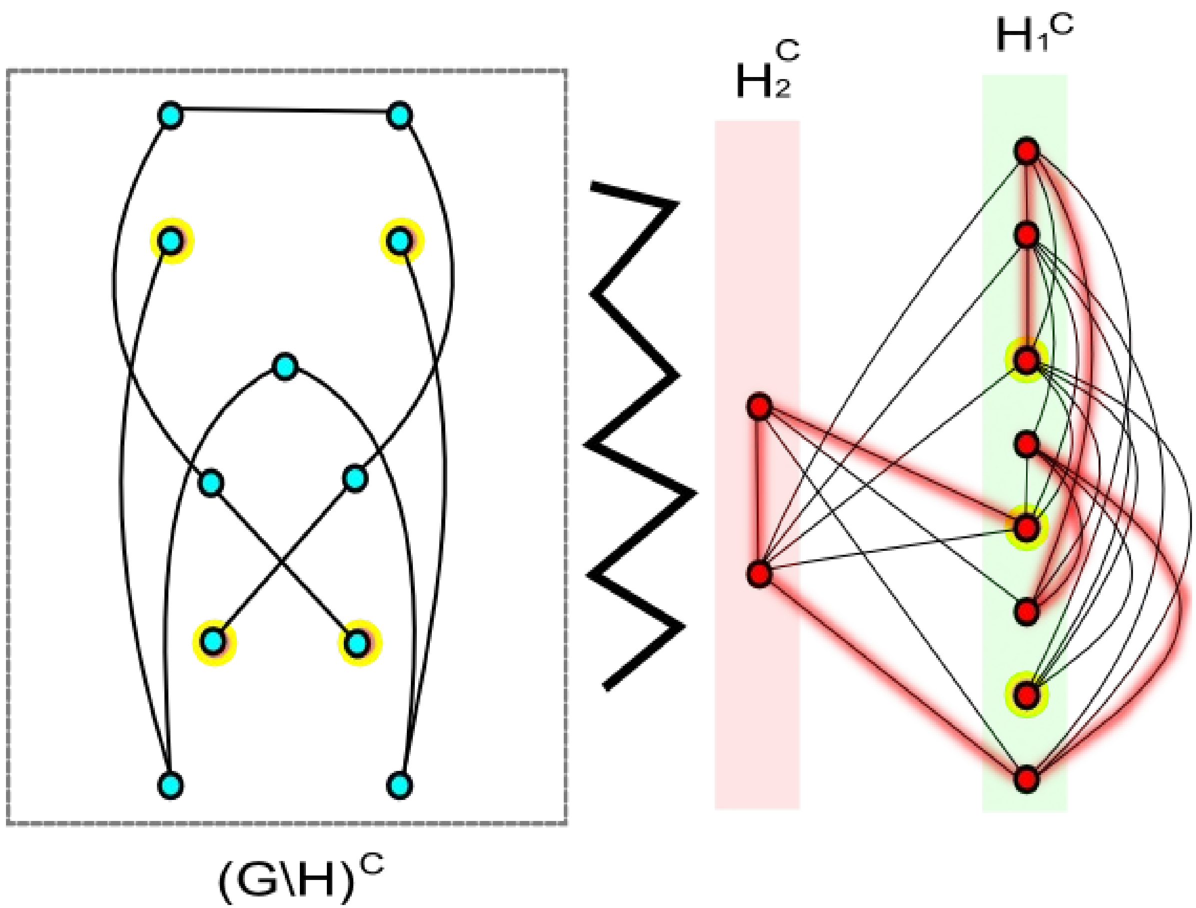
Example



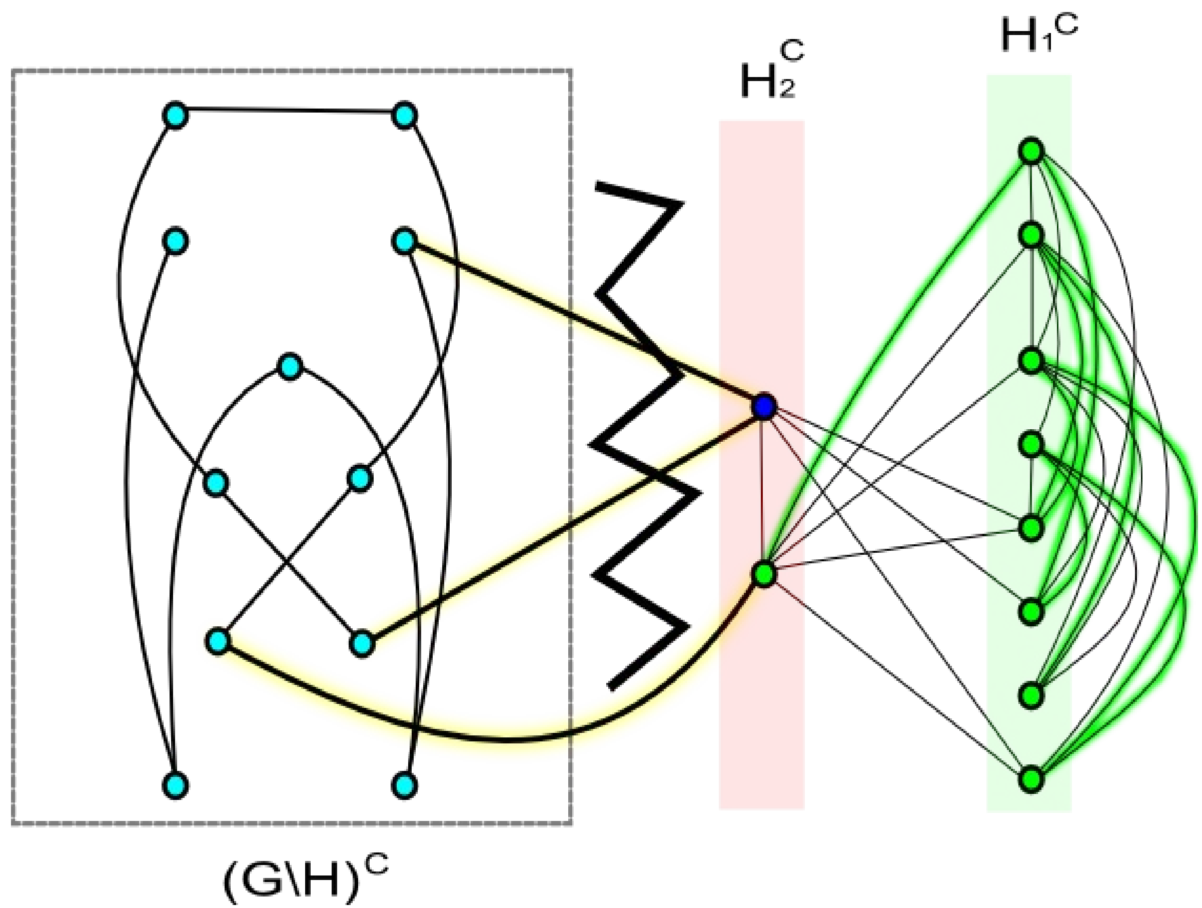
Example



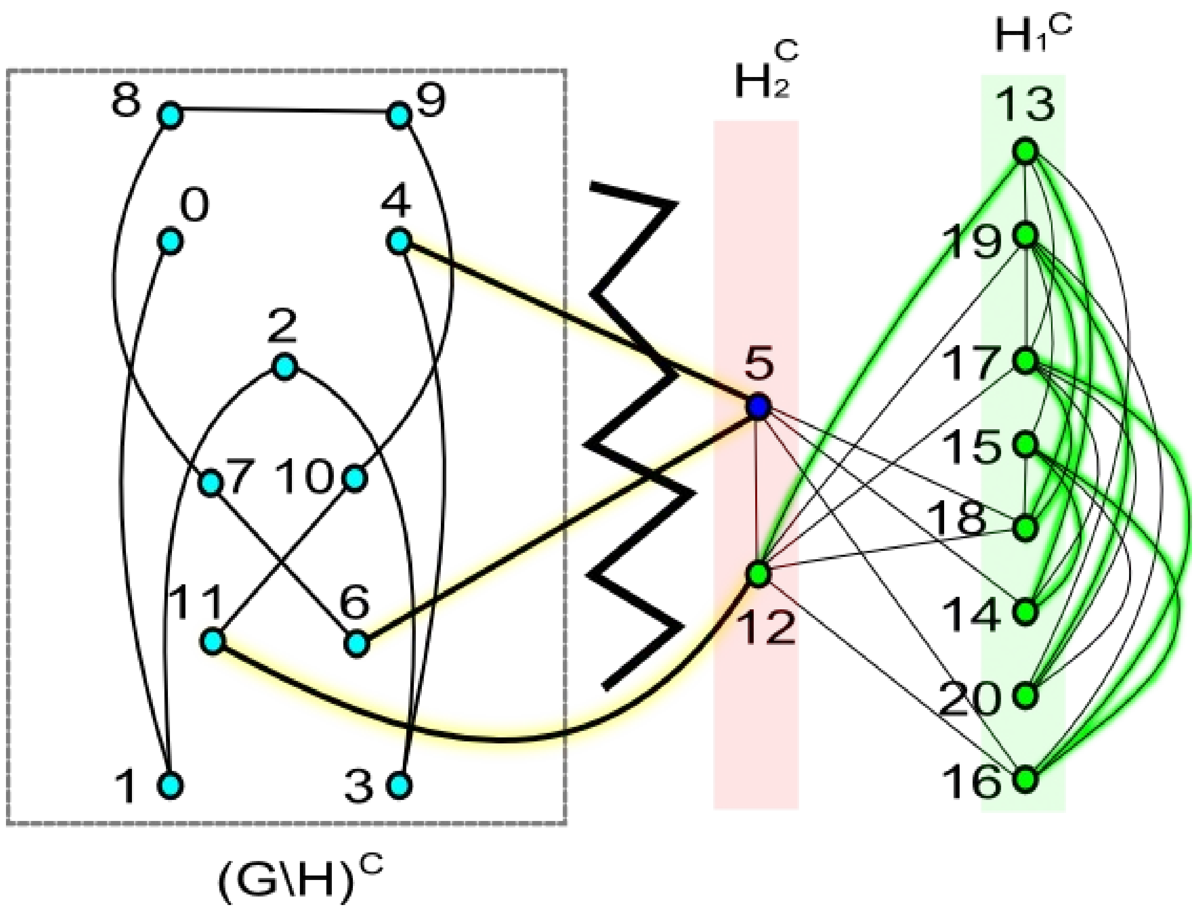
Example



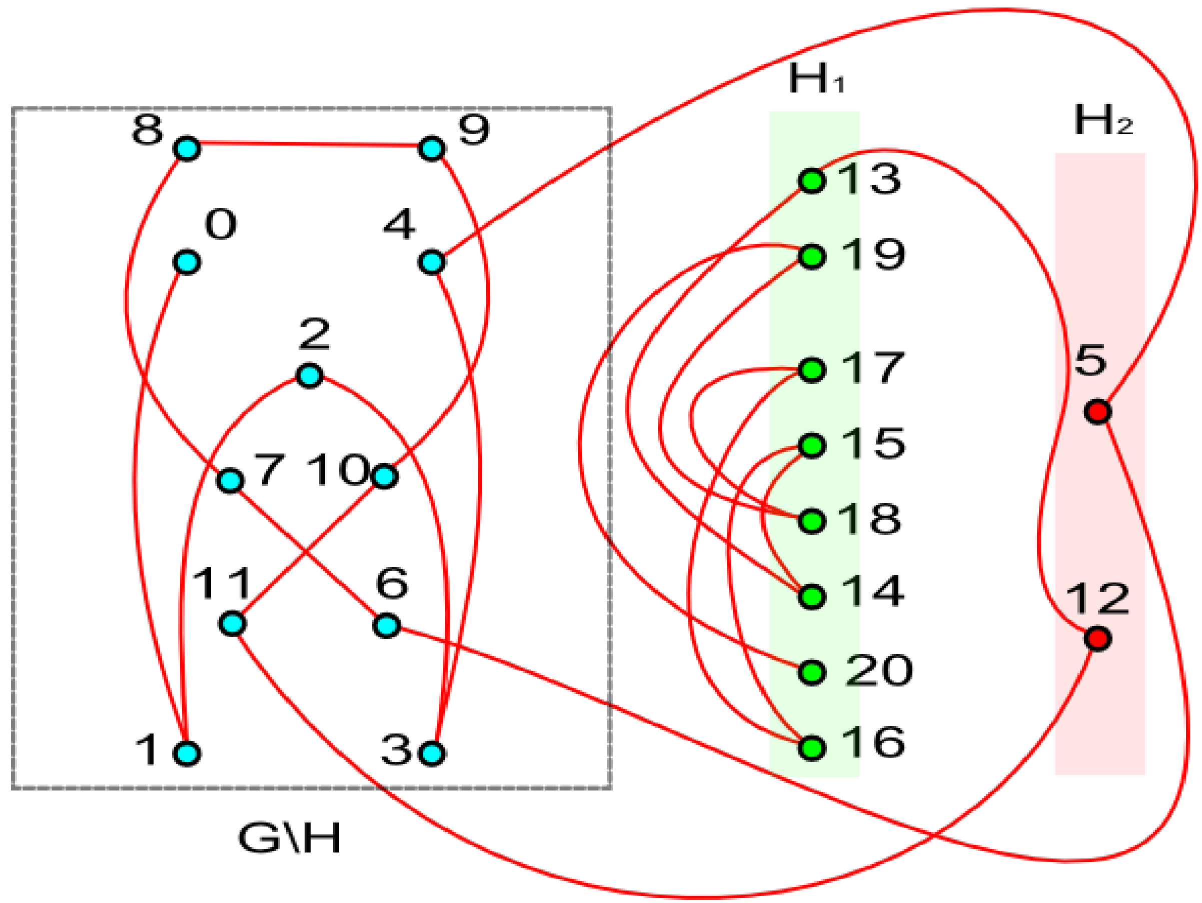
Example



