Long Alternating Paths Exist

Wolfgang Mulzer

FU Berlin
Berlin

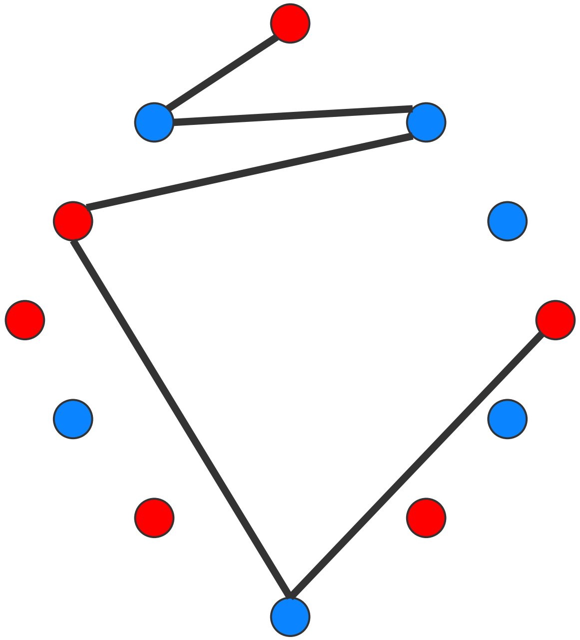
Pavel Valtr

Charles-University
Prague

The Problem

Given: 2n points, convex, n red, n blue

Want: (noncrossing) alternating path: alternate between red and blue, every point used at most once, no crossings



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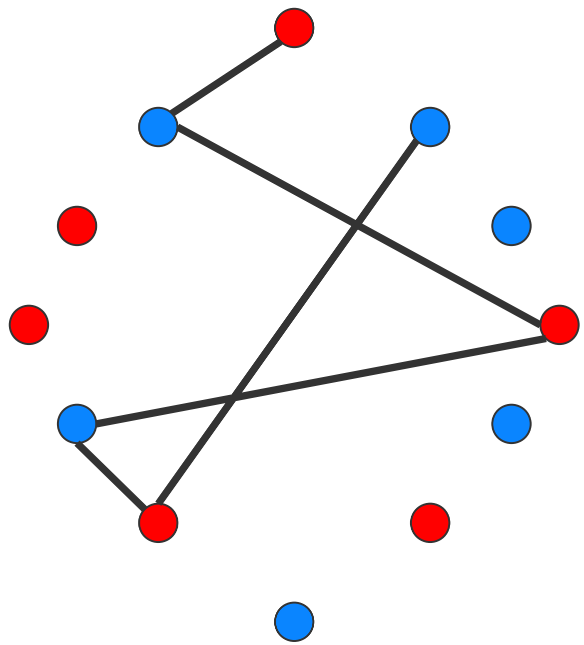

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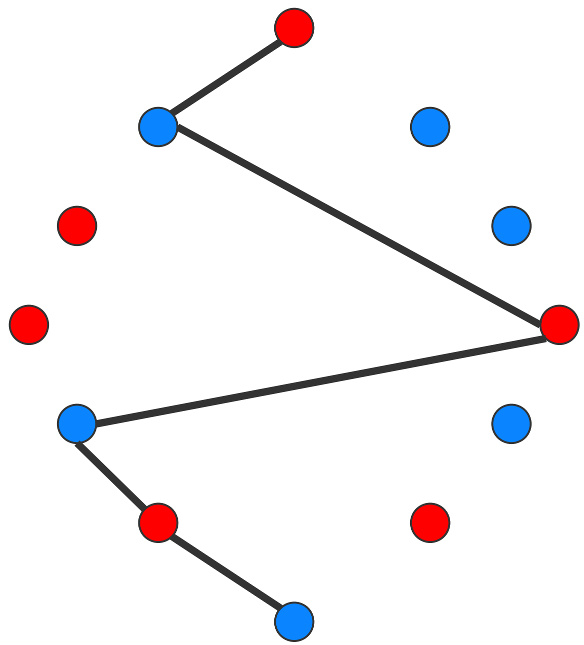


Yes!

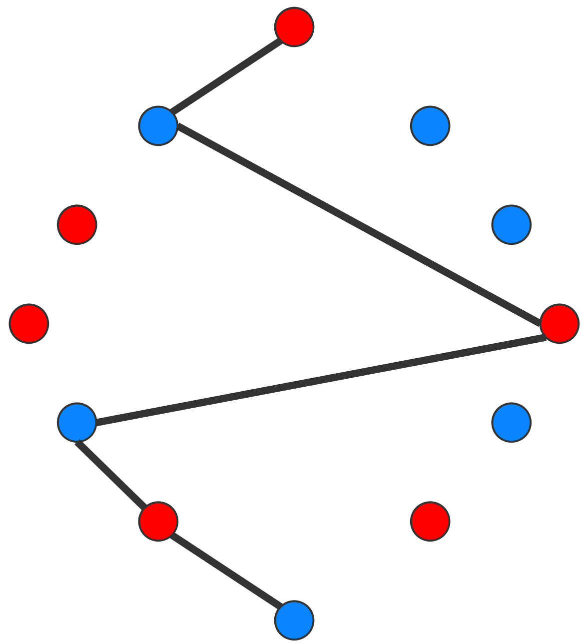
The Problem

Given: 2n points, convex, n red, n blue

Want: (noncrossing) alternating path: alternate between red and blue, every point used at most once, no crossings

Question: What is the longest alternating path?

algorithmically easy (dynamic programming)

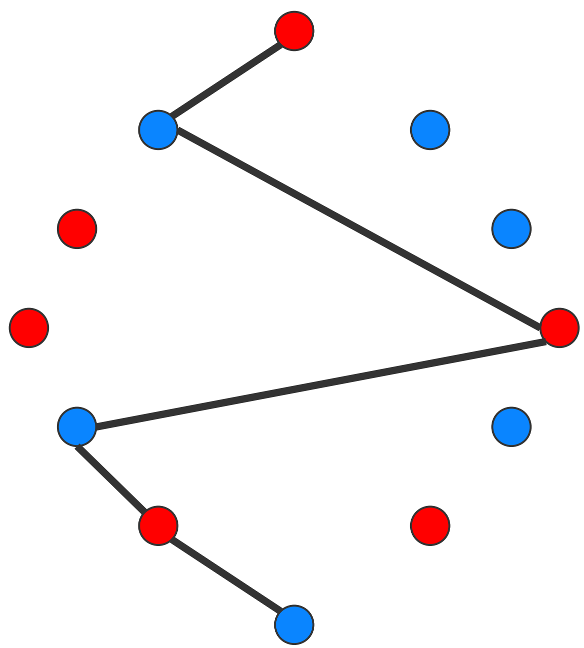


The Problem

Given: 2n points, convex, n red, n blue

Want: (noncrossing) alternating path: alternate between red and blue, every point used at most once, no crossings

Question: What is the longest alternating path as a function of n, alt(n)? (min over all colorings)



Easy Lower Bound (Erdős, 1980s)

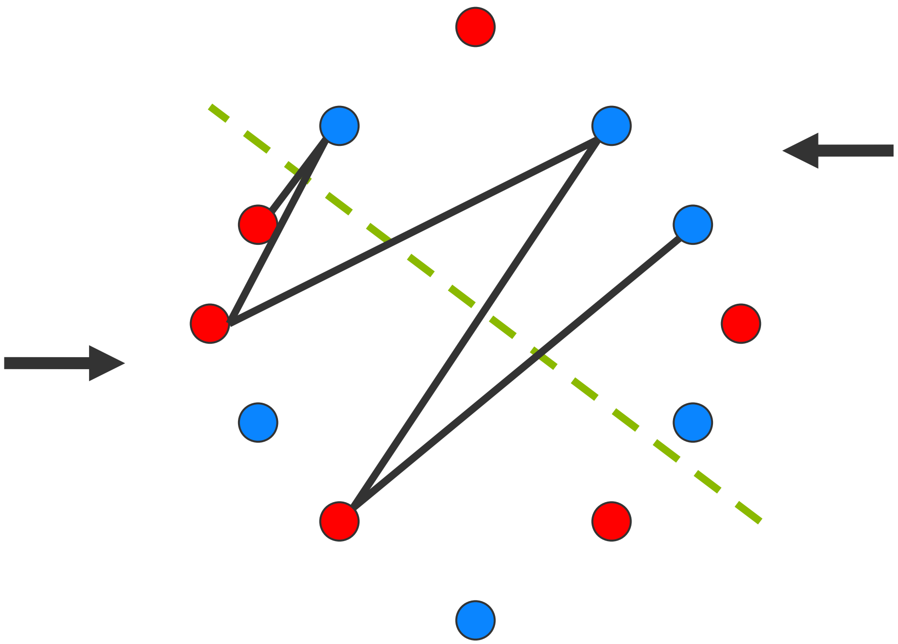
Take any halving line.

