Asymptotics for graphically divergent series

Khaydar Nurligareev (joint with Sergey Dovgal)

LiB, University of Burgundy

Seminar CALIN

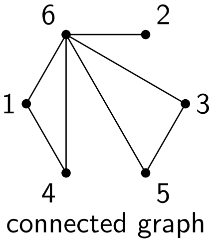
September 19, 2023

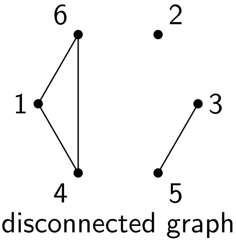
Table of content

Outline

Graphs (labeled undirected graphs)

- $\mathfrak{g}_n = \#\{\text{graphs with } n \text{ vertices}\}$
- $\mathfrak{cg}_n = \#\{\text{connected graphs with } n \text{ vertices}\}$



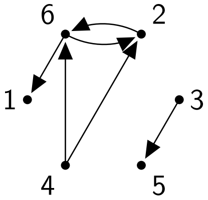


$$\mathfrak{g}_n = 2^{\binom{n}{2}}$$

$$(\mathfrak{cg}_n) = 1, 1, 4, 38, 728, 26704, 1866256, \ldots$$

Digraphs (labeled directed graphs)

- $\mathfrak{d}_n = \#\{\text{digraphs with } n \text{ vertices}\}$

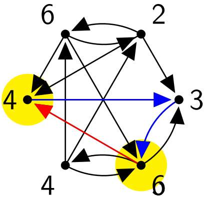


digraph

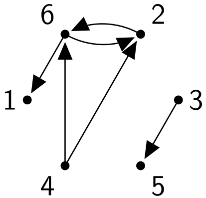
$$\mathfrak{d}_n = 2^{2\binom{n}{2}}$$

Digraphs (labeled directed graphs)

- $\mathfrak{d}_n = \#\{\text{digraphs with } n \text{ vertices}\}$
- $\mathfrak{scd}_n = \#\{\text{strongly connected digraphs with } n \text{ vertices}\}$



strongly connected digraph



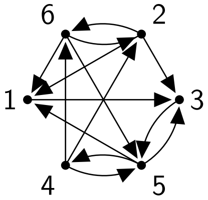
digraph

$$\mathfrak{d}_n = 2^{2\binom{n}{2}}$$

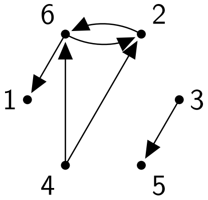
$$(\mathfrak{scd}_n) = 1, 1, 18, 1606, 565080, 734774776, \ldots$$

Digraphs (labeled directed graphs)

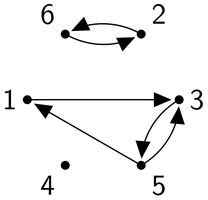
- $\mathfrak{d}_n = \#\{\text{digraphs with } n \text{ vertices}\}$
- $\mathfrak{scd}_n = \#\{\text{strongly connected digraphs with } n \text{ vertices}\}$
- $\mathfrak{ssd}_n = \#\{\text{semi-strong digraphs with } n \text{ vertices}\}$



strongly connected digraph



digraph



semi-strong digraph

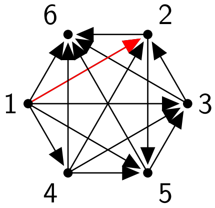
$$\mathfrak{d}_n = 2^{2\binom{n}{2}}$$

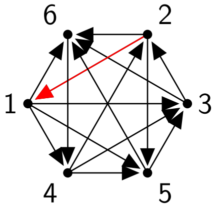
$$(\mathfrak{scd}_n) = 1, 1, 18, 1606, 565080, 734774776, \ldots$$

$$(\mathfrak{ssd}_n) = 1, 2, 22, 1688, 573496, 738218192, \ldots$$

Tournaments (labeled tournaments)

- $\mathfrak{t}_n = \#\{\text{tournaments with } n \text{ vertices}\}$

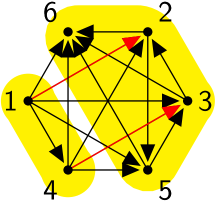




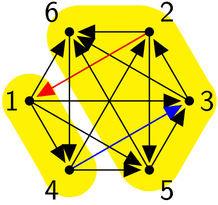
$$\mathfrak{t}_n = 2^{\binom{n}{2}}$$

Tournaments (labeled tournaments)

- $\mathfrak{t}_n = \#\{\text{tournaments with } n \text{ vertices}\}$
- $\mathfrak{it}_n = \#\{\text{irreducible tournaments with } n \text{ vertices}\}$



reducible tournament



irreducible tournament

$$\mathfrak{t}_n = 2^{\binom{n}{2}}$$

$$(\mathfrak{it}_n) = 1, 0, 2, 24, 544, 22320, 1677488, \ldots$$

Tournaments (labeled tournaments)