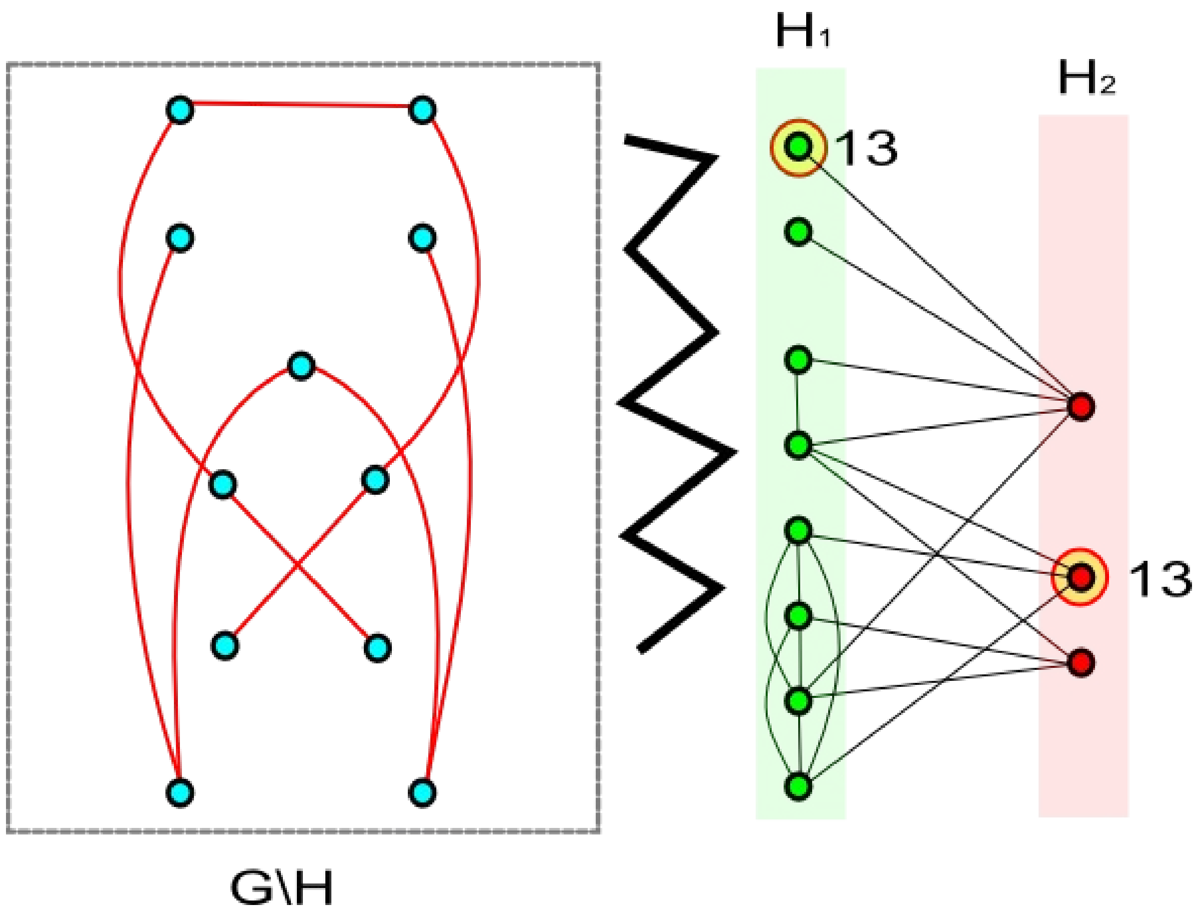
Example



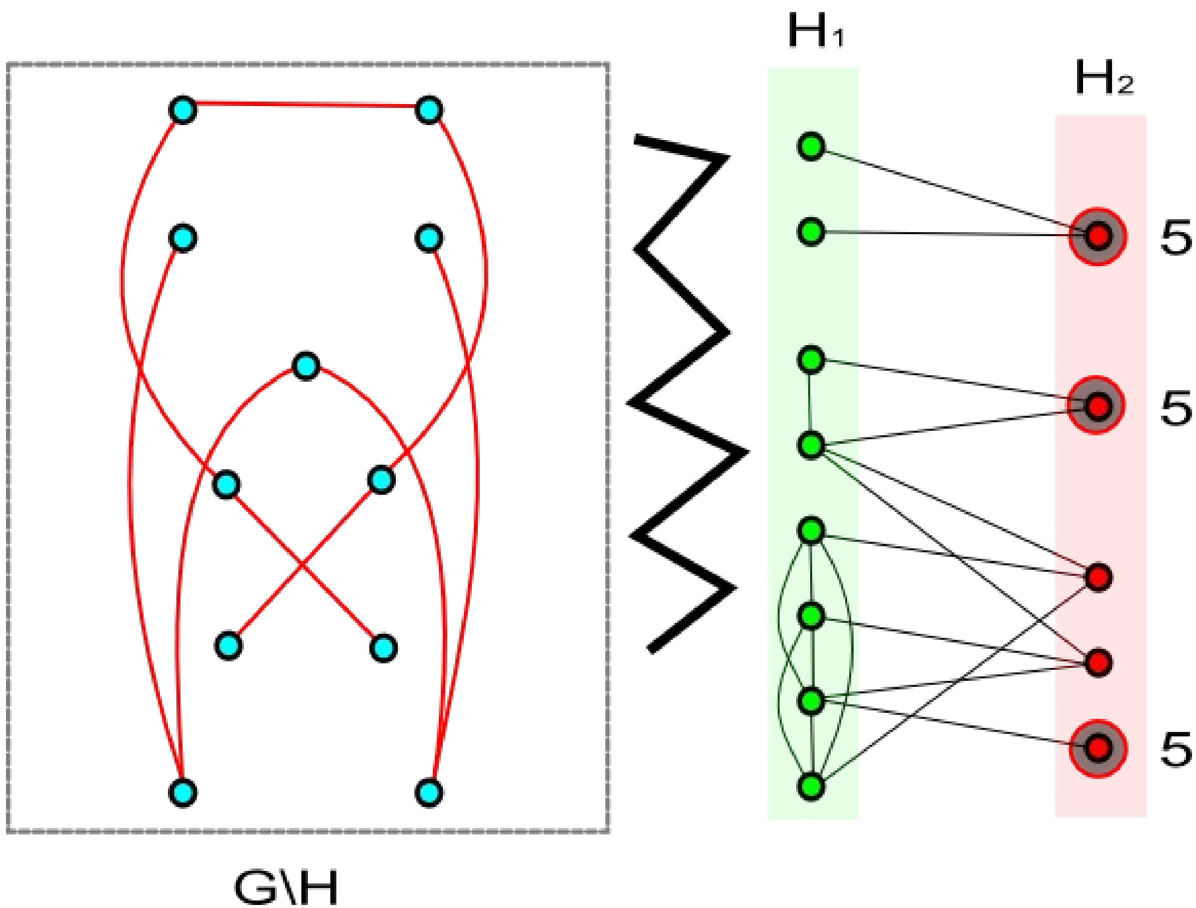
Example



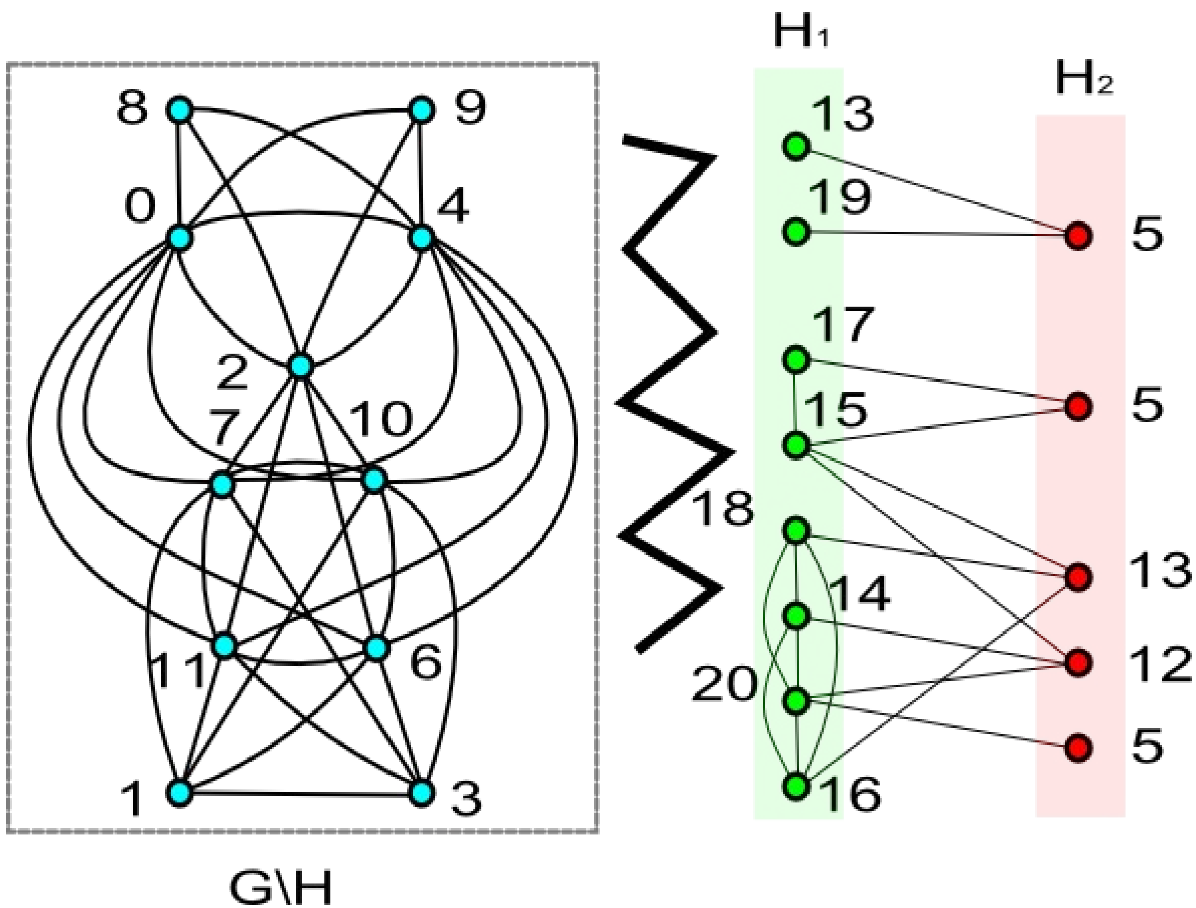
Example



Example

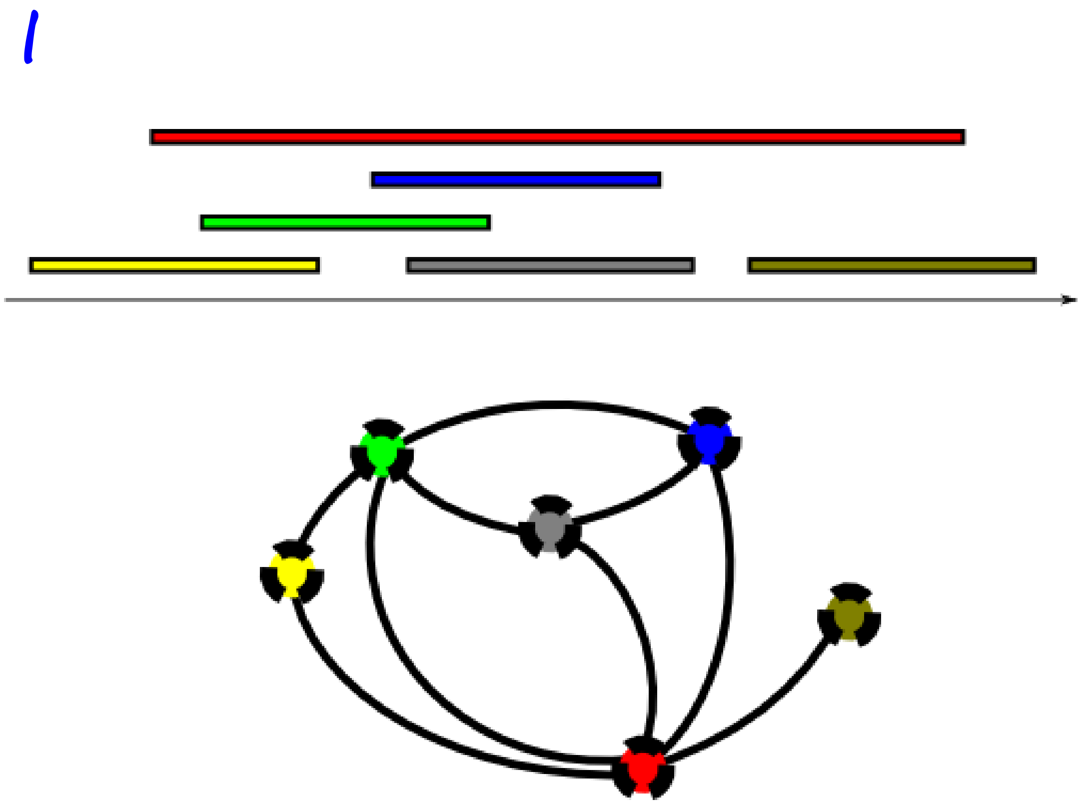


Example



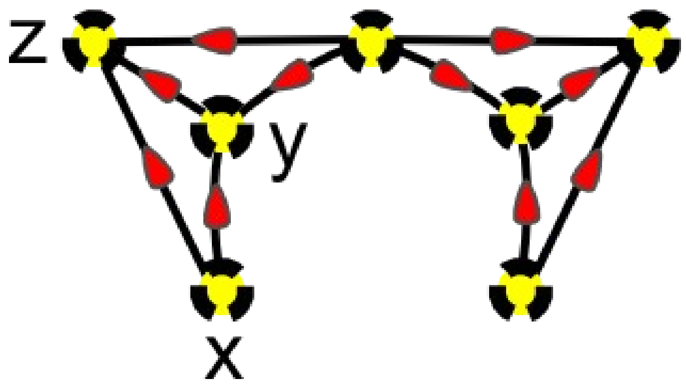
interval graph

- **Interval graph:** G = Ω(*I*)



comparability graph

- **Comparability graph: ∃ transitive orientation** of the edges of the graph.