One side has $\geq n/2$ red points.

Other side has $\geq n/2$ blue points.

Connect into an alternating path with n points.

Thus: $alt(n) \geq n$



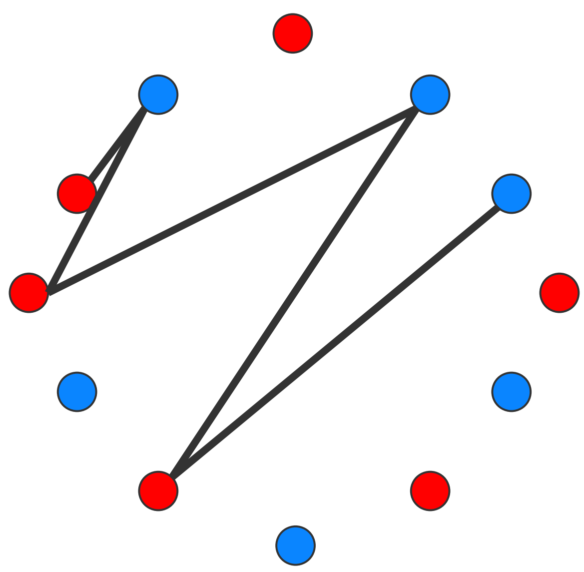
Better Lower Bounds

run: maximal sequence of consecutive points of the same color

Theorem [Kynčl, Pach and Tóth ‘08]: $\mathrm{alt}(n) \geq n + \#\mathrm{runs}/2 - 1$

Theorem [Mészáros‘11]: $\mathrm{alt}(n) \geq n + \lfloor (n - 1) / \#\mathrm{runs} \rfloor$

Corollary: $\mathrm{alt}(n) \geq n + \Omega(\sqrt{n})$

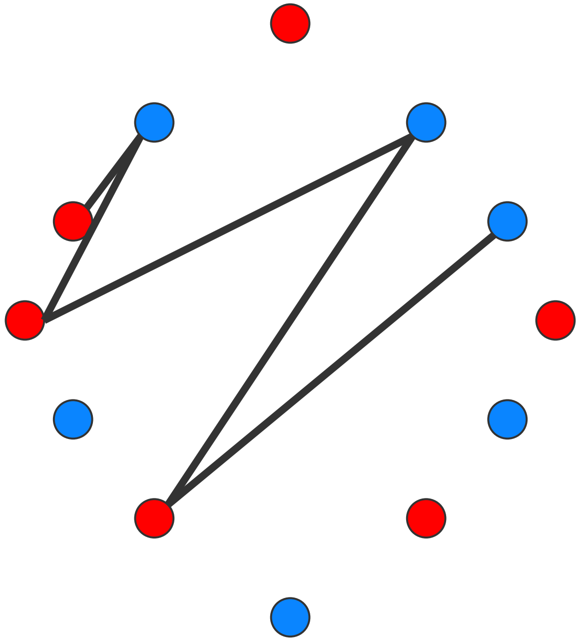


Our Result

Theorem: $\exists\ \varepsilon > 0$: alt(n) $\geq (1+ \varepsilon)$n

Remark: also for monochromatic matchings

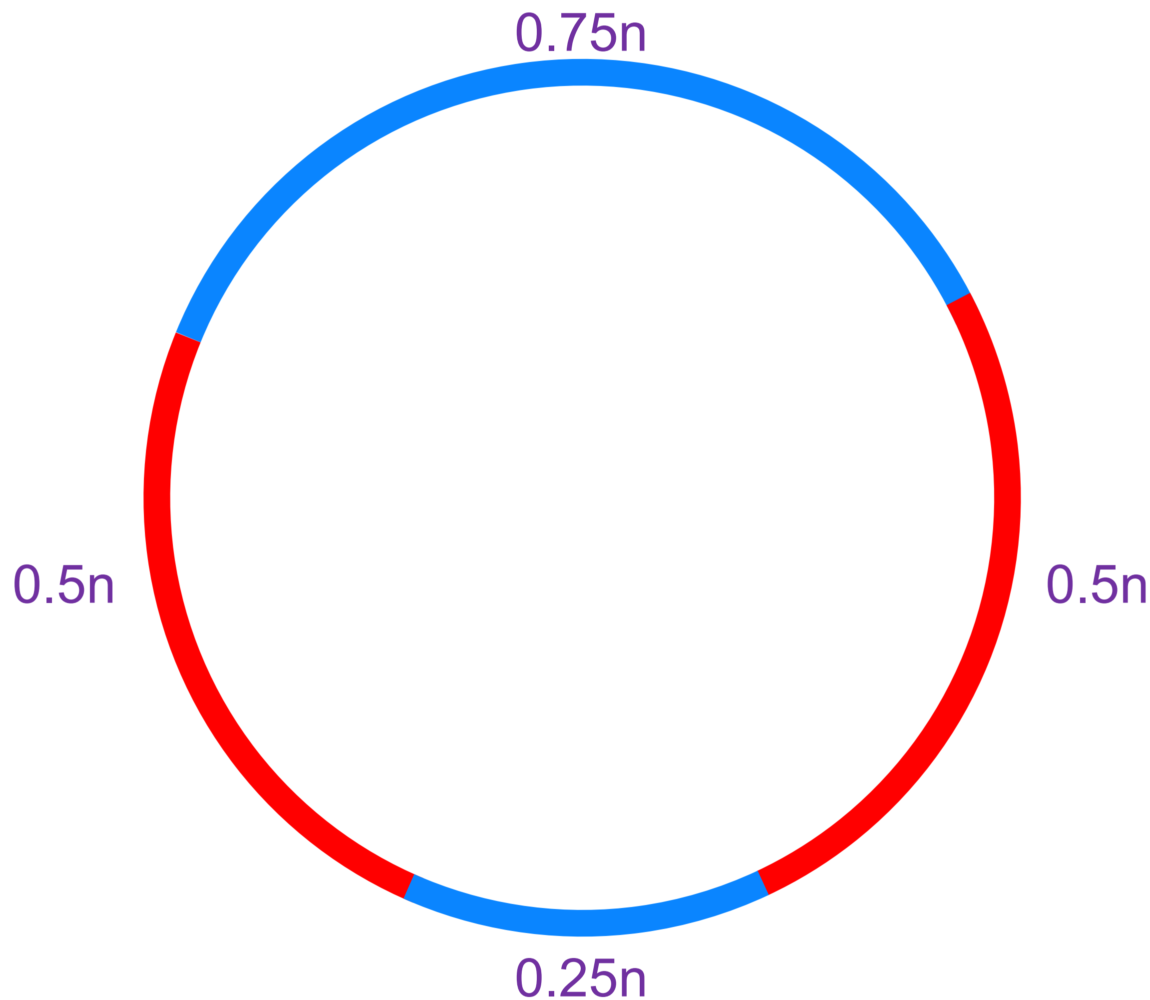
can also interpreted as a statement about (anti)palindromic subsequences in circular words.



