Partitions d'entiers et groupes de Coxeter

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Plan

1. Nekrasov-Okounkov type formulas
2. Generalizations through Littlewood decomposition
3. Coxeter groups and automata theory

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Partitions

A partition λ of n is a non-increasing sequence $(\lambda_1, \lambda_2, \dots, \lambda_k)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. We represent a partition by its Ferrers diagram.

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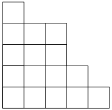

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4	2	1		
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7	5	4	1	
9	7	6	3	1

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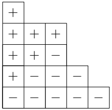

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Set $\varepsilon_h = \begin{cases} +1 & \text{if } h \text{ is stricly above the diagonal} \\ -1 & \text{else} \end{cases}$

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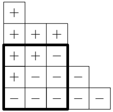

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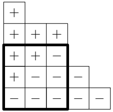

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$\mathcal{H}_t(\lambda)$ the multi-set of hook lengths which are multiple of t

t-cores

Let $t \geq 2$ be an integer. A partition is a *t-core* if its hook lengths set does not contain t. It is equivalent to the fact that the hook lengths set does not contain any integral multiple of t, *i.e.* $\mathcal{H}_t(\lambda) = \emptyset$.

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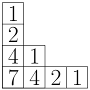

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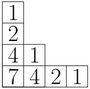

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Self-conjugate and doubled distinct partitions

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$SC_{(t)}$: subset of self-conjugate t-cores.

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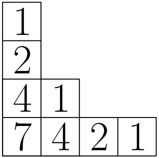

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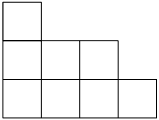

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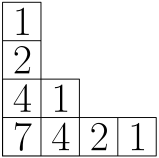

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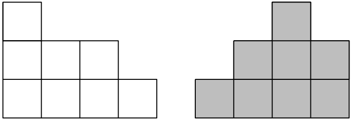

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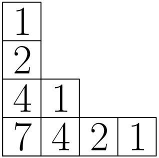

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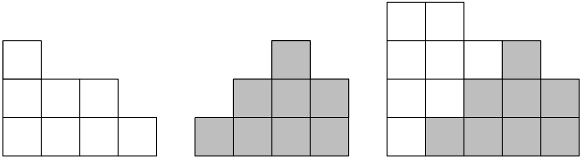

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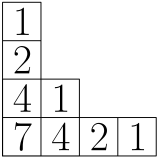

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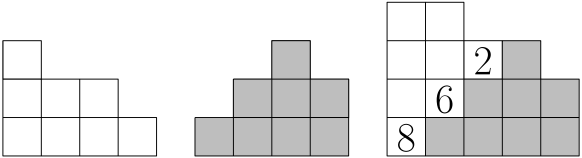

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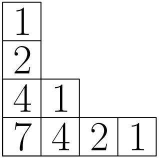

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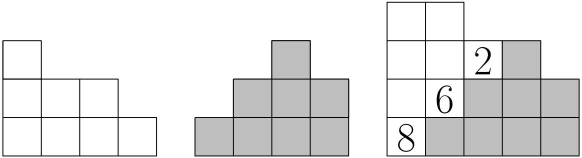

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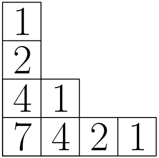

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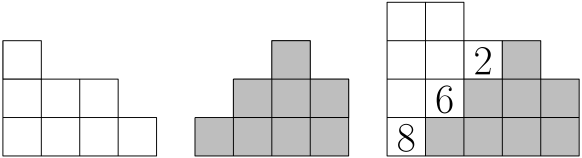

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$$\begin{pmatrix} b_1 + 1 & \dots & b_k + 1 \\ b_1 & \dots & b_k \end{pmatrix}$$

Dedekind η function

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Lehmer's conjecture (1947)

Coefficents of the expansion of η^{24} are nonzero.