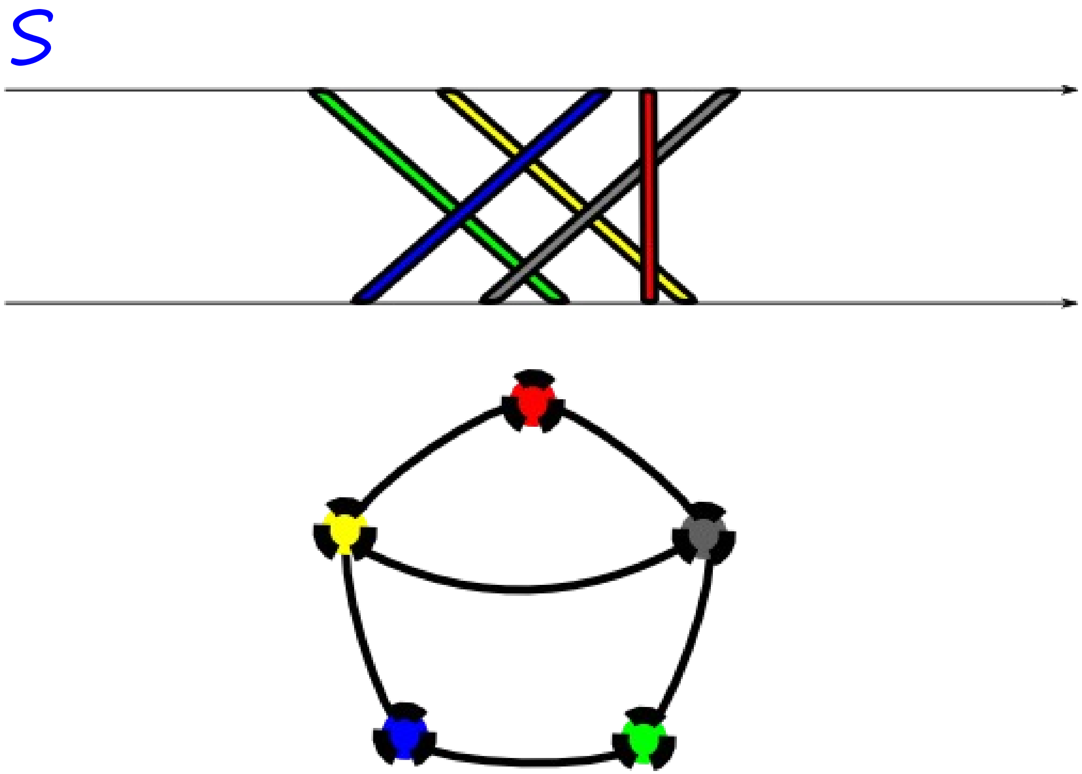


$$\overrightarrow{xy}\ ,\ \overrightarrow{yz} \Rightarrow \overrightarrow{xz}$$

- **Cocomparability graph: G^c** is a comparability graph.

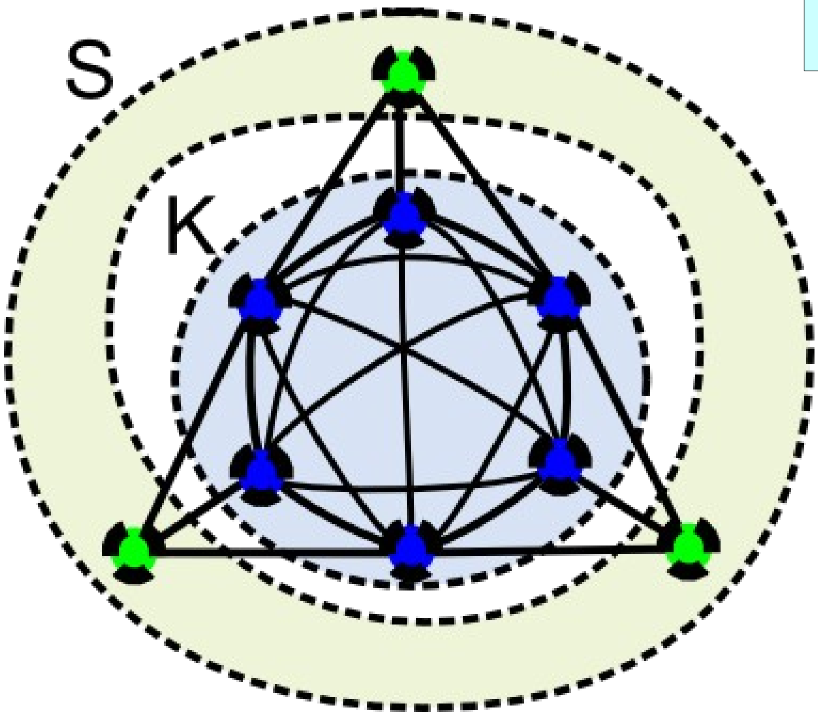
permutation graph

- **Permutation graph:** G = Ω(S)



split graph

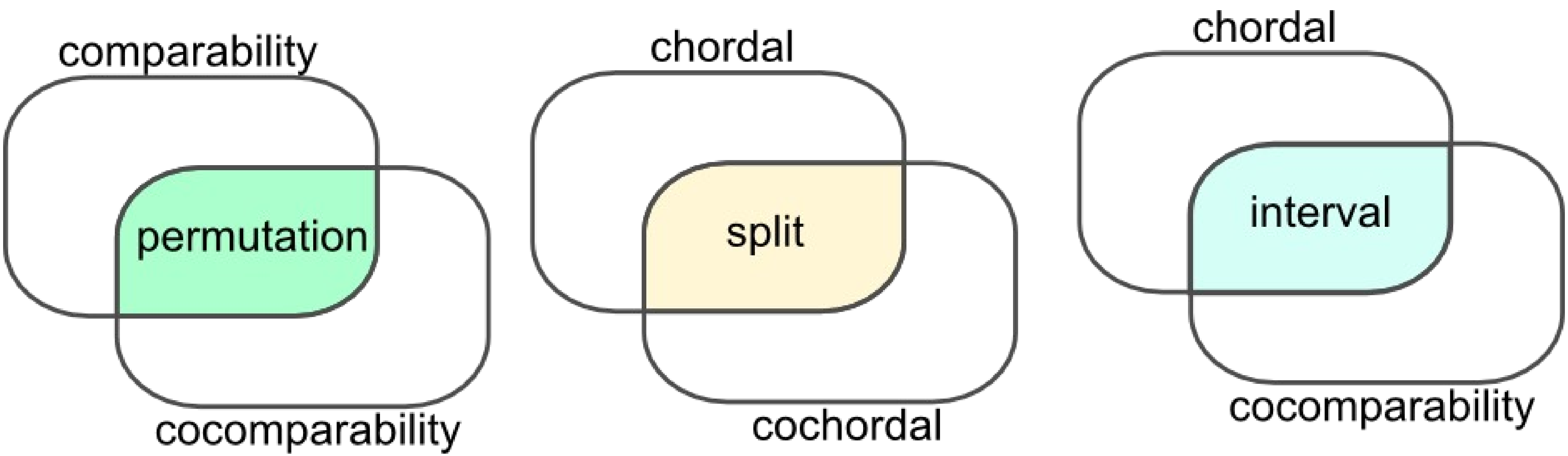
- **Split graph: G = (V, E), V = S ∪ K.**



K is a **clique**
S is a **stable set**

split permutation graphs

- **Split permutation graph:** **G** is **split** and **permutation**.