More Background: Upper Bounds

[Erdős, 1980s]

alt(n) $\leq$ 1.5n + 2



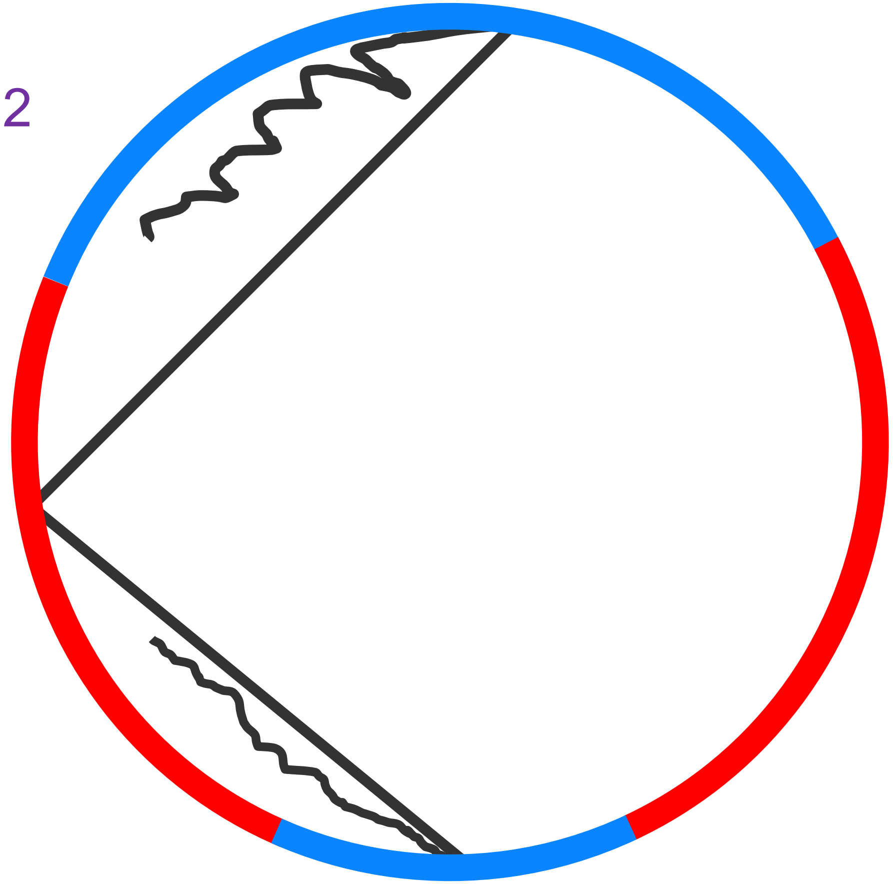
More Background: Upper Bounds

[Erdős, 1980s]

alt(n) ≤ 1.5n + 2

Assume alt(n) > 1.5n + 2

≤0.5n red points.



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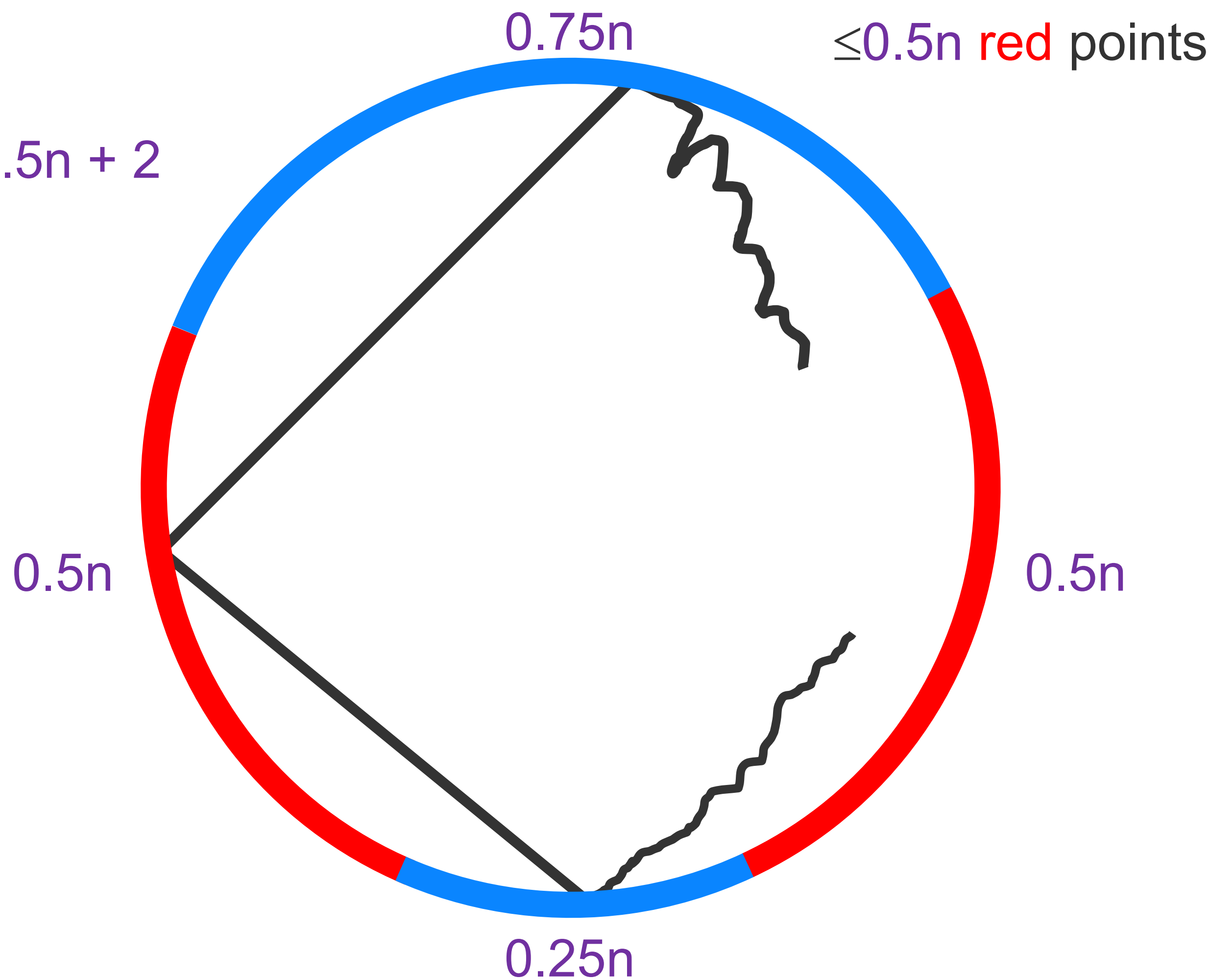