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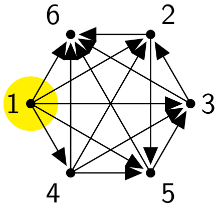
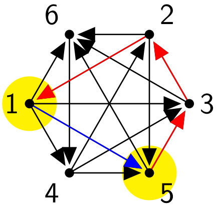


reducible tournament



irreducible tournament

$$\mathfrak{t}_n = 2^{\binom{n}{2}}$$

$$(\mathfrak{it}_n) = 1, 0, 2, 24, 544, 22320, 1677488, \ldots$$

Probability of a graph to be connected

Question. What is the probability $p_n = \frac{\mathfrak{cg}_n}{\mathfrak{g}_n}$ that a random graph with n vertices is connected, as $n \to \infty$?

Probability of a graph to be connected

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- Wright, 1970:

$$p_n = 1 - \binom{n}{1}\frac{2}{2^n} - 2\binom{n}{3}\frac{2^6}{2^{3n}} - 24\binom{n}{4}\frac{2^{10}}{2^{4n}} + O\left(\frac{n^5}{2^{5n}}\right)$$

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- Monteil, N., 2021:

$$p_n = 1 - \sum_{k=1}^{r-1} \mathfrak{it}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right).$$

Probability of a tournament to be irreducible

Question. What is the probability $q_n = \frac{\mathfrak{it}_n}{\mathfrak{t}_n}$ that a random tournament with n vertices is irreducible as $n \to \infty$?

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- Wright, 1970:

$$q_n = 1 - \binom{n}{1}\frac{2^2}{2^n} + \binom{n}{2}\frac{2^4}{2^{2n}} - \binom{n}{3}\frac{2^8}{2^{3n}} - \binom{n}{4}\frac{2^{15}}{2^{4n}} + O\left(\frac{n^5}{2^{5n}}\right)$$

- Monteil, N., 2021:

$$q_n = 1 - \sum_{k=1}^{r-1} \left(2\mathfrak{it}_k - \mathfrak{it}_k^{(2)}\right) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where $\mathfrak{it}_k^{(2)} = \#\{\text{tournaments of size } n \text{ with 2 irreducible parts}\}$.

Probability of a digraph to be strongly connected

Question. What is the probability r_n that a random directed graph with n vertices is strongly connected, as $n \to \infty$?

Probability of a digraph to be strongly connected

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Wright, 1971: $$r_n = \sum_{k=0}^{r-1} \frac{\omega_k(n)}{2^{kn}} \cdot \frac{n!}{(n+[k/2]-k)!} + O\left(\frac{n^r}{2^{rn}}\right),$$

where

$$\omega_k(n) = \sum_{\nu=0}^{[k/2]} \gamma_\nu \xi_{k-2\nu} \frac{2^{k(k+1)/2}}{2^{\nu(k-\nu)}}(n+[k/2]-k)\dots(n+\nu+1-k),$$

$$\gamma_0 = 1,\ \gamma_\nu = \sum_{s=0}^{\nu-1} \frac{\gamma_s \eta_{\nu-s}}{(\nu-s)!},\quad \sum_{\nu=0}^{\infty} \xi_\nu z^\nu = \left(1 - \sum_{s=0}^{\infty} \frac{\eta_s}{2^{s(s-1)/2}} \frac{z^s}{s!}\right)^2,$$

$$\eta_1 = 1,\quad \eta_s = 2^{s(s-1)} - \sum_{t=1}^{s-1} \binom{s}{t} 2^{(s-1)(s-t)} \eta_t.$$

Probability of a digraph to be strongly connected

Question. What is the probability r_n that a random directed graph with n vertices is strongly connected, as $n \to \infty$?

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$$\eta_1 = 1,\quad \eta_s = 2^{s(s-1)} - \sum_{t=1}^{s-1} \binom{s}{t} 2^{(s-1)(s-t)} \eta_t.$$

Motivation

Summary. The probability r_n has an expansion of the form

$$r_n = \sum_{m=0}^{r-1} \frac{1}{2^{mn}} \sum_{\ell=0}^{\ell_m} n^{\underline{\ell}} a^{\circ}_{m,\ell} + O\left(\frac{n^r}{2^{rn}}\right),$$

where $n^{\underline{\ell}} = n(n-1)\dots(n-\ell+1)$ are falling factorials.

Observation. The array of coefficients $(a^{\circ}_{m,\ell})_{m,\ell=0}^{\infty}$ can be assembled into a (bivariate) generation function.

