Forbidden subgraph characterizations of some classes of intersection graphs

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Outline

Definitions

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Definitions

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- A family of graphs is hereditary if given any $\mathcal{F}$-graph, then all its induced subgraphs are $\mathcal{F}$-graphs.
- A graph is minimally non-$\mathcal{F}$ (or minimal forbidden subgraph for the class $\mathcal{F}$) if it is not an $\mathcal{F}$–graph and all its induced are $\mathcal{F}$- graph.

Interval graphs

- Let $\mathcal{F}$ be a finite family of non-empty sets. The intersection graph of $\mathcal{F}$ is obtained by representing each set by a vertex, two vertices being connected by an edge if and only if the corresponding sets intersect.

Interval graphs

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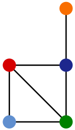
- An interval graph is the intersection graph of a finite family of open intervals on the real line (such a family of intervals is called an interval model for the graph).



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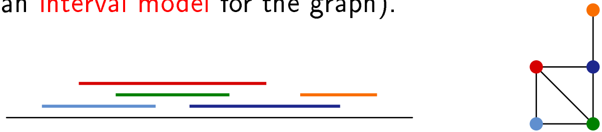
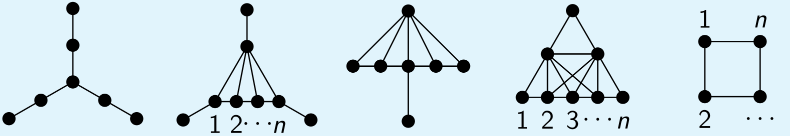


- The class of interval graphs is a hereditary class.

Characterization by minimal forbidden subgraphs

Theorem (Boland and Lekkerkerker, 1962)

A graph G is an interval graph if and only if G does not contain any of the following graphs as induced subgraphs:



bipartite claw $\quad$ n-net, $n \geq 2$ $\quad$ umbrella $\quad$ n-tent, $n \geq 3$ $\quad$ C_n, $n \geq 4$

Unit interval graphs

- A unit interval is an interval graph having an interval model with all its intervals having the same length, such an interval model is called unit interval model. The class of unit interval graph is denoted by $\mathcal{U}$

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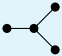
Theorem (Roberts, 1969)

Let G be an interval graph. G is proper interval if and only if G does not contain an induced claw .

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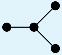
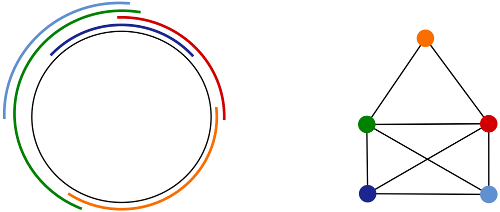
Let G be an interval graph. G is proper interval if and only if G does not contain an induced claw .

Corollary

A graph is a unit interval graph if and only if it contains no induced claw, 2-net, 3-tent, or C_n for any $n \geq 4$.

Circular-arc graphs

- A circular-arc graph (CA graph) is the intersection graph of a finite family of arcs on a circle (such a family of arcs is called a circular-arc model of the graph).



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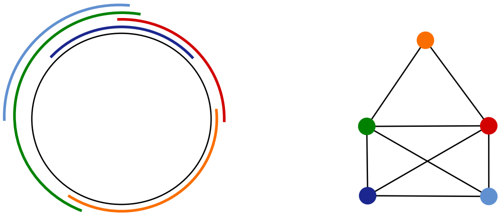

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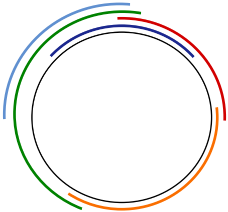

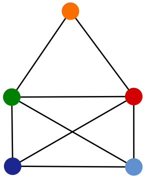

- Circular-arc graphs are a generalization of interval graphs.
- They can be recognized in linear time (McConnell, 2003).

Known partial characterizations

- Tucker (1974) characterized the following subclasses of CA graphs by minimal forbidden subgraphs:
    - proper CA graphs (ie, those that have a CA model in which no arc contains another), and
    - unit CA graphs (ie, those that have a CA model with all arcs of equal length).

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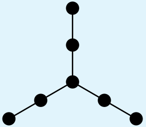
    - proper CA graphs (ie, those that have a CA model in which no arc contains another), and
    - unit CA graphs (ie, those that have a CA model with all arcs of equal length).
- Trotter and Moore (1976) characterized, by minimal forbidden subgraphs, those CA graphs that are complements of bipartite graphs.