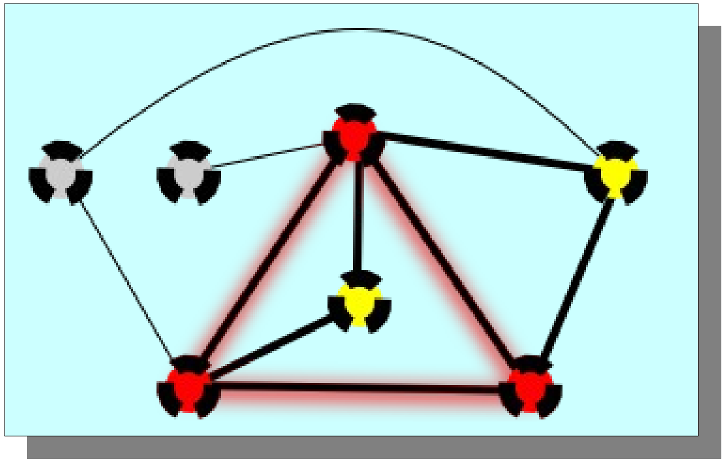
- **[Brandstädt, Bang Le and Spinrad-99]**

split permutation graphs

- There are $\theta\left(\frac{4^n}{\sqrt{n}}\right)$ split permutation graphs.
  [Guruswami-99]

- **Split permutation $\subset$ clique Helly**
  [ISGCI]

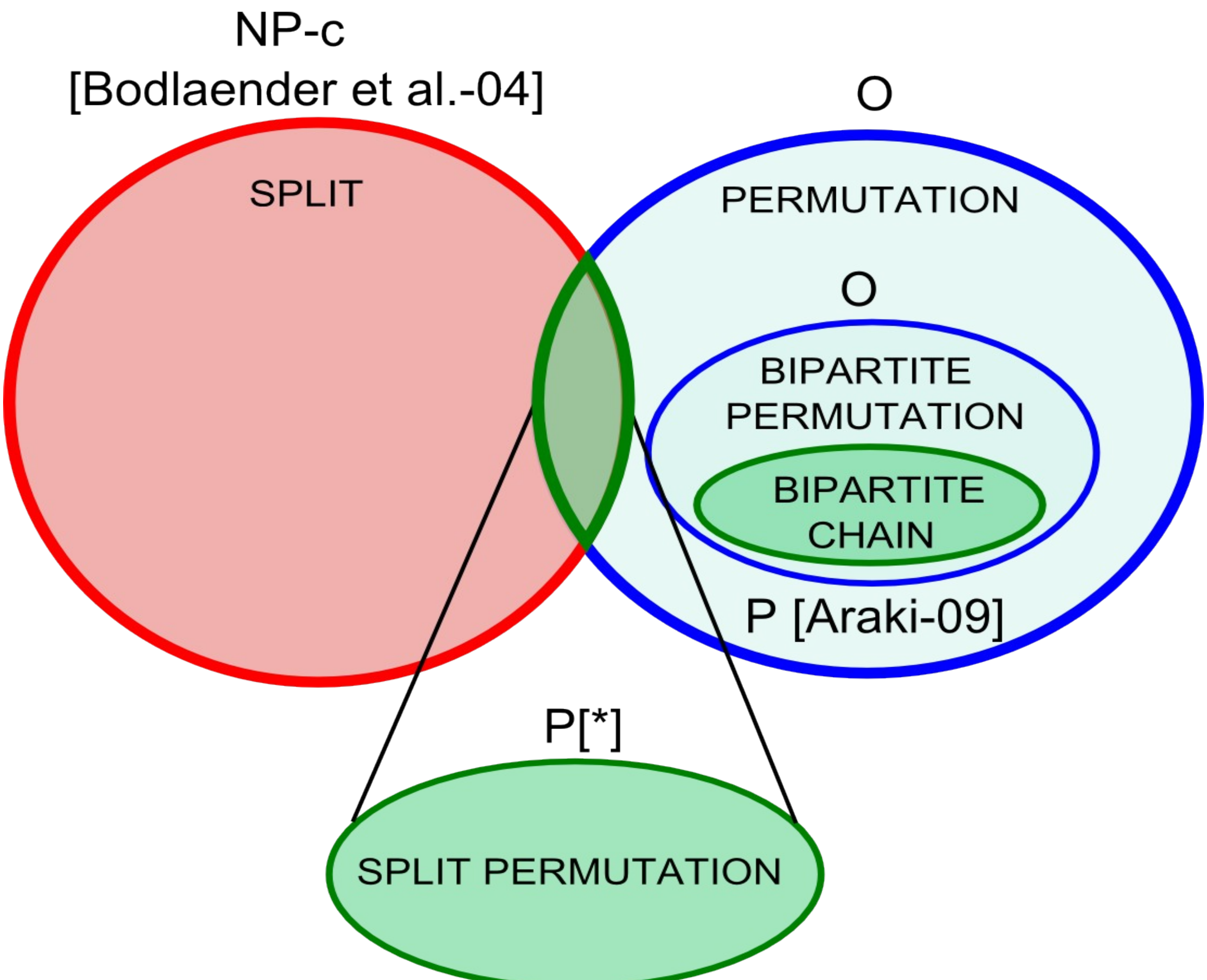
- **Extended triangle**



- **G** is **clique-Helly** $\Leftrightarrow$ every **extended triangle** of G has an **universal vertex**

  [Szwarcfiter-97]

split permutation graphs



split permutation graphs

- **Our work:**

For a **split permutation** graph **G**,

$$\lambda(G) = \max\{ \lambda(G_R), \lambda(G_L) \}$$

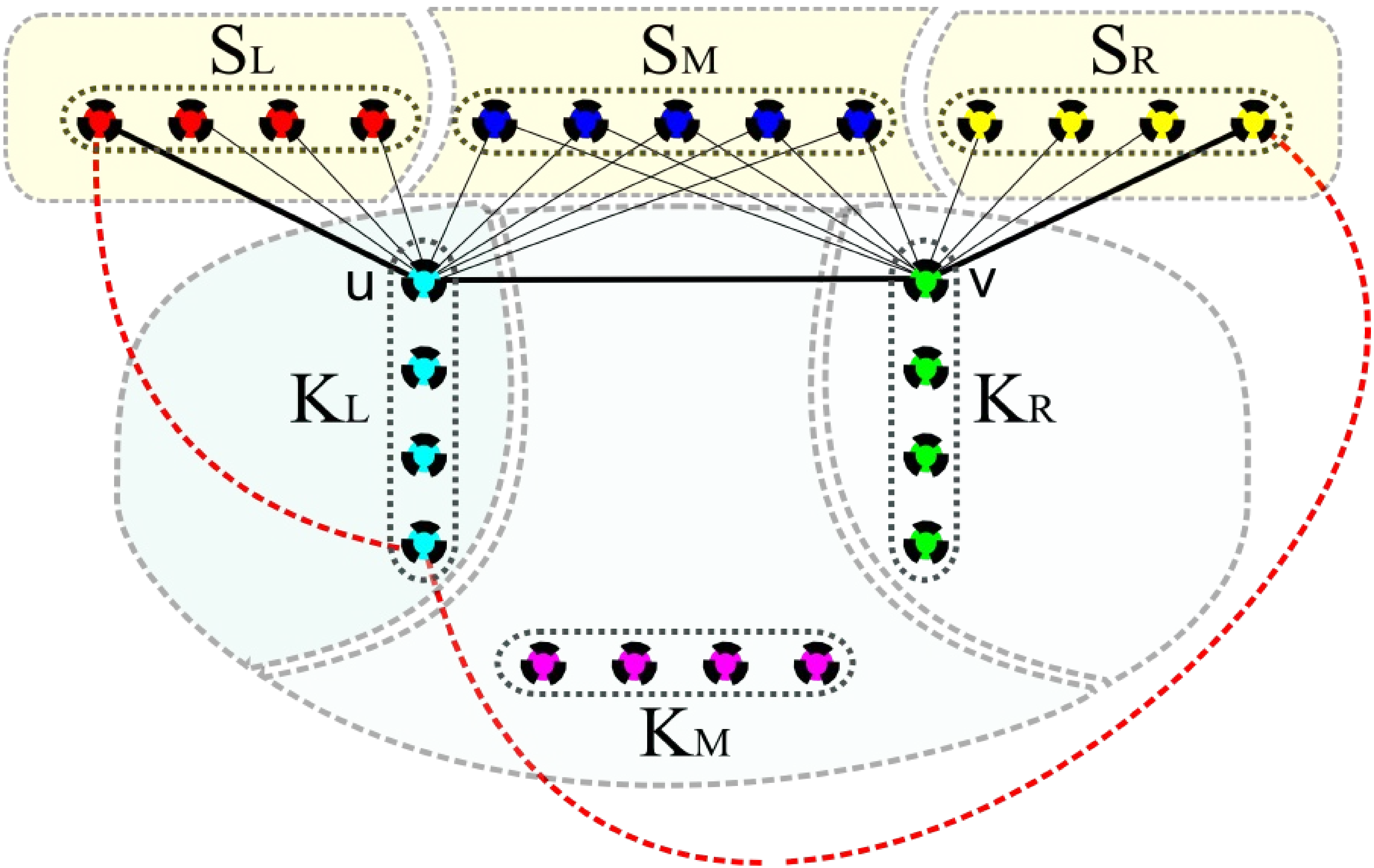
$G_L = G \setminus S_R$

$G_R = G \setminus S_L$

⇒ $\lambda(G)$ can be **computed** in linear time.