Questions. Can we express this bivariate generating function explicitly in terms of other known generating functions?

What is the structure of the asymptotics?

Outline

Graphically divergent series

- $\alpha \in \mathbb{R}_{>1}$ and $\beta \in \mathbb{Z}_{>0}$,
- $\mathfrak{G}_\alpha^\beta$ is the set of **graphically divergent series**, i.e.

$$A(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$$

such that

$$a_n \approx \alpha^{\beta\binom{n}{2}} \left[\sum_{m \geqslant M} \frac{1}{\alpha^{mn}} \sum_{\ell=0}^{\infty} n^{\underline{\ell}} a^{\circ}_{m,\ell} \right],$$

where

- $M \in \mathbb{Z}$,
- $n^{\underline{\ell}} = n(n-1)\ldots(n-\ell+1)$ are falling factorials,
- the support of $(a^{\circ}_{m,\ell})_{\ell=0}^{\infty}$ is finite for each $m \in \mathbb{Z}_{\geqslant M}$.

Coefficient generating function

- If $A \in \mathfrak{G}_\alpha^\beta$ with

$$a_n \approx \alpha^{\beta\binom{n}{2}} \left[\sum_{m \geqslant M} \frac{1}{\alpha^{mn}} \sum_{\ell=0}^{\infty} n^{\ell} a^{\circ}_{m,\ell} \right],$$

then the associated **coefficient generating function** of type (α, β) is

$$A^{\circ}(z, w) = \sum_{m=M}^{\infty} \sum_{\ell=0}^{\infty} a^{\circ}_{m,\ell} \frac{z^m}{\alpha^{\frac{1}{\beta}\binom{m}{2}}} w^{\ell}.$$

- $\mathfrak{C}_\alpha^\beta$ is the set of corresponding coefficient generating functions.
- $\mathcal{Q}_\alpha^\beta \colon \mathfrak{G}_\alpha^\beta \to \mathfrak{C}_\alpha^\beta$ is the mapping of the form

$$\mathcal{Q}_\alpha^\beta A = A^{\circ}.$$

First examples

- Graphs:

$$G(z) = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{z^n}{n!}, \qquad \mathfrak{g}_n = 2^{\binom{n}{2}}.$$

Its coefficient generating function of type $(2, 1)$ is

$$G^{\circ}(z, w) = (\mathcal{Q}_2^1 G)(z, w) = 1.$$

First examples

- Graphs:

$$G(z) = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{z^n}{n!}, \qquad \mathfrak{g}_n = 2^{\binom{n}{2}}.$$

Its coefficient generating function of type $(2,1)$ is

$$G^{\circ}(z,w) = (\mathcal{Q}_2^1 G)(z,w) = 1.$$

- Digraphs:

$$D(z) = \sum_{n=0}^{\infty} 2^{2\binom{n}{2}} \frac{z^n}{n!}, \qquad \mathfrak{d}_n = 2^{2\binom{n}{2}}.$$

Its coefficient generating function of type $(2,2)$ is

$$D^{\circ}(z,w) = (\mathcal{Q}_2^2 D)(z,w) = 1.$$

Properties, part I

1. The set $\mathfrak{G}_\alpha^\beta$ forms a ring with

$$\big(\mathcal{Q}_\alpha^\beta(A+B)\big)(z,w) = (\mathcal{Q}_\alpha^\beta A)(z,w) + (\mathcal{Q}_\alpha^\beta B)(z,w)$$

and

$$\begin{aligned}\big(\mathcal{Q}_\alpha^\beta(A\cdot B)\big)(z,w) =& A\big(\alpha^{\frac{\beta+1}{2}} z^\beta w\big)\cdot(\mathcal{Q}_\alpha^\beta B)(z,w) + \\ & B\big(\alpha^{\frac{\beta+1}{2}} z^\beta w\big)\cdot(\mathcal{Q}_\alpha^\beta A)(z,w)\,.\end{aligned}$$

2. Derivation:

$$\big(\mathcal{Q}_\alpha^\beta A'\big)(z,w) = \alpha^{-\frac{\beta+1}{2}} z^{-\beta}\left(\big(\mathcal{Q}_\alpha^\beta A\big)(z,w) + \frac{\partial}{\partial w}\big(\mathcal{Q}_\alpha^\beta A\big)(z,w)\right).$$

3. Integration:

$$\left(\mathcal{Q}_\alpha^\beta \int A\right)(z,w) = \alpha^{\frac{\beta+1}{2}} z^\beta \left(\sum_{k=0}^{\infty}(-1)^k \frac{\partial^k}{\partial w^k}\big(\mathcal{Q}_\alpha^\beta A\big)(z,w)\right).$$