Nekrasov-Okounkov type formulas

Theorem (Nekrasov-Okounkov, 2006; Westbury, 2006 ; Han, 2009 ; P., 2015)

For all complex number z, we have :

$$\prod_{k\geq 1}(1-x^k)^{z-1} = \sum_{\lambda\in\mathcal{P}} x^{|\lambda|} \prod_{h\in\mathcal{H}(\lambda)} \left(1-\frac{z}{h^2}\right)$$

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$$\left(\prod_{k\geq 1}\frac{(1-x^{2k})^{z+1}}{1-x^k}\right)^{2z-1} = \sum_{\lambda\in SC} \delta_\lambda\, x^{|\lambda|} \prod_{h\in\mathcal{H}(\lambda)} \left(1-\frac{2z}{h\,\varepsilon_h}\right)$$

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Generalization of the type $\tilde{C}$ Nekrasov-Okounkov formula

Theorem (P., 2015)

Let t be a positive integer. For any complex numbers y and z we have

$$\sum_{\lambda \in DD} \delta_\lambda \, x^{|\lambda|/2} \prod_{h \in \mathcal{H}_t(\lambda)} \left(y - \frac{tyz}{h \, \varepsilon_h} \right)$$

$$= \begin{cases} \displaystyle\prod_{k \geq 1} (1-x^k)(1-x^{kt})^{t'-1} \left(1 - x^{kt} y^{2k}\right)^{(z-1)(zt+t-3)/2} & \text{if } t = 2t'+1 \\ \displaystyle\prod_{k \geq 1} \frac{(1-x^k)(1-x^{kt})^{t'-1}}{1-x^{kt'}} \left(\frac{(1-y^{2k}x^{kt})^{zt'-1+t'}}{1-y^k x^{kt'}} \right)^{z-1} & \text{if } t = 2t' \end{cases}$$

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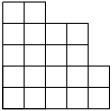
Exemple : $\lambda = (5, 5, 4, 4, 2)$



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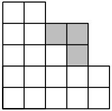

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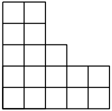

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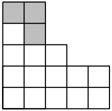

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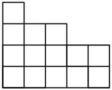

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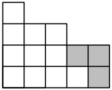

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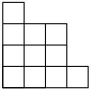

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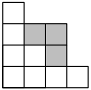

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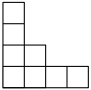
has for 3-core 

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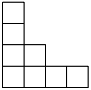

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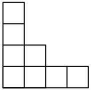
has for 3-core 

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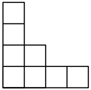
has for 3-core 

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The t-core of a doubled-distinct partition is doubled-distinct

Littlewood decomposition

Theorem (Littlewood, 1951, probably)

The Littlewood decomposition maps a partition λ to $(\tilde{\lambda}, \lambda^0, \lambda^1, \ldots, \lambda^{t-1})$ such that:

(i) $\tilde{\lambda}$ *is the t-core of* λ *and* $\lambda^0, \lambda^1, \ldots, \lambda^{t-1}$ *are partitions;*

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(ii) $|\lambda| = |\tilde{\lambda}| + t(|\lambda^0| + |\lambda^1| + \dots + |\lambda^{t-1}|)$

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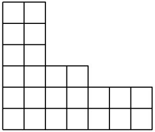

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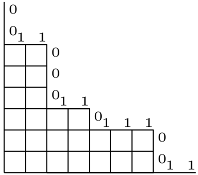

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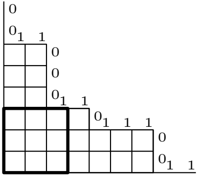

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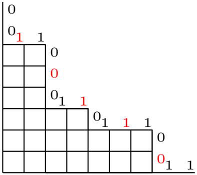

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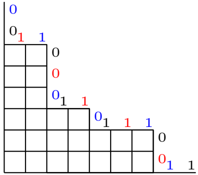

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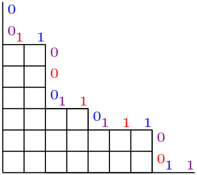

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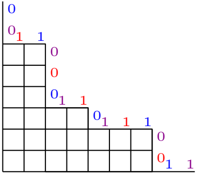

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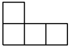
$\lambda^0 =$ 