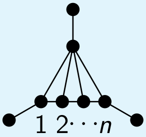
Basic minimally non-CA graphs

Lemma (Trotter and Moore, 1976)

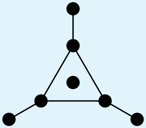
The following are minimally non-CA graphs



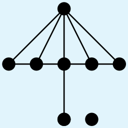
bipartite claw



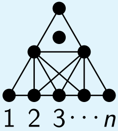
n-net, $n \geq 3$



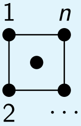
net $\cup K_1$



umbrella $\cup K_1$



n-tent $\cup K_1$, $n \geq 3$

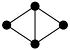
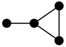
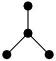
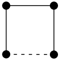


$C_n \cup K_1$, $n \geq 4$

We refer to these graphs as basic minimally non-CA graphs.

New partial characterizations

In this work we present new characterizations of circular-arc graphs by minimal forbidden subgraphs for graphs that belong to one of the following classes:

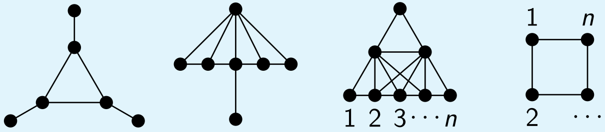
- diamond-free graphs 
- cographs (ie, P_4-free graphs) 
- paw-free graphs 
- claw-free chordal graphs  

Nonbasic minimally non-CA graphs

Proposition

Let G be a minimally non-CA graph. Then at least one of the following conditions hold:

1. G is a basic minimally non-CA graph, or
2. G contains at least one induced subgraph H isomorphic to one of the following graphs



net umbrella n-tent, $n \geq 3$ C_n, $n \geq 4$

Moreover, all vertices v of $G - H$ are adjacent to at least one vertex of H.

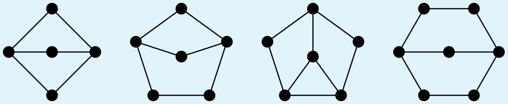
Holes in minimally non-CA graphs

A hole is a chordless cycle of length ≥ 4.

Theorem

Let G be a minimalyl non-CA graph. Then exactly one of the following conditions hold:

1. For each hole H of G and for each vertex v of $G - H$, either v is complete to H or $N_H(v)$ induces a non-empty path in H, or
2. G is isomorphic to $C_j \cup K_1$ for some $j \geq 4$, or to one of the following graphs



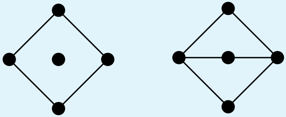
Cographs

1. Cographs are those graphs not containing P_4 as induced subgraph.

Theorem

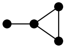
Let G be a cograph. The following conditions are equivalent:

1. G is a CA graph,
2. G is $\{C_4 \cup K_1, K_{2,3}\}$-free graph.



$C_4 \cup K_1$ $K_{2,3}$

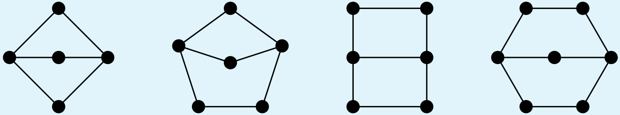
Paw-free graphs

The graph  is called a paw. A graph is paw-free if it does not contain an induced paw.

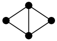
Theorem

Let G be a paw-free graph. The following conditions are equivalent:

1. G is a CA graph,
2. G contains neither an induced $C_j \cup K_1$ for any $j \geq 4$, nor a bipartite claw, nor any of the following graphs as induced subgraphs:



Diamond-free graphs

The graph  is called a diamond. A graph is diamond-free it does not contain an induced diamond.

Theorem

Let G be a diamond-free graph. The following are equivalent:

- G is a CA graph,
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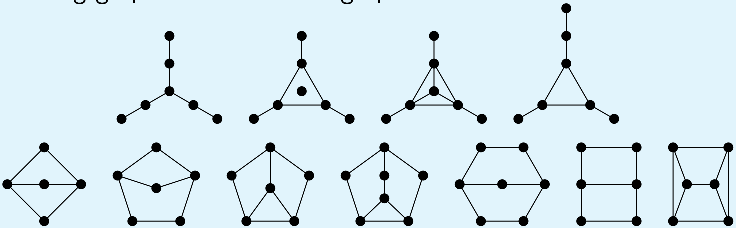
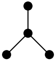


Claw-free chordal graphs

The graph  is called a claw. A graph is claw-free if it does not contain an induced claw.