Properties, part II

3. Composition (interpretation of Bender's theorem): if
   - F is analytic in a neighbourhood of the origin,
   - $a_0 = 0$,
   - $H(z) = \frac{\partial}{\partial x} F(x) \Big|_{x=A(z)}$,

   then $F \circ A \in \mathfrak{G}_\alpha^\beta$ and

$$\big(\mathcal{Q}_\alpha^\beta (F \circ A)\big)(z, w) = H\big(\alpha^{\frac{\beta+1}{2}} z^\beta w\big) \cdot (\mathcal{Q}_\alpha^\beta A)(z, w).$$

4. Powers: if $m \in \mathbb{Z}_{\geqslant 0}$ (or $m \in \mathbb{Q}$ and $a_0 = 1$), then

$$\big(\mathcal{Q}_\alpha^\beta A^m\big)(z, w) = m \cdot A^{m-1}\big(\alpha^{\frac{\beta+1}{2}} z^\beta w\big) \cdot (\mathcal{Q}_\alpha^\beta A)(z, w).$$

Outline

Connected graphs

Theorem (Monteil, N., 2021)

For every $r \geqslant 1$, the probability p_n that a random labeled graph of size n is connected satisfies

$$p_n = 1 - \sum_{k=1}^{r-1} \mathrm{it}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right).$$

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type $(2,1)$ of connected graphs satisfies

$$(\mathcal{Q}_2^1 \, \mathrm{CG})(z,w) = \frac{1}{\mathrm{G}(2zw)} = 1 - \mathrm{IT}(2zw).$$

Key ideas: $\mathrm{CG}(z) = \log\left(G(z)\right), \quad \frac{1}{\mathrm{G}(z)} = \frac{1}{\mathrm{T}(z)} = 1 - \mathrm{IT}(z).$

Irreducible tournaments

Theorem (Monteil, N., 2021)

For every $r \geqslant 1$, the probability q_n that a random tournament of size n is irreducible satisfies

$$q_n = 1 - \sum_{k=1}^{r-1} \left(2\mathrm{it}_k - \mathrm{it}_k^{(2)}\right) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right).$$

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type $(2,1)$ of irreducible tournaments satisfies

$$(\mathcal{Q}_2^1 \,\mathrm{IT})(z, w) = \big(1 - \mathrm{IT}(2zw)\big)^2.$$

Key ideas: $\mathrm{IT}(z) = 1 - \dfrac{1}{\mathrm{T}(z)}, \quad \dfrac{1}{\mathrm{T}^2(z)} = \big(1 - \mathrm{IT}(z)\big)^2.$

Fixed number of connected components in a graph

Observation: $\mathrm{G}(z; t) = \exp\big(t \cdot \mathrm{CG}(z)\big)$,
where t marks the number of connected components.

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type $(2, 1)$ of graphs with the marking variable t satisfies

$$(\mathcal{Q}_2^1\, G)(z, w; t) = t \cdot \mathrm{G}(2zw; t-1) = t \cdot \mathrm{G}(2zw; t) \cdot \big(1 - \mathrm{IT}(2zw)\big).$$

In particular,

$$[t^{m+1}](\mathcal{Q}_2^1\, G)(z, w; t) = \frac{\mathrm{CG}^m(2zw)}{m!} \cdot \big(1 - \mathrm{IT}(2zw)\big)$$

is the coefficient generating function for graphs with $(m+1)$ connected components, $m \in \mathbb{Z}_{\geqslant 0}$.

Fixed number of irreducible parts in a tournament

Observation: $\mathsf{T}(z;t) = \dfrac{1}{1 - t \cdot \mathsf{IT}(z)}$,

where t marks the number of irreducible parts.

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type $(2,1)$ *of tournaments with the marking variable* t *satisfies*

$$(\mathcal{Q}_2^1\, T)(z,w;t) = t \cdot \Big(\mathsf{T}(2zw;t) \cdot (1 - \mathsf{IT}(2zw))\Big)^2.$$

In particular,

$$[t^{m+1}](\mathcal{Q}_2^1\, T)(z,w;t) = (m+1) \cdot \mathsf{IT}^m(2zw) \cdot \big(1 - \mathsf{IT}(2zw)\big)^2$$

is the coefficient generating function for tournaments with $(m+1)$ *irreducible parts,* $m \in \mathbb{Z}_{\geqslant 0}$.

The Erdős-Rényi model $G(n, p)$, part I

Fix $\quad p \in (0,1), \qquad q = 1 - p, \qquad \rho = p/q.$

Consider a random labeled graph G:

- p is the probability of edge presence;
- $q = 1 - p$ is the probability of edge absence.

$$\mathbb{P}(G) = p^{|E(G)|} q^{\binom{n}{2} - |E(G)|} = \frac{\rho^{|E(G)|}}{(\rho + 1)^{\binom{n}{2}}}.$$

The Erdős-Rényi model $G(n,p)$, part I

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$$\mathbb{P}(G) = p^{|E(G)|} q^{\binom{n}{2} - |E(G)|} = \frac{\rho^{|E(G)|}}{(\rho+1)^{\binom{n}{2}}}.$$

Denote:

- $\alpha = \rho + 1 = q^{-1}$.