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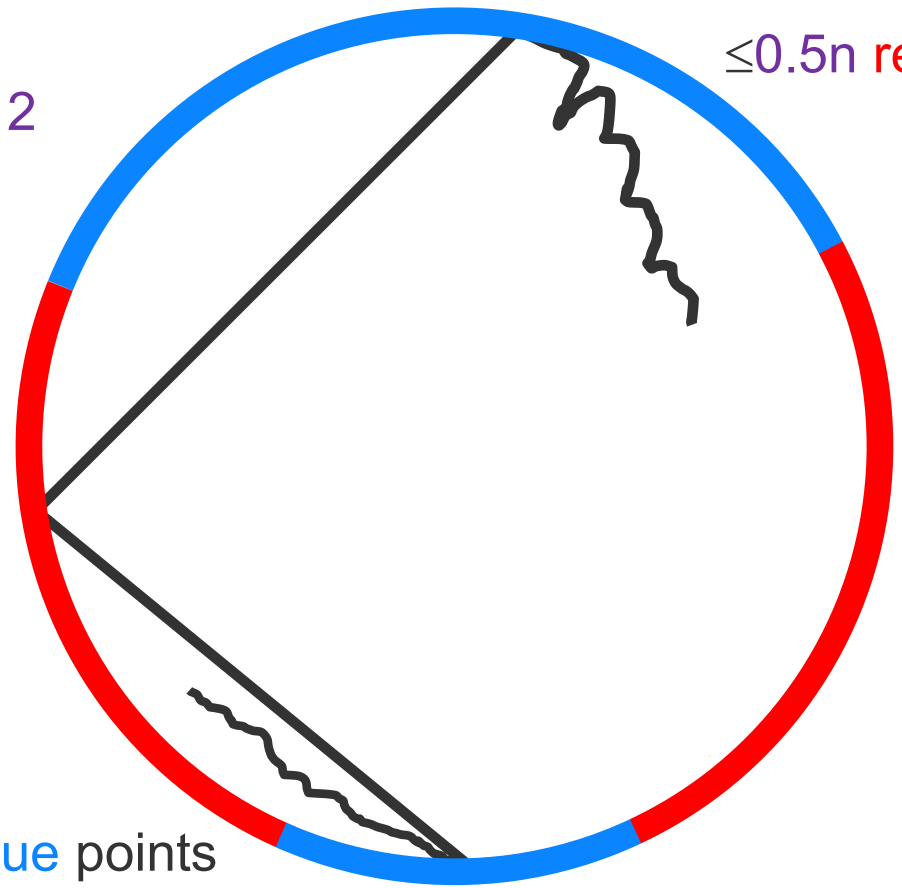

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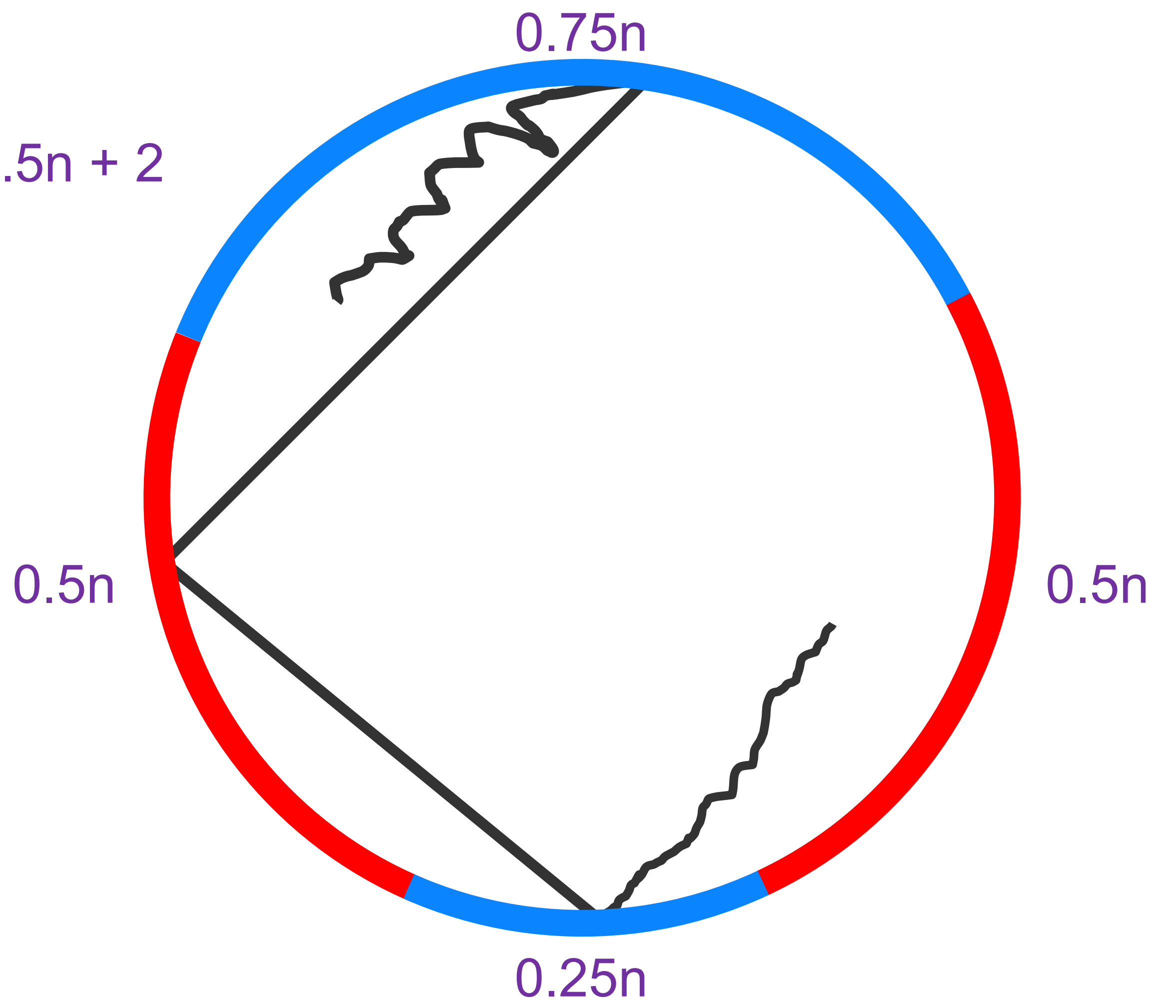

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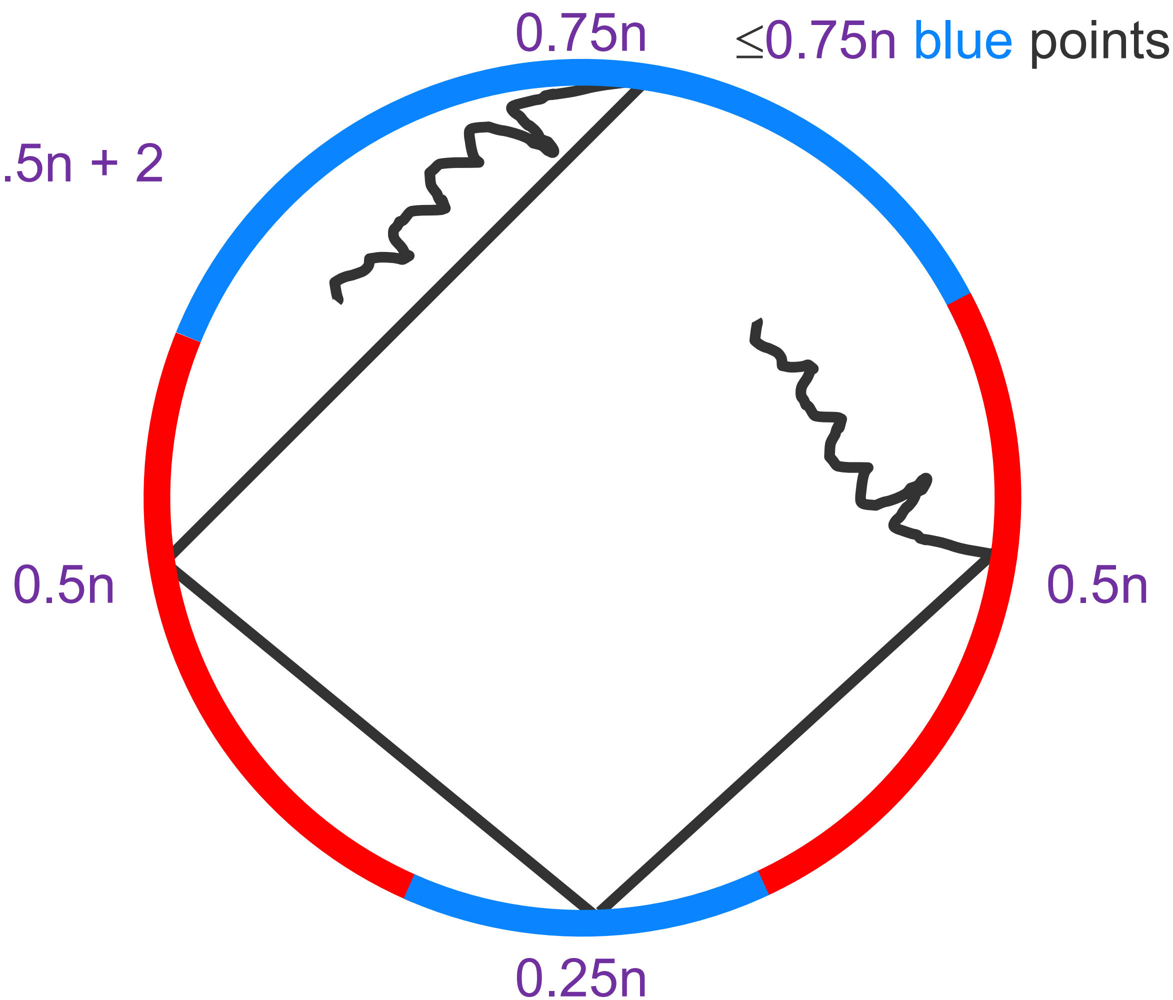