Theorem

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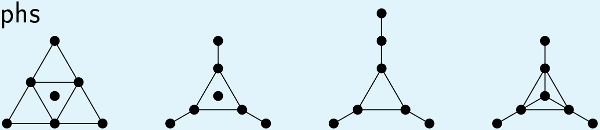
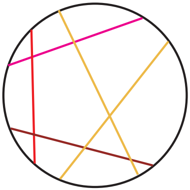


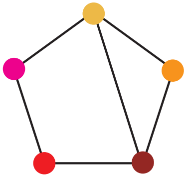
Circle Graphs

- A circle graph is the intersection graph of a family $\{L_v\}_{v \in V}$ of chords of a circle; i.e., v and w are adjacent if and only if $L_v \cap L_w \neq \emptyset$. The family $\{L_v\}_{v \in V}$ is called a circle model of G.

Example:



Circle model

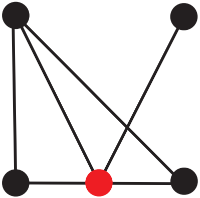


Circle Graph

Local Complementation

- The local complement of a graph $G = (V, E)$ with respect to a vertex $u \in V$ is the graph $G * u$ that arises from G by replacing the induced subgraph $G[N_G(u)]$ by its complement.

Example:



Local Complementation

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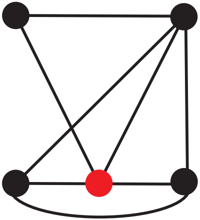
- Two graphs G and H are locally equivalent if and only if G arises from H by a sequence of local complementations.

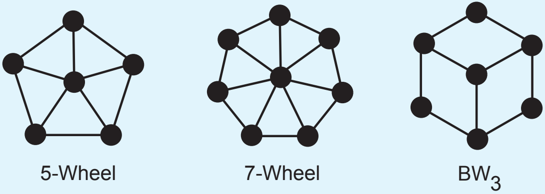
Example:



Bouchet's Characterization

Theorem, Bouchet (1994)

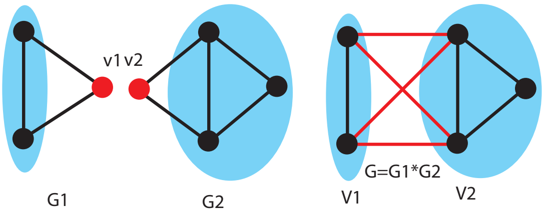
Let G be a graph. Then G is a circle graph if and only if no graph locally equivalent to G contains W_5, W_7 or BW_3 as induced subgraph.



Split Decomposition

Let G_1 and G_2 be two graphs such that $|V(G_i)| \geq 3$, $i = 1, 2$. Let $v_i \in G_i$ (mark vertex of G_i), $i = 1, 2$. The split composition with respect to v_1 and v_2 is the graph $G_1 * G_2$, where $V(G_1 * G_2) = (V(G_1) \cup V(G_2)) \setminus \{v_1, v_2\}$ and $E(G_1 * G_2) = E(G_1 - \{v_1\}) \cup E(G_2 - \{v_2\}) \cup \{uv : u \in N_{G_1}(v_1) \text{ and } v \in N_{G_2}(v_2)\}$.

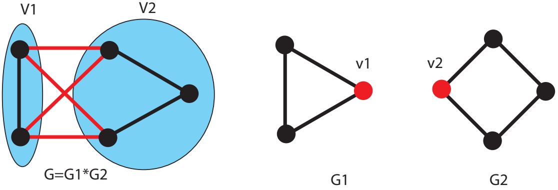
Example:



Split Decomposition

- We say that G has a split decomposition if there exist two graphs G_1 and G_2 with $|V(G_i)| \geq 3$, $i = 1, 2$, such that $G = G_1 * G_2$. G_1 and G_2 are called the factors of the split decomposition. Those graphs that do not have a split decomposition are called prime graphs.

Example:



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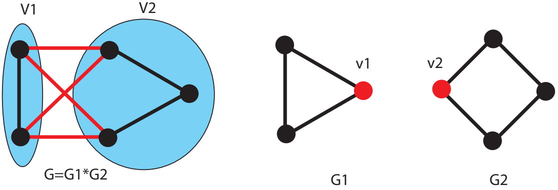
- If any of the factors of a split decomposition has a split decomposition we can continue the process until every factor is prime, a star or a complete.

Example:

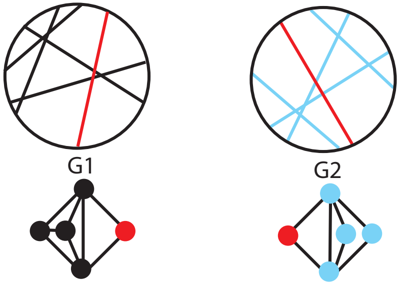


Split Decomposition on Circle Graphs

Theorem (Bouchet, 1987)

- Let G be a graph such that $G = G_1 * G_2$. Then, G is a circle graph if and only if G_1 and G_2 are circle graphs.