Then

$$G(z) = \sum_{n=0}^{\infty} (\rho+1)^{\binom{n}{2}} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \alpha^{\binom{n}{2}} \frac{z^n}{n!}.$$

The Erdős-Rényi model $G(n,p)$, part II

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type $(2,1)$ *of connected graphs in the Erdős-Rényi model satisfies*

$$(\mathcal{Q}_2^1\,\mathrm{CG})(z,w) = \frac{1}{\mathrm{G}(2zw)} = \exp\big(-\mathrm{CG}(2zw)\big).$$

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type $(2,1)$ *of graphs in the Erdős-Rényi model with the marking variable* t *for the number of strongly connected components satisfies*

$$(\mathcal{Q}_2^1\,G)(z,w;t) = t\cdot \mathrm{G}(2zw;t-1).$$

In particular,

$$[t^{m+1}](\mathcal{Q}_2^1\,G)(z,w;t) = \frac{\mathrm{CG}^m(2zw)}{m!}\cdot\exp\big(-\mathrm{CG}(2zw)\big).$$

Outline

Transitions, part I

Theorem (Dovgal, de Panafieu, 2019)

The exponential generating function of strongly connected digraphs satisfies

$$\mathrm{SCD}(z) = -\log\left(G(z) \odot \frac{1}{G(z)}\right).$$

- Exponential Hadamard product:

$$\left(\sum_{n=0} a_n \frac{z^n}{n!}\right) \odot \left(\sum_{n=0} b_n \frac{z^n}{n!}\right) = \left(\sum_{n=0} a_n b_n \frac{z^n}{n!}\right).$$

- Exponential Hadamard product (with $G(z)$) changes:
  - the rate of convergence,
  - the type of coefficient generating function.

Transitions, part I

Theorem (Dovgal, de Panafieu, 2019)

The exponential generating function of strongly connected digraphs satisfies

$$\mathsf{SCD}(z) = -\log\left(G(z) \odot \frac{1}{G(z)}\right).$$

- If $\beta > 1$, then

$$\Delta_\alpha \colon \mathfrak{G}_\alpha^\beta \to \mathfrak{G}_\alpha^{\beta-1}$$

  is defined by

$$\Delta_\alpha\left(\sum_{n=0}^{\infty} f_n \frac{z^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{f_n}{\alpha^{\binom{n}{2}}} \frac{z^n}{n!}.$$

- $F(z) \odot G(z) = \Delta_2^{-1} F(z)$.

Transitions, part II

- If $\alpha \in \mathbb{R}_{>1}$ and $\beta, \gamma \in \mathbb{Z}_{>0}$, then

$$\Phi_\alpha^{\beta,\gamma} \colon \mathfrak{C}_\alpha^\beta \to \mathfrak{C}_\alpha^\gamma$$

is defined as

$$\Phi_\alpha^{\beta,\gamma}\left(\sum_{m=M}^{\infty}\sum_{\ell=0}^{\infty} a^\circ_{m,\ell}\frac{z^m}{\alpha^{\frac{1}{\beta}\binom{m}{2}}}w^\ell\right) = \sum_{m=M}^{\infty}\sum_{\ell=0}^{\infty} a^\circ_{m,\ell}\frac{z^m}{\alpha^{\frac{1}{\gamma}\binom{m}{2}}}w^\ell .$$

- The following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{G}_\alpha^\beta & \xrightarrow{\mathcal{Q}_\alpha^\beta} & \mathfrak{C}_\alpha^\beta \\ {\scriptstyle \Delta_\alpha^{\beta-\gamma}}\big\downarrow & & \big\downarrow{\scriptstyle \Phi_\alpha^{\beta,\gamma}} \\ \mathfrak{G}_\alpha^\gamma & \xrightarrow{\mathcal{Q}_\alpha^\gamma} & \mathfrak{C}_\alpha^\gamma \end{array}$$

Strongly connected directed graphs, part I

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type $(2,2)$ *of strongly connected digraphs satisfies*

$$(\mathcal{Q}_2^2\,\mathrm{SCD})(z,w) = \mathrm{SSD}(2^{3/2}z^2w)\cdot \Phi_2^{1,2}\big(1-\mathrm{IT}(2zw)\big)^2 .$$

where $\mathrm{SSD}(z)$ *is the exponential generating function of semi-strong digraphs.*