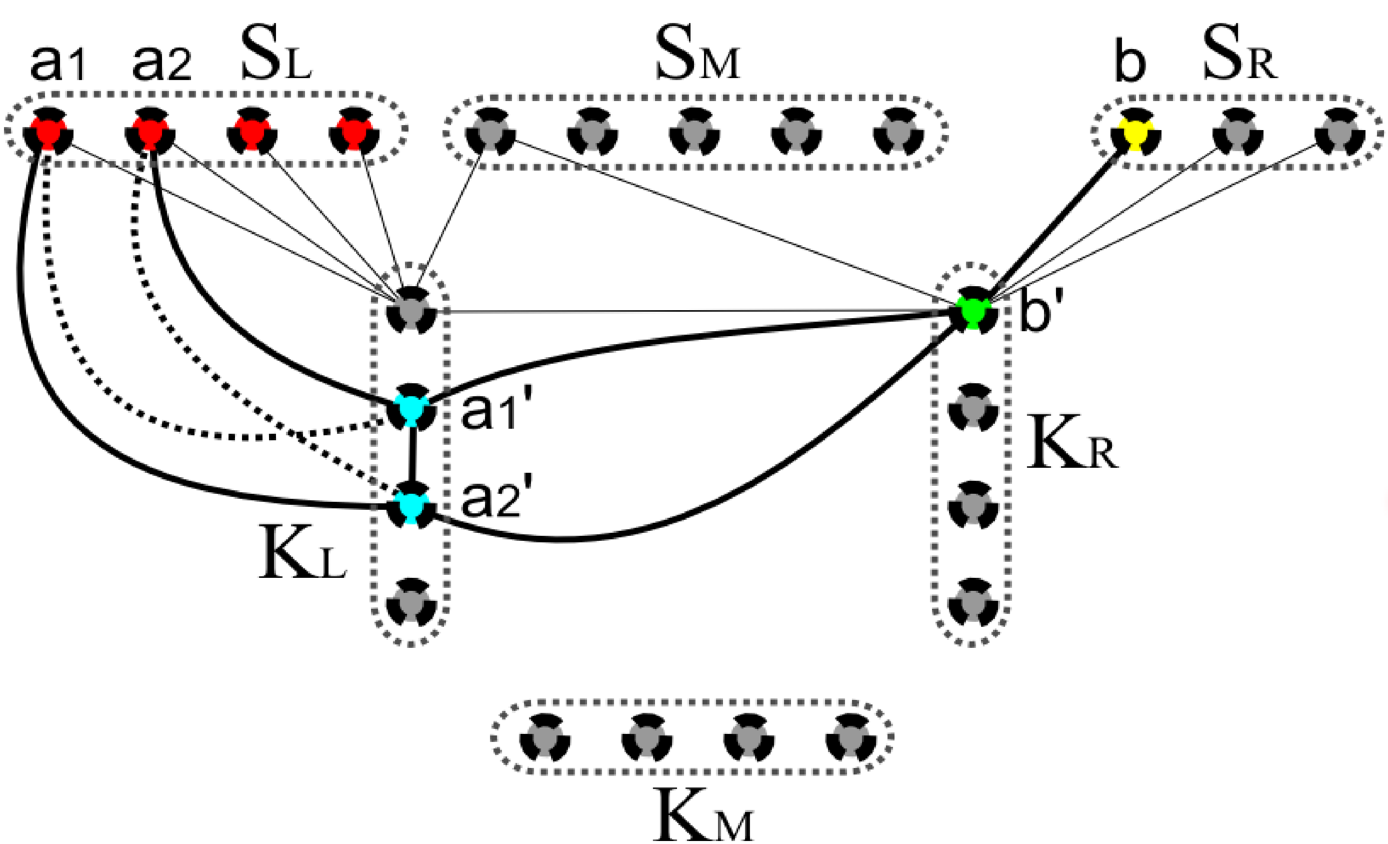
O(n^2) algorithm that obtain an **λ-coloring** with **this span**

split permutation graphs

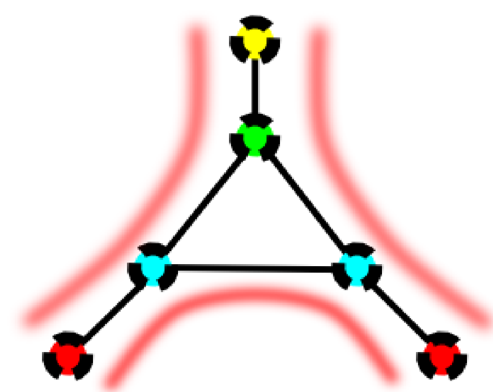


split permutation graphs

$\exists$ Chain ordering $a_1 < a_2 < \ldots < a_{L+M}$
such that: $N(a_1) \subseteq N(a_2) \subseteq \ldots \subseteq N(a_{L+M})$



Asteroidal triple (AT)



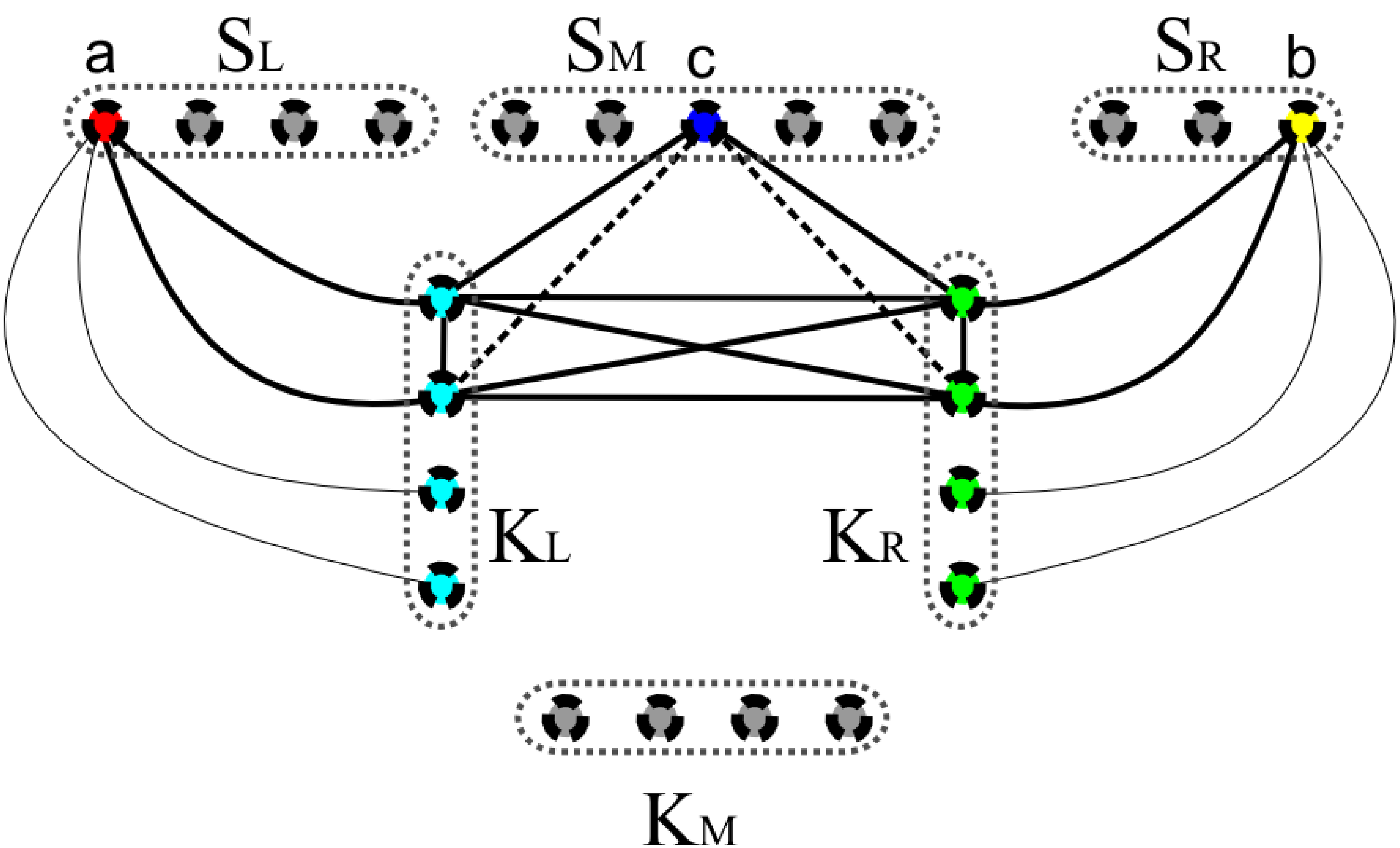
interval

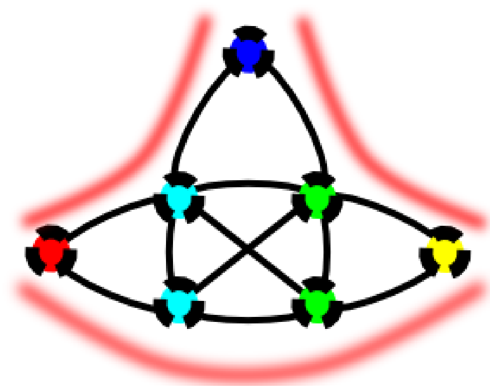
(AT-free)

C

split permutation graphs

$\forall\, c \in S_M \Rightarrow K_L \subseteq N(c)$ or $K_R \subseteq N(c)$.