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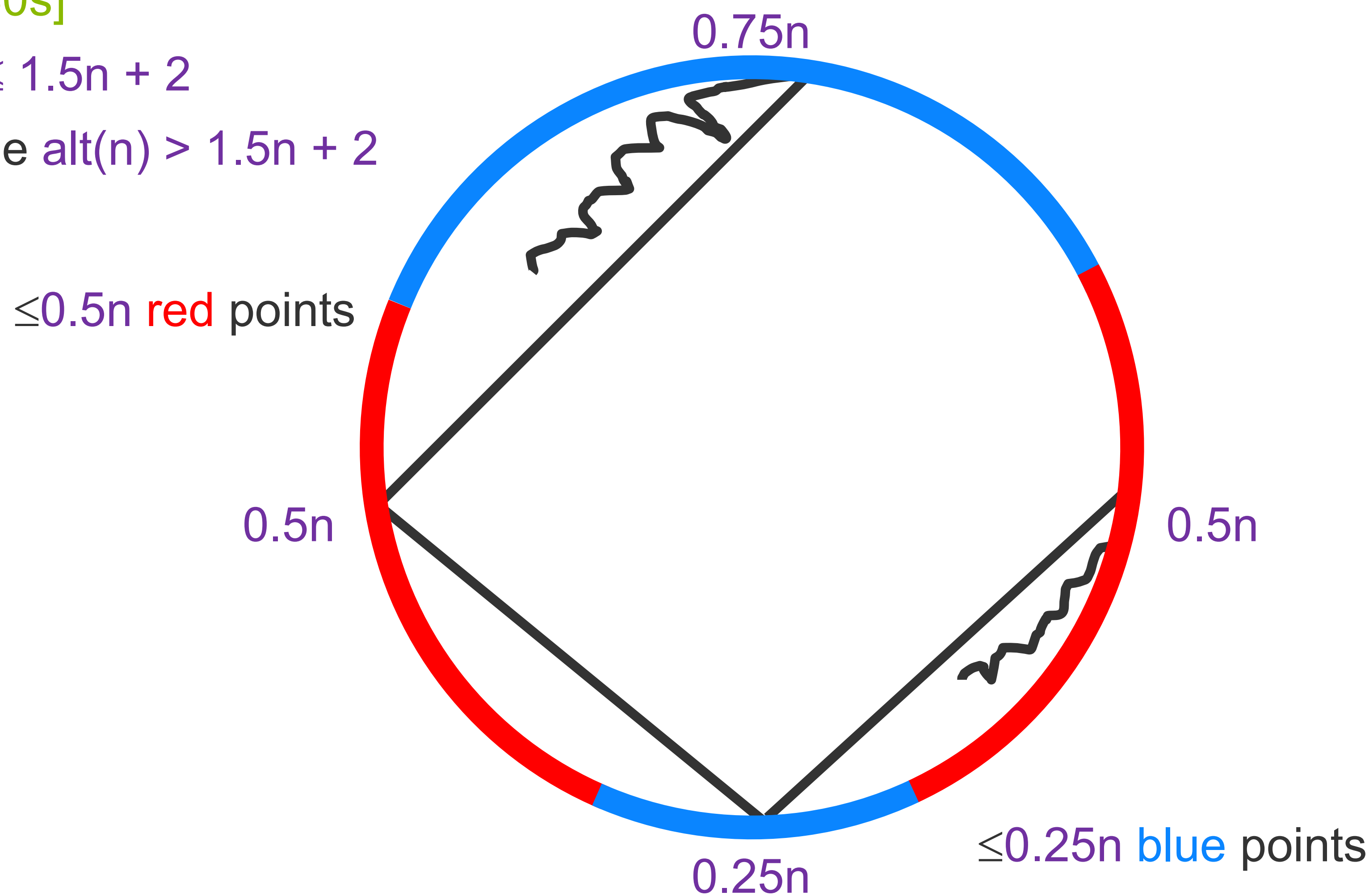


More Background: Upper Bounds

[Erdős, 1980s]

$alt(n) \leq 1.5n + 2$

[Abellanas, Garcia, Hurtado, and Tejel '03; Kynčl, Pach and Tóth '08; Mészáros '11]

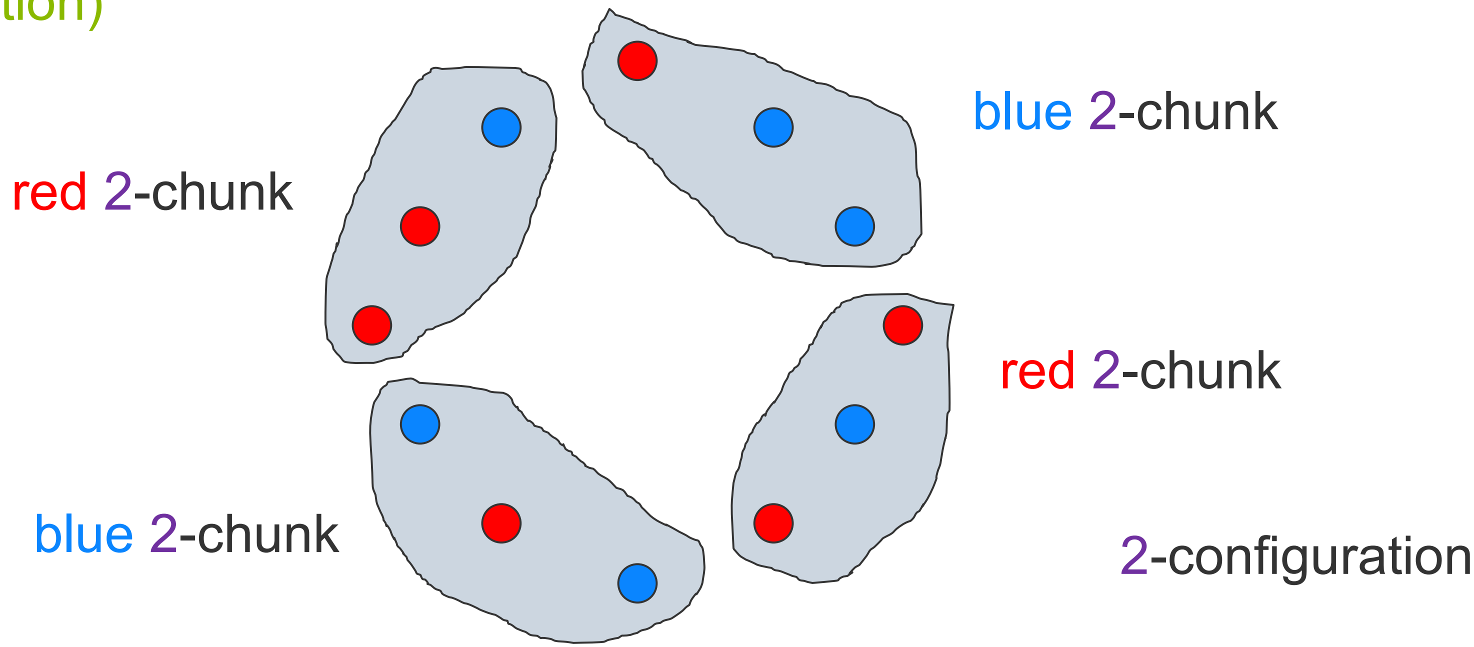
$alt(n) \leq 4n/3 \approx 1.33n$

[Csóka, Blázsik, Király and Lenger '20]