Key ideas (Dovgal, de Panafieu, 2019; Monteil, N., 2021):

- $\mathrm{SCD}(z) = -\log\left(G(z)\odot\frac{1}{G(z)}\right) = -\log\left(1-\Delta_2^{-1}\mathrm{IT}(z)\right),$
- $\mathrm{SSD}(z) = \left(G(z)\odot\frac{1}{G(z)}\right)^{-1} = \frac{1}{1-\Delta_2^{-1}\mathrm{IT}(z)}.$

Strongly connected directed graphs, part II

Corollary

For every $r \geqslant 1$, the probability r_n that a random labeled digraph of size n is strongly connected satisfies

$$r_n = \sum_{m=0}^{r-1} \frac{1}{2^{nm}} \sum_{\ell=\lceil m/2 \rceil}^{m} n^{\underline{\ell}} \mathfrak{scd}^{\circ}_{m,\ell} + O\left(\frac{n^r}{2^{rn}}\right),$$

where

- $\mathfrak{scd}^{\circ}_{m,\ell} = \dfrac{2^{m(m+1)/2}}{2^{\ell(m-\ell)}} \dfrac{\mathfrak{ssd}_{m-\ell}}{(m-\ell)!} \dfrac{\mathbb{I}_{m=2\ell} - 2\mathfrak{it}_{2\ell-m} + \mathfrak{it}^{(2)}_{2\ell-m}}{(2\ell-m)!}$,
- $\mathfrak{ssd}_k$ *is the number of semi-strong digraphs of size* k,
- $\mathfrak{it}_k$ *is the number of irreducible tournaments of size* k,
- $\mathfrak{it}^{(2)}_k$ *is the number of tournaments of size* k *with two irreducible components.*

Strongly connected directed graphs, part II

Corollary

For every $r \geqslant 1$, the probability r_n that a random labeled digraph of size n is strongly connected satisfies

$$r_n = \sum_{m=0}^{r-1} \frac{1}{2^{nm}} \sum_{\ell=\lceil m/2 \rceil}^{m} n^{\ell}\, \mathfrak{scd}^{\circ}_{m,\ell} + O\left(\frac{n^r}{2^{rn}}\right),$$

where

- $\mathfrak{scd}^{\circ}_{m,\ell} = \dfrac{2^{m(m+1)/2}}{2^{\ell(m-\ell)}} \dfrac{\mathfrak{ssd}_{m-\ell}}{(m-\ell)!} \dfrac{\mathbb{I}_{m=2\ell} - 2\mathfrak{it}_{2\ell-m} + \mathfrak{it}^{(2)}_{2\ell-m}}{(2\ell-m)!}$,
- *Interpretation of Wright's coefficients:*

$$\eta_k = 2^{\binom{k}{2}}\mathfrak{it}_k, \qquad \gamma_k = \frac{\mathfrak{ssd}_k}{k!}, \qquad \xi_k = \frac{\mathbb{I}_{k=0} - 2\mathfrak{it}_k + \mathfrak{it}^{(2)}_k}{k!}.$$

Fixed number of strongly connected components, part I

Observation: $\mathrm{SSD}(z;t) = \exp\big(t \cdot \mathrm{SCD}(z)\big)$,
where t marks the number of connected components.

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type $(2,2)$ of semi-strong digraphs with the marking variable t satisfies

$$(\mathcal{Q}_2^2\, SSD)(z,w;t) = t \cdot \mathrm{SSD}(2^{3/2}z^2w; t+1) \cdot \Phi_2^{1,2}\big(1 - \mathrm{IT}(2zw)\big)^2.$$

In particular,

$$[t^{m+1}](\mathcal{Q}_2^2\, SSD)(z,w;t) = \frac{\mathrm{SCD}^m(2^{3/2}z^2w)}{m!} \cdot (\mathcal{Q}_2^2\, SCD)(z,w)$$

is the coefficient generating function for semi-strong digraphs with $(m+1)$ strongly connected components, $m \in \mathbb{Z}_{\geqslant 0}$.

Fixed number of strongly connected components, part II

Observation (Robinson, 1973):

$$\mathrm{D}(z;t) = \Delta_2^{-1}\left(\frac{1}{\Delta_2\, e^{-t\cdot \mathrm{SCD}(z)}}\right) = \Delta_2^{-1}\left(\frac{1}{\Delta_2\, \mathrm{SSD}(z;-t)}\right),$$

where t marks the number of connected components.

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type $(2,2)$ *of digraphs with the marking variable* t *satisfies*

$$(\mathcal{Q}_2^2\, D)(z,w;t) = -\Phi_2^{1,2}\left(\frac{\Phi_2^{2,1}\big((\mathcal{Q}_2^2\, SSD)(z,w;-t)\big)}{\big(\Delta_2\, SSD(2zw;-t)\big)^2}\right).$$

Fixed number of strongly connected components, part III

- u marks purely source-like components,
- v marks purely sink-like components,
- y marks isolated components,
- t marks all components.