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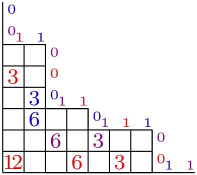

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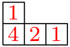
$\lambda^0 =$ 

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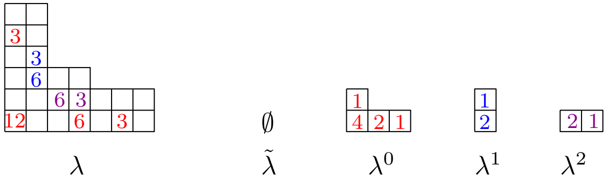

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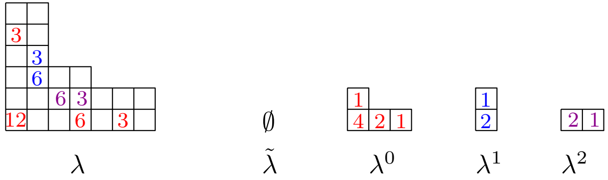

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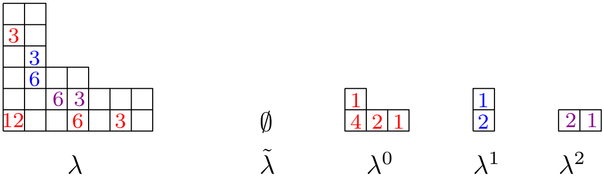

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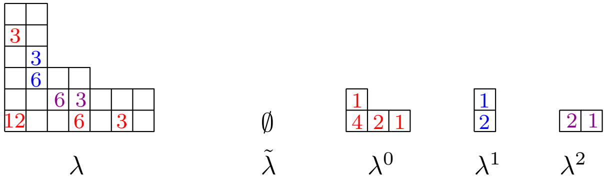

Proof of our generalization

- Fix $t = 2t' + 1$ an integer, $\lambda \in DD$ and its Littlewood decomposition $(\tilde{\lambda}, \lambda^0, \lambda^1, \ldots, \lambda^{t-1})$.

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- And sum over all doubled distinct partitions.

And for self-conjugate partitions ?

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Theorem (P., 2015)

Let t be a positive integer. For all complex numbers y and z, we have :

$$\sum_{\lambda \in SC} \delta_\lambda \, x^{|\lambda|} \prod_{h \in \mathcal{H}_t(\lambda)} \left(y - \frac{yzt}{h\, \varepsilon_h} \right)$$

$$= \begin{cases} \displaystyle\prod_{k \geq 1} \frac{1-x^k}{1-x^{2k}} \left(1 - x^{2kt}\right)^{t'} (1 - y^{2k} x^{2kt})^{(z^2-1)t'} & \text{if } t = 2t' \\ \displaystyle\prod_{k \geq 1} \frac{1-x^k}{1-x^{2k}} \frac{(1-x^{2kt})^{t'+1}}{1-x^{kt}} \frac{(1 - y^{2k} x^{2kt})^{(tz^2+z-t-1)/2}}{(1-y^k x^{kt})^{z-1}} & \text{if } t = 2t'+1 \end{cases}$$

Some consequences

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New hook formulas

Corollary

if t is odd,

$$\sum_{\substack{\lambda \in DD,\ |\lambda|=2tn \\ \#\mathcal{H}_t(\lambda)=2n}} \delta_\lambda \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h\,\varepsilon_h} = \frac{(-1)^n}{n!2^n t^n}$$

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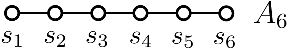

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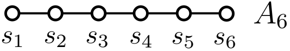

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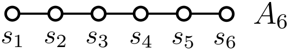

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Stembridge's property

Theorem (Stembridge, 1995)

A reduced expression correspond to a fully commutative element if and only if it does not contain up to commutation a subword $stst\ldots$ of length m_{st}.

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- conclude as generating series of languages recognizable by finite state automata are rational

Corollary

Let W be a Coxeter group. The generating function of CFC element is algorithmically computable.

Thank you for your attention!