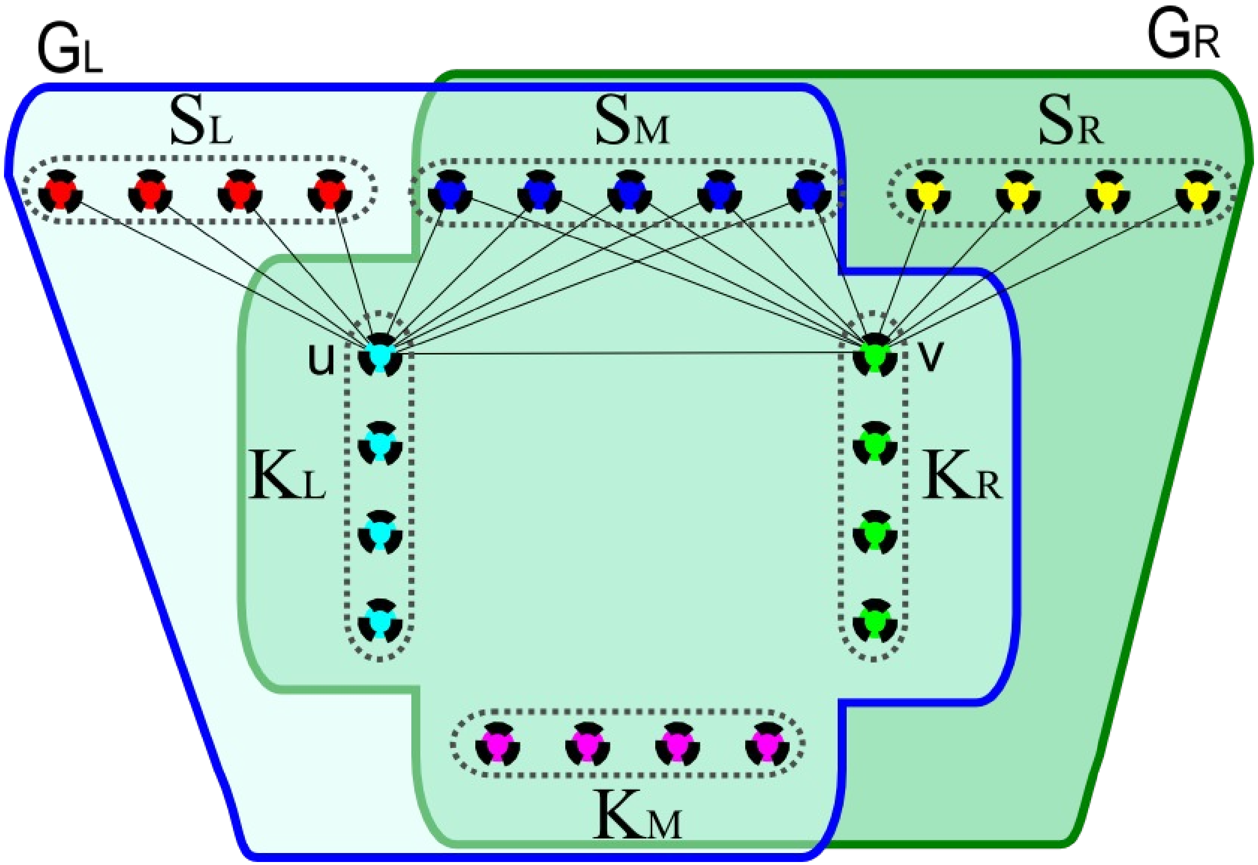




interval

split permutation graphs

For a split permutation graph **G**,
λ(G) ≥ max{λ(G$_R$), λ(G$_L$) }



diameter of **G$_L$** is **2**

⇓

$\lambda(G_L) = n_L + pv(G_L^c) - 2$

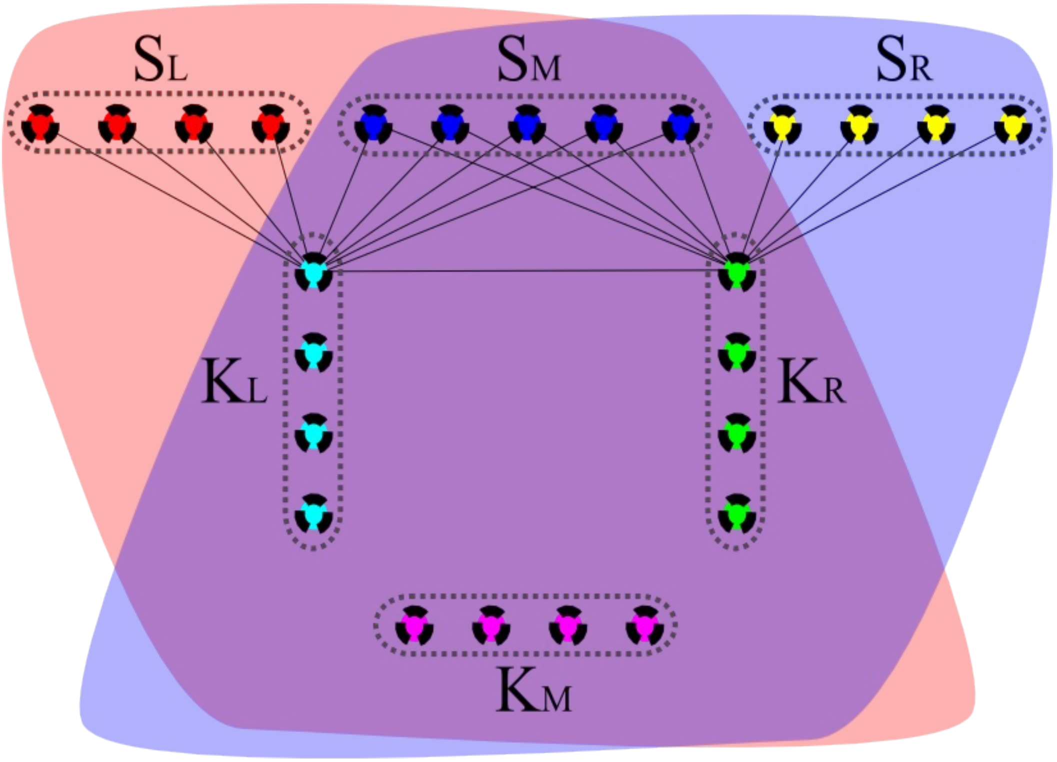
G$_L$ subgraph of G

⇓

$\lambda(G) \geq \lambda(G_L)$.

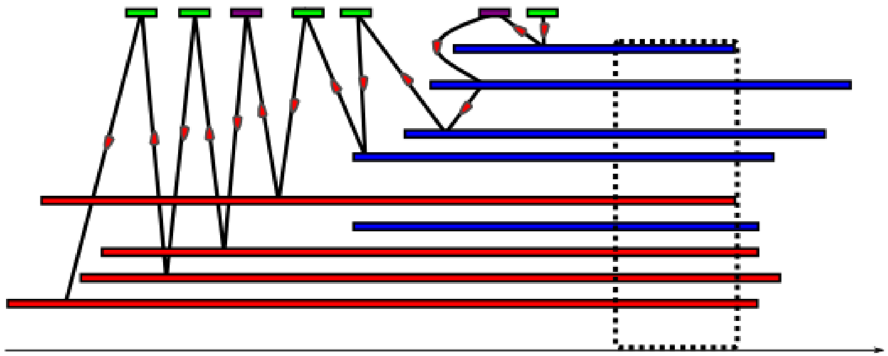
split permutation graphs

- $P(R)$?