Example: The following figure shows two circle graphs and their circle models.



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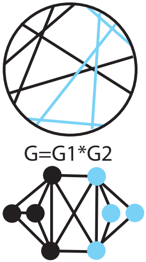
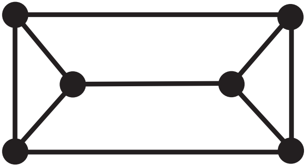


Edge Subdivision

Theorem

Let G be a graph. If G is not a circle graph, then any graph H that arises from G by edge subdivisions is not a circle graph.

Steps of the proof:



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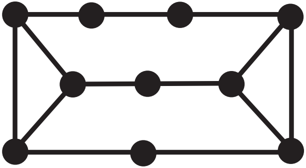

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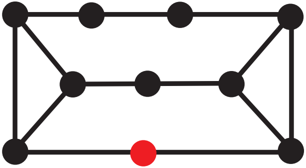

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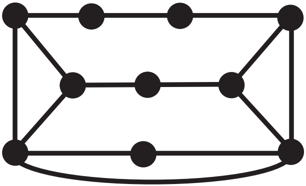

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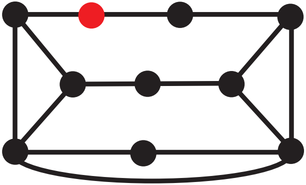

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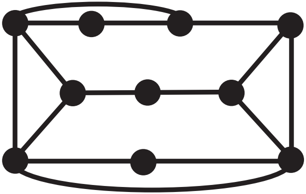

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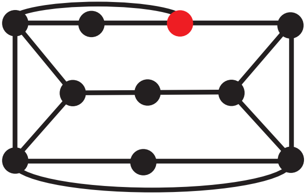

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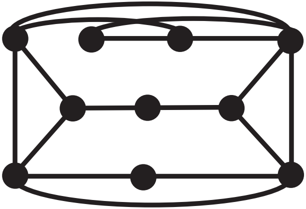

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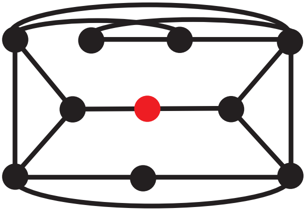

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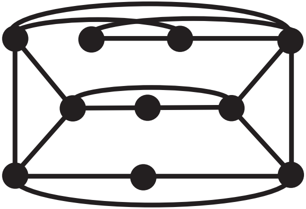

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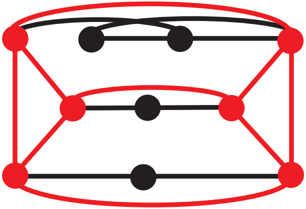
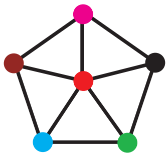
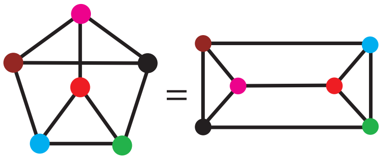


The Prisms

- $\overline{C_6}$ is called a prism.


W5


Complement of W5

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- $\overline{C_6}$ is called a prism.
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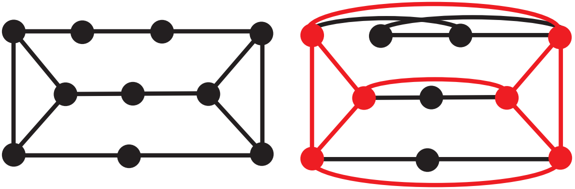
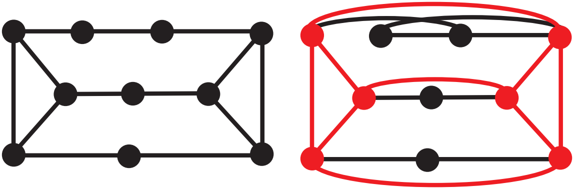


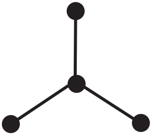
The Prisms

- $\overline{C_6}$ is called a prism.
- If the edges linking the triangles of $\overline{C_6}$ are subdivided, then the resulting graph is also called a prism.
- Since $\overline{C_6}$ is locally equivalent to W_5, it is not a circle graph. So, prisms are not circle graphs.