$alt(n) \leq (4 - 2\sqrt{2})n \approx 1.17n$

Our Approach – Chunks

k-chunk	k points of one color and <k points of other color
k-configuration	partition into k-chunks
index (chunk)	#points minority color/#points majority color
index (configuration)	average index over all chunks

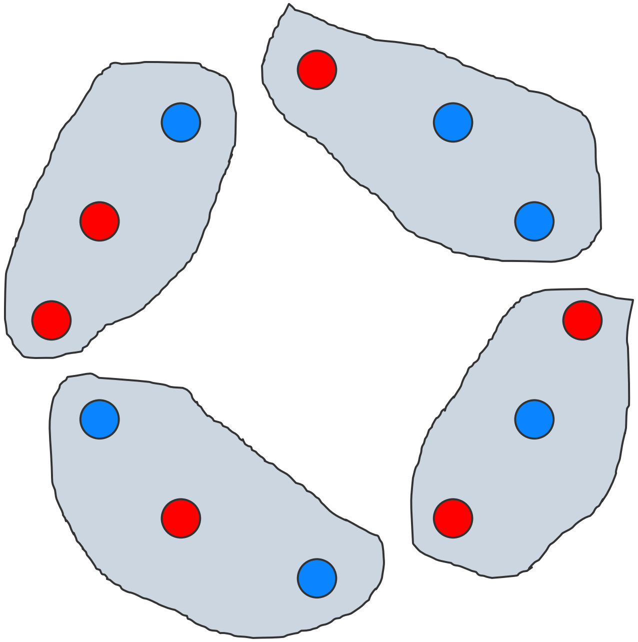


Our Approach – Configurations

Suppose: For every k, we can find a canonical k-configuration Γ_k on P

Observation 1: If Γ_{1000} has index ≥ 0.1, a long alternating path exists.

Reason: There must be many runs.



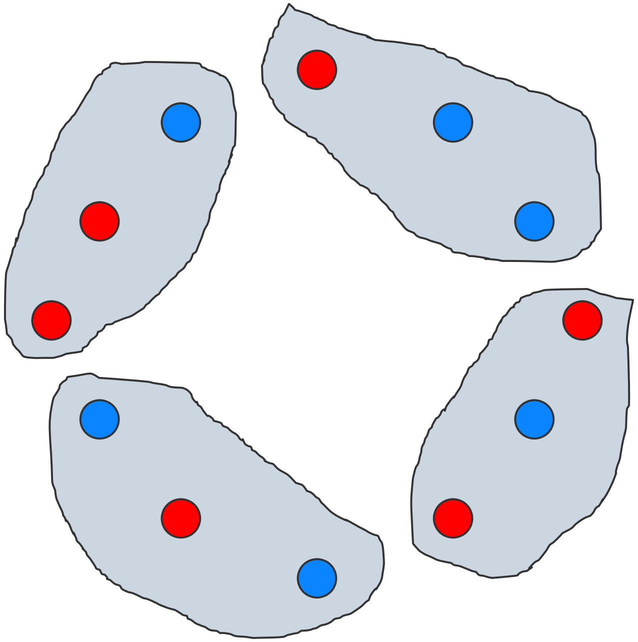
Our Approach – Configurations

Suppose: For every k, we can find a canonical k-configuration Γ_k on P

Observation 2: If $\Gamma_{n/1000}$ has index <0.1, a long alternating path exists.

Reason: There must be a large unbalanced chunk.

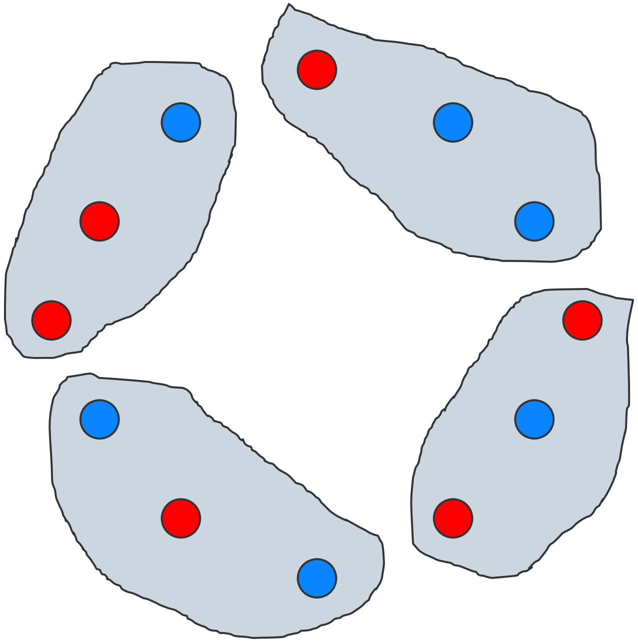
→ Kynčl, Pach and Tóth



Our Approach – Configurations

Suppose: For every k, we can find a canonical k-configuration Γ_k on P

Thus: We can focus on a canonical 3k-configuration Γ_{3k} with $1000 < 3k < n/1000$ and index ≈ 0.1

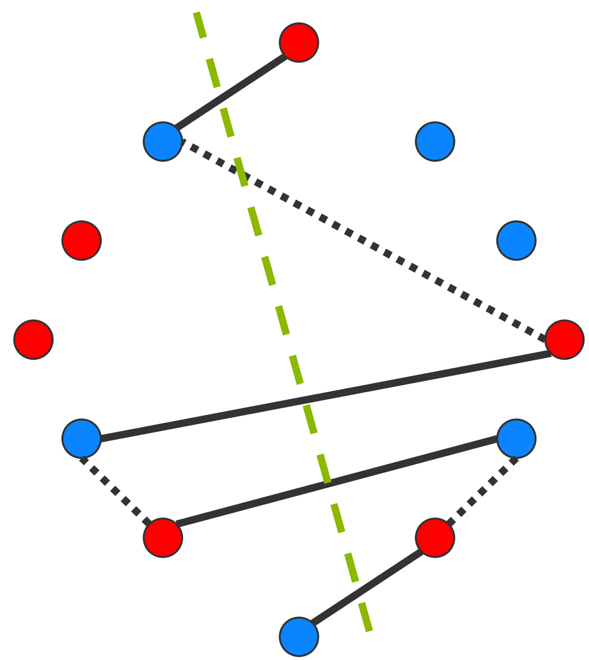


Our Approach – Separated Matchings

We now look at separated matchings.

separated matching: plane bichromatic matching, all segments intersected by one line

Obvious: separated matching with k edges → alternating path with 2k points



Our Approach – Separated Matchings

We look at separated matchings.

separated matching: plane bichromatic matching, all segments intersected by one line