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type $(2, 2)$ *of digraphs with the above marking variables satisfies*

$$(\mathcal{Q}_2^2\, D)(z, w; u, v, y, t) = \mathrm{D}_1^\circ + \mathrm{D}_{20}^\circ \cdot \Phi_2^{1,2}\Big(\mathrm{D}_{21}^\circ + \mathrm{D}_{22}^\circ + \mathrm{D}_{23}^\circ\Big),$$

where

$$\mathrm{D}_1^\circ(z, w; u, v, y, t) = (y - u - v + 1)t \cdot \mathrm{D}(2^{3/2}z^2w; u, v, y, t) \cdot (\mathcal{Q}_2^2\, \mathrm{SCD})(z, w)\,,$$

$$\mathrm{D}_{20}^\circ(z, w; u, v, y, t) = \mathrm{SSD}\big(2^{3/2}z^2w; (y - u - v + 1)t\big)\,,$$

$$\mathrm{D}_{21}^\circ(z, w; u, v, y, t) = \widehat{\mathrm{D}}(2zw; u, t) \cdot \Phi_2^{2,1}\Big((\mathcal{Q}_2^2\, \mathrm{SSD})\big(z, w; (v - 1)t\big)\Big)\,,$$

$$\mathrm{D}_{22}^\circ(z, w; u, v, y, t) = \widehat{\mathrm{D}}(2zw; v, t) \cdot \Phi_2^{2,1}\Big((\mathcal{Q}_2^2\, \mathrm{SSD})\big(z, w; (u - 1)t\big)\Big)\,,$$

$$\mathrm{D}_{23}^\circ(z, w; u, v, y, t) = \widehat{\mathrm{D}}(2zw; u, t) \cdot \widehat{\mathrm{D}}(2zw; v, t) \cdot \Phi_2^{2,1}\Big((\mathcal{Q}_2^2\, \mathrm{SSD})(z, w; -t)\Big)$$

and

$$\widehat{\mathrm{D}}(z; s, t) = \frac{\Delta_2\, SSD\big(2zw; (s - 1)t\big)}{\Delta_2\, SSD\big(2zw; -t\big)}\,.$$

Outline

2-CNF formulae

- $x_1, \ldots, x_n$ are **Boolean variables**,
- $c_{ij} \in \{x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n\}$ are **literals**,
- **2-conjunctive normal form** (2-CNF) formula:

$$\bigwedge_{i=1}^{m} (c_{i1} \vee c_{i2}),$$

  - n Boolean variables and m **clauses**,
  - $(x \vee x)$ and $(x \vee \bar{x})$ are forbidden,
  - repetitions are forbidden,
- $\mathfrak{cnf}_n = \#\{\text{2-CNF with } n \text{ Boolean variables}\}$,

$$\mathfrak{cnf}_n = 2^{4\binom{n}{2}},$$

- a formula is **satisfiable** iff it can be made TRUE by assigning appropriate values to its variables.

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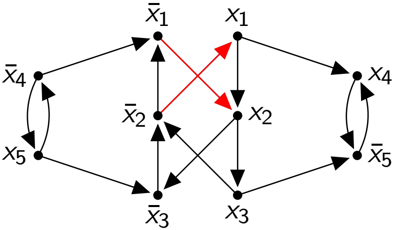
Implication digraph (of a 2-CNF formula)

- Vertices: $x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n$,
- Clause $x \vee y \quad \rightsquigarrow \quad$ edges $\bar{x} \to y$ and $\bar{y} \to x$.

$$\begin{cases} x_1 \vee x_2 \\ \bar{x}_1 \vee x_2 \\ \bar{x}_1 \vee x_4 \\ \bar{x}_2 \vee x_3 \\ \bar{x}_2 \vee \bar{x}_3 \\ \bar{x}_3 \vee \bar{x}_5 \\ x_4 \vee x_5 \\ \bar{x}_4 \vee \bar{x}_5 \end{cases} \quad \rightsquigarrow$$



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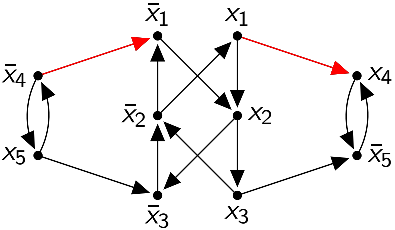