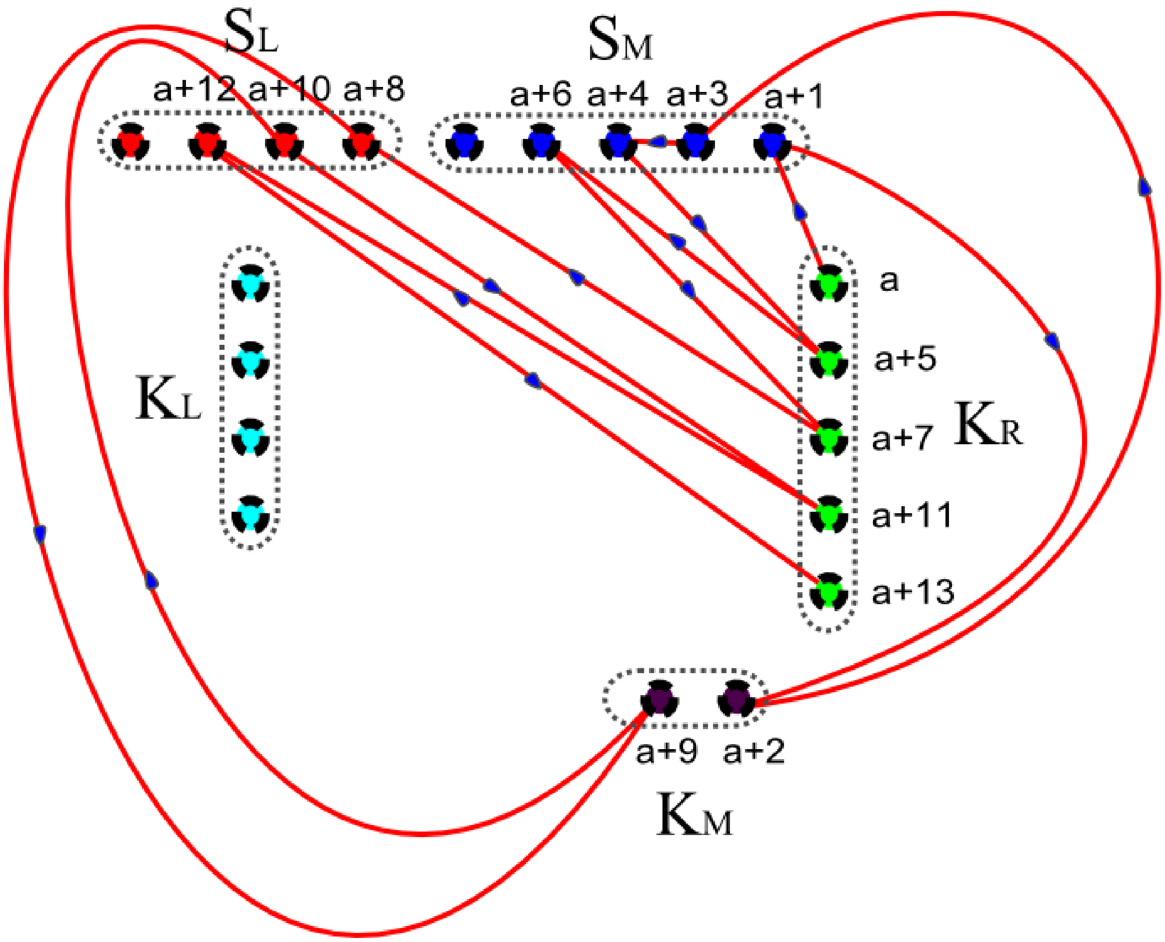


$$\lambda(G_R) \geq P(R)$$
$$\Downarrow$$
$$\lambda(G) = \max\{\lambda(G_R), \lambda(G_L)\}$$

split permutation graphs



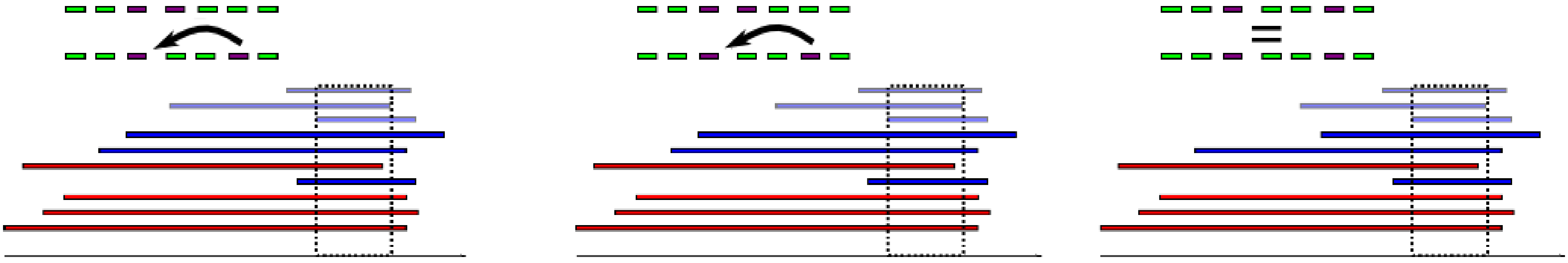
G^c



cointerval

split permutation graphs

- **Interval model** has to be **modified**:

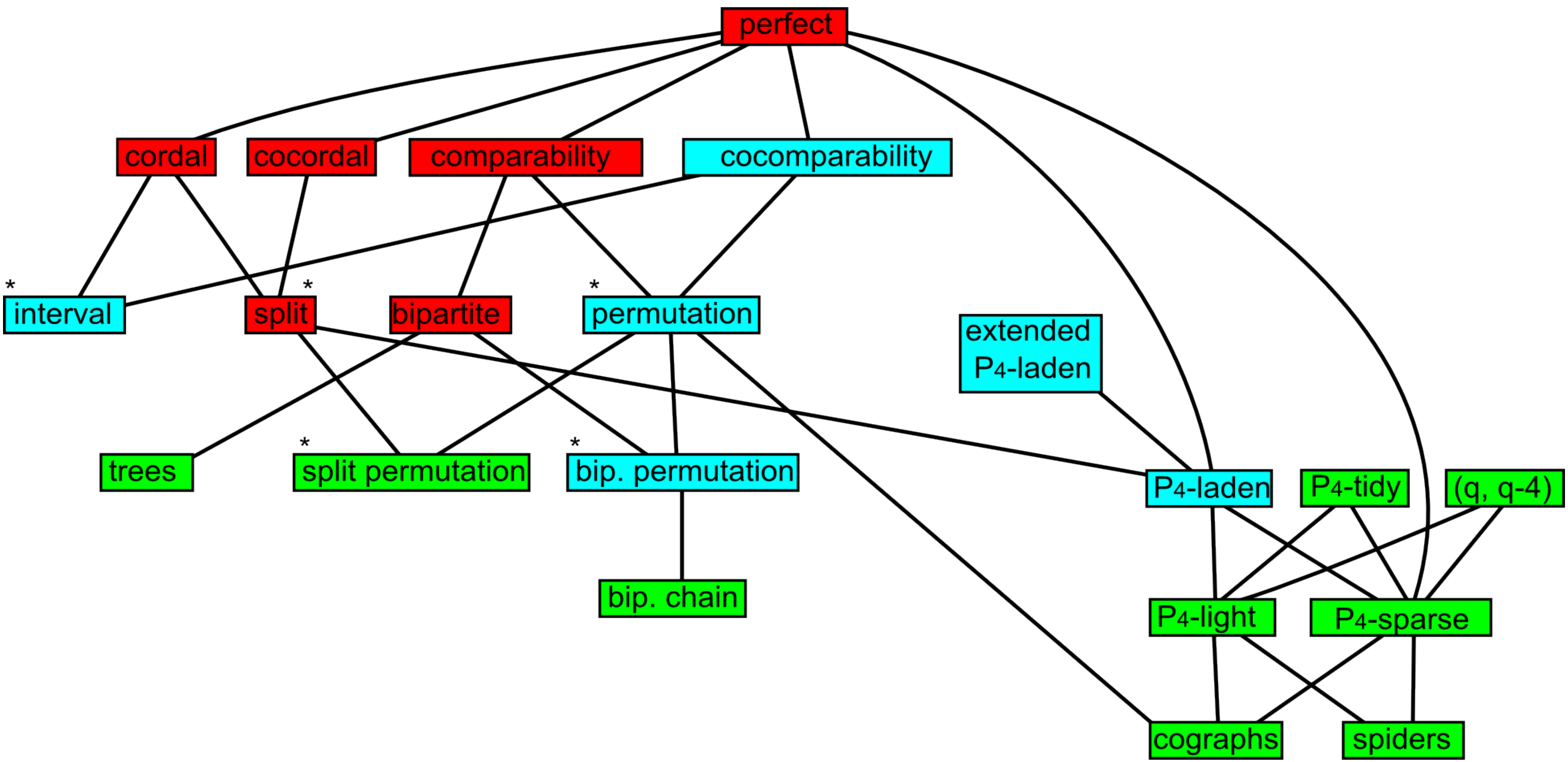


(a) (b) (c)

split permutation graphs

- A **linear-time algorithm** to find $\lambda(G_R)$ and $\lambda(G_L)$
- For split permutation graphs $\lambda(G) = \max\{\lambda(G_R), \lambda(G_L)\}$.
- This proof also gives a $O(n^2)$ algorithm to find an **optimum λ-coloring** of graphs on this class.

Open Problems



Open Problems

- **Griggs and Yeh Conjecture**:

$$\lambda \leq \Delta^2$$

only proved for a few classes of graphs,

and it is still open for bipartite graphs.

- $\lambda \leq \Delta^2 + \Delta - 2$
[Gonçalves 06]

Upper bounds in λ

Class	Upper bound
diameter 2	Δ^2 **[Griggs and Yeh]**
regular grids	**Δ+2 [Calamoneri et al.]**
cocomparability	**4Δ -1 [Calamoneri et al.]**
cograph	n + pv(G^c) -2 **[Chang e Kuo]**, 2Δ **[C. and P.]**
planar	**2Δ + 25 [van den Heuvel and McGuinness]**
bipartite permutation	wb(G)+1 **[Araki]**
weakly chordal	Δ^2 **[C. and P.]**
split	$0.385\Delta^{1.5} + 2\Delta + \Delta^{0.5} - 2$**[C. and P.]**
interval	2Δ **[Calamoneri et al.]**