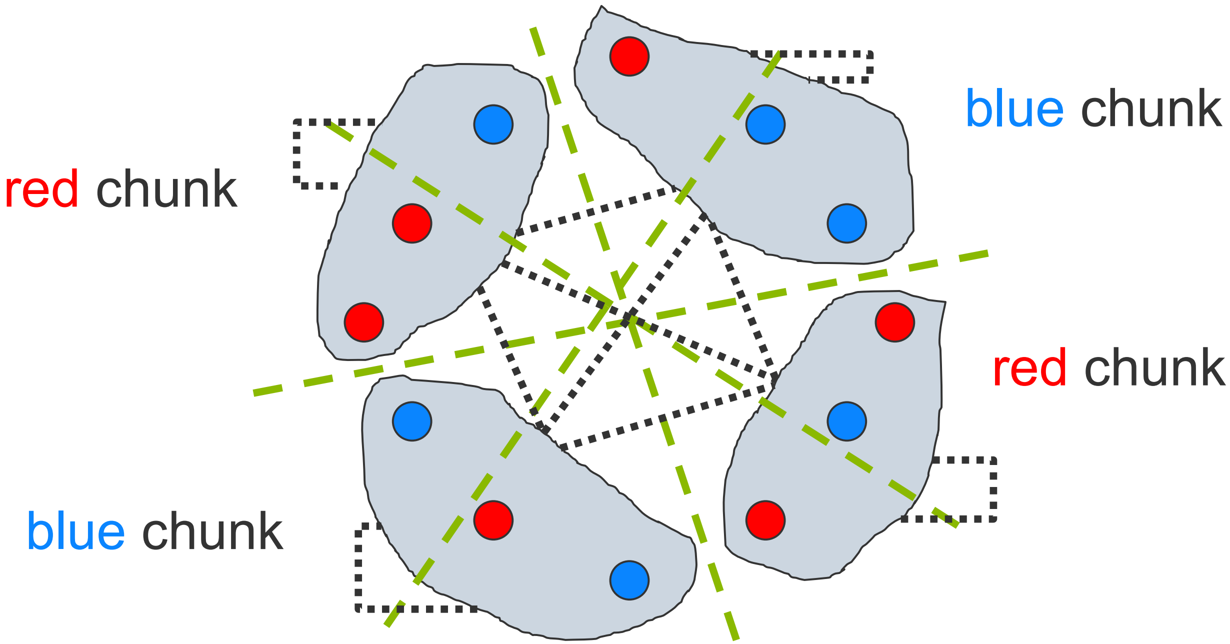
Obvious: separated matching with k edges $\rightarrow$ alternating path with $2k$ points

We show: $\exists\ \varepsilon > 0\ \forall$ suitable Γ_{3k} $\exists$ sep. matching of $(1/2+\varepsilon)n$ edges

Our Approach – Chunk Matchings

chunk matching: match 3k-chunks in Γ_{3k} along a chunk-halving-line

random chunk matching: pick chunk-halving-line uniformly at random



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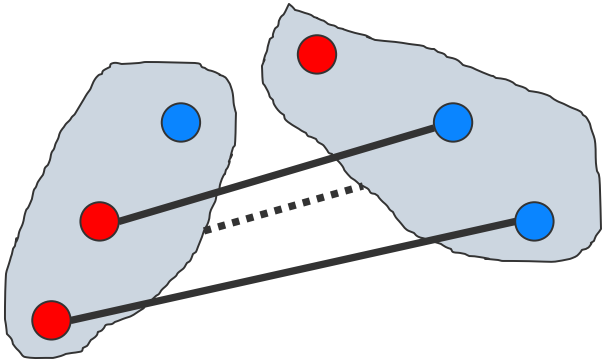
Our Approach – Chunk Matchings

Observation: chunk matching → separated matching



red 3k-chunk

blue 3k-chunk

3k edges

Our Approach – Chunk Matchings

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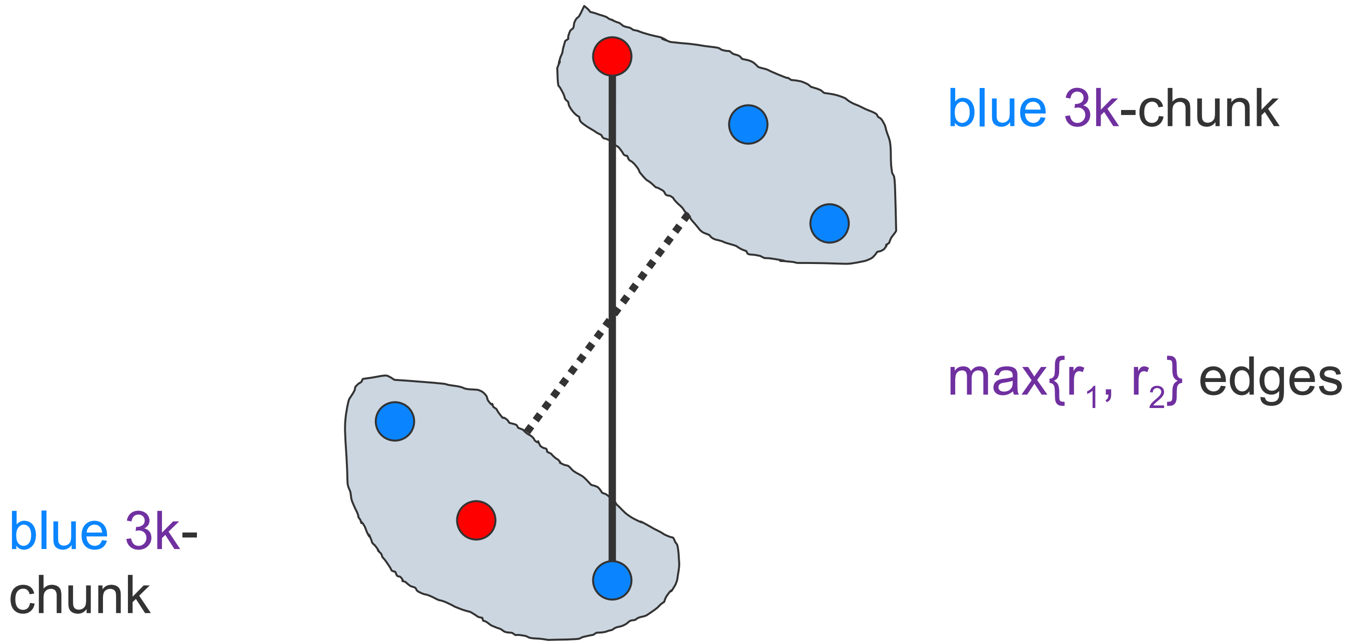