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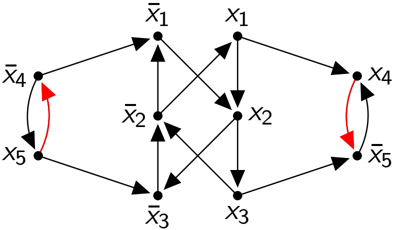
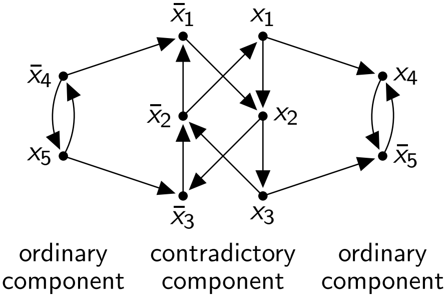


Implication digraph (of a 2-CNF formula)

- Vertices: $x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n$,
- Clause $x \vee y \quad \rightsquigarrow \quad$ edges $\bar{x} \to y$ and $\bar{y} \to x$.

$$\begin{cases} x_1 \vee x_2 \\ \bar{x}_1 \vee x_2 \\ \bar{x}_1 \vee x_4 \\ \bar{x}_2 \vee x_3 \\ \bar{x}_2 \vee \bar{x}_3 \\ \bar{x}_3 \vee \bar{x}_5 \\ x_4 \vee x_5 \\ \bar{x}_4 \vee \bar{x}_5 \end{cases} \quad \rightsquigarrow$$



- **Contradictory component** contain x and $\bar{x}$ at the same time.
- Fact: formula is satisfiable iff there is no contradictory component.

Asymptotics of 2-SAT formulae

Implication generating function of 2-SAT formulae:

$$\ddot{\mathrm{SAT}}(z) = \sum_{n=0}^{\infty} \mathfrak{sat}_n \frac{z^n}{2^{n^2} n!} .$$

Observation (Dovgal, de Panafieu, Ravelomanana, 2023):

$$\ddot{\mathrm{SAT}}(z) = G(z) \cdot \Delta_2^2 \left(G(z) \odot \frac{1}{G(z)} \right)^{1/2} .$$

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type $(2,1)$ *of 2-SAT formulae satisfies*

$$(\mathcal{Q}_2^1 \ddot{\mathrm{SAT}})(z,w) = \frac{\ddot{\mathrm{SAT}}(2zw)}{\mathrm{G}(2zw)} = \ddot{\mathrm{SAT}}(2zw)(1 - \mathrm{IT}(2zw)) .$$

Asymptotics of contradictory components

Observation (Dovgal, de Panafieu, Ravelomanana, 2023):

$$\mathrm{CSC}(z) = \frac{1}{2}\mathrm{SCD}(2z) + \log\left(D(z) \odot \frac{D(z)}{G(2z)}\right).$$

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type $(2, 4)$ *of contradictory strongly connected implication digraphs satisfy*

$$(\mathcal{Q}_2^4\,\mathrm{CSC})(z, w) = \exp\left(\frac{1}{2}\mathrm{SCD}(2^{7/2}z^4w) - \mathrm{CSC}(2^{5/2}z^4w)\right) \cdot \Phi_2^{2,4}(1 - \mathrm{IT}(2^{5/2}z^2zw)).$$

Fixed number of strongly connected components

Let

- s marks contradictory components,
- t marks ordinary components.

Observation (Dovgal, de Panafieu, Ravelomanana, 2023):

$$\ddot{\mathrm{CNF}}(z; s, t) = \Delta_2\big(D(z; t)\big) \cdot \Delta_2^2 \left(e^{s \cdot \mathrm{CSC}(z/2) - t/2 \cdot \mathrm{SCD}(z)}\right) .$$

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type $(2,2)$ *of implication digraphs with the above marking variables satisfies*

$$(\mathcal{Q}_2^2\, \ddot{\mathrm{CNF}})(z, w; s, t) = s \cdot \Delta_2\Big(\mathrm{D}(2^{3/2} z^2 w; t)\Big) \cdot \Phi_2^{4,2}\left[z \cdot S^\circ \cdot \Phi_2^{2,4}\Big(1 - \mathrm{IT}(4z^2 w)\Big)\right],$$

where

$$S^\circ(z, w; s, t) = \exp\left((s-1) \cdot \mathrm{CSC}(2^{3/2} z^4 w) + \frac{(1-t)}{2} \cdot \mathrm{SCD}(2^{5/2} z^4 w)\right).$$

Conclusion

1. We have constructed a tool for manipulating coefficients of asymptotic expansions.

2. Transfers extend to graphic families with marked patterns: any family with a fixed number of components:
   - strongly connected components in digraphs, contradictory components in 2-sat,
   - source-like, sink-like, isolated components, ...
   - any graphically divergent series with marking variables.

3. Bonus: combinatorial explanations of the expansion coefficients.

Thank you for your attention!

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