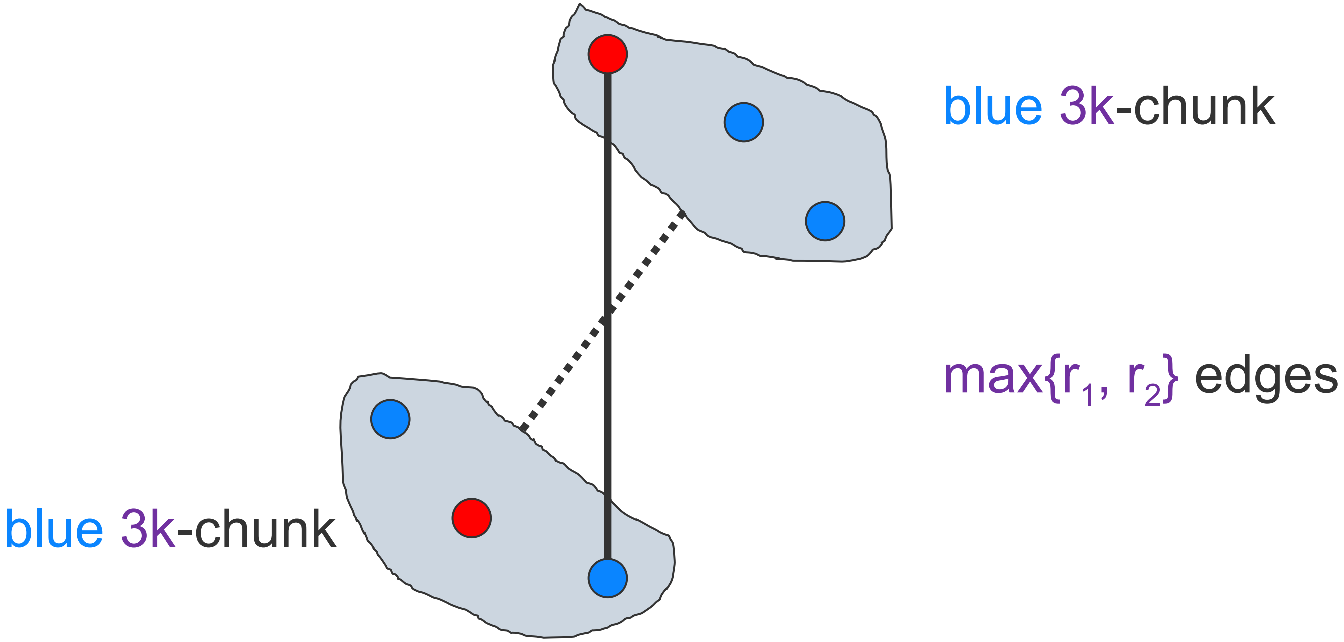
Our Approach – Chunk Matchings

Observation: chunk matching → separated matching

Fact: A random chunk matching yields a separated matching of expected size n/2 (# edges).

Proof: Brute-force calculation.

Crucial: bound $\max\{r_1, r_2\} \geq (r_1 + r_2)/2$

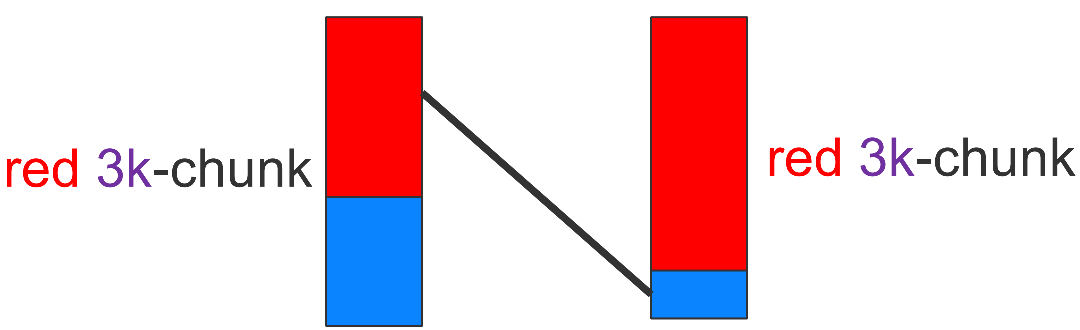


Our Approach – Proof Strategy

Suppose: 3k-configuration Γ_{3k} of index ≈ 0.1 is at hand

Consider: random chunk matching in Γ_{3k}

Lemma: If the individual chunk indices in Γ_{3k} have "large variance", we get a separated matching with $(1/2 + \varepsilon)n$ edges in expectation.



$\max\{b_1, b_2\} >> (b_1 + b_2)/2$ edges

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Our Approach – Proof Strategy

Suppose: 3k-configuration Γ_{3k} of index ≈ 0.1 is at hand

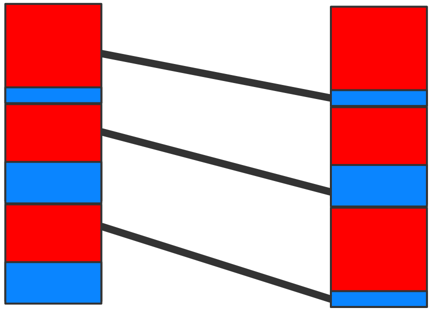
Consider: random chunk matching in Γ_{3k}

Lemma: If Γ_{3k} has "large variance", we get a separated matching with $(1/2 + \varepsilon)n$ edges in expectation.

Otherwise: Consider refined k-configuration Γ_k for Γ_{3k} (it exists).

Lemma: If Γ_k has "large variance", we get a separated matching with $(1/2 + \varepsilon)n$ edges in expectation.

3 red k-chunks



3 red k-chunks

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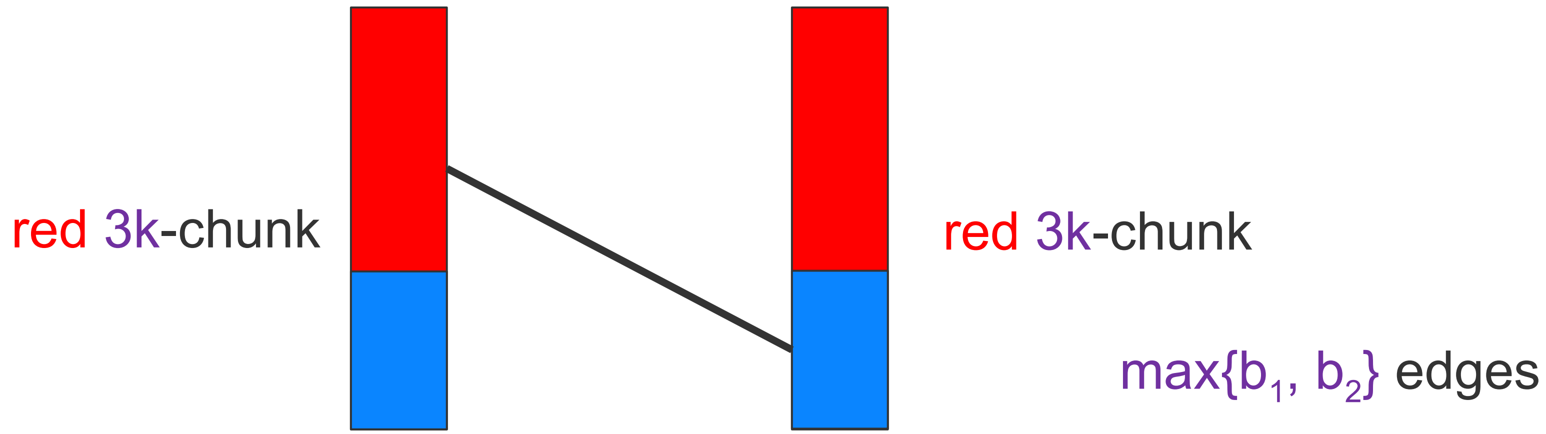
Our Approach – Proof Strategy

Remains: 3k-configuration Γ_{3k} and refined k-configuration Γ_k with "uniform" chunks.

Main trick: gain when matching two 3k-chunks of the same color!



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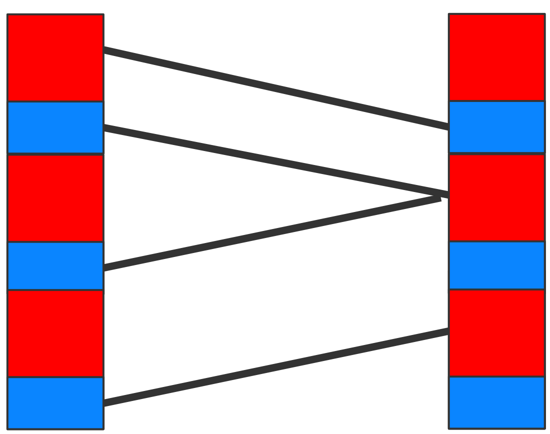

$\approx(4/3)\max\{b_1, b_2\}$ edges

Conclusion

very technical

very small ε

What is the right bound for $alt(n)$?

Questions?