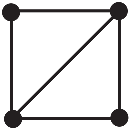


Linear Domino Graphs

1. The graph G is domino if each of its vertices belong to at most two cliques. In addition, if each of its edges belongs to at most one clique, G is linear domino. Linear domino graphs coincide with {claw,diamond}-free graphs



Claw



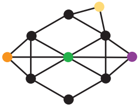
Diamond

Linear Domino Prime Graphs

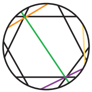
Theorem

Let G be a linear domino connected prime graph. Then, G is a circle graph if and only if G does not contain prisms as induced subgraphs.

Example of linear domino prime graph



Prime linear domino graph



Circle Model

Linear Domino Prime Graphs

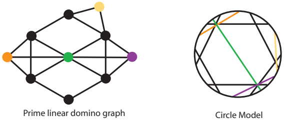
Theorem

Let G be a linear domino connected prime graph. Then, G is a circle graph if and only if G does not contain prisms as induced subgraphs.

Corollary

Let G be a linear domino graph. Then, G is a circle graph if and only if G contains no induced prism.

Example of linear domino prime graph



Permutation graphs

- Comparability graphs were characterized by Gallai in 1967. This characterization implies the characterization by forbidden induced subgraphs for permutation graphs.

Permutation graphs

- Comparability graphs were characterized by Gallai in 1967. This characterization implies the characterization by forbidden induced subgraphs for permutation graphs.
- Given two graphs G and H. The join of G and H is the graph denoted by $G + H$ whose vertex set is $V(G) \cup V(H)$ and whose edge set is $E(G) \cup E(H) \cup \{vw : v \in V(G),\ w \in V(H)\}$.

Lemma

The join $G = G_1 + G_2$ is a circle graph if and only if both G_1 and G_2 are permutation graphs.

Superclasses of cographs

- Let G be a graph and let A be a vertex set inducing a P_4 in G. A vertex v of G is said a partner of A if $G[A \cup \{v\}]$ contains at least two induced P_4's. Finally, G is called *P_4-tidy* if each vertex set A inducing a P_4 in G has at most one partner.

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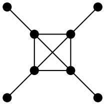
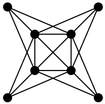
- Tree-cographs are a generalization of cographs. They are defined recursively as follows: trees are tree-cographs; the disjoint union of tree-cographs is a tree-cograph; and the complement of a tree-cograph is also a tree-cograph.

P_4–tidy

- A spider H is a graph whose vertex set can be partitioned into three sets S, C, and R, where $S = \{s_1, \ldots, s_k\}$ ($k \geq 2$) is a stable set; $C = \{c_1, \ldots, c_k\}$ is a complete set; s_i is adjacent to c_j if and only if $i = j$ (a thin spider, denoted by thin$_k(H[R])$), or s_i is adjacent to c_j if and only if $i \neq j$ (a thick spider, denoted by thick$_k(H[R])$); R is allowed to be empty and if it is not, then all the vertices in R are adjacent to all the vertices in C and nonadjacent to all the vertices in S. The triple (S, C, R) is called the spider partition. A fat spider is obtained from a spider by adding a true or false twin of a vertex $v \in S \cup C$.



thin$_4$



thick$_4$

Characterization of P_4-tidy graphs

Theorem (V. Giakoumakis et al, 1997).

Let G be a P_4-tidy graph with at least two vertices. Then, exactly one of the following conditions holds:

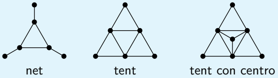
1. G is disconnected.
2. $\overline{G}$ is disconnected.
3. G is isomorphic to P_5, $\overline{P_5}$, C_5, a spider, or a fat spider.

Superclasses of cographs

- G^+ stands for the graph G plus a universal vertex.

Theorem

Let G be a P_4-tidy graph. Then, G is a circle graph if and only if G contains no W_5, net$^+$, tent$^+$, or tent-with-center as induced subgraph.



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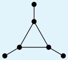
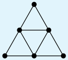
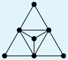
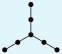

net

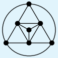

tent


tent con centro

Theorem

Let G be a tree-cograph. Then, G is a circle graph if and only if G contains no induced bipartite-claw$^+$ and no induced co-(bipartite-claw).


bipartite claw


$\overline{\text{bipartite claw}}$

Definitions

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- A graph is unit circle if it has a circle model such that every chord has the same length.
- A graph is unit Helly circle (**UHC**) if it has a circle model such that every chord has the same length and every subset of chords pairwise intersecting has a point in common.

Characterization of Unit Helly Circle Graphs

Theorem

Let G be a graph. Then the following assertions are equivalent:

1. G is a unit Helly circle graph.
2. G contains no induced paw, no induced claw, no induced diamond and no induced $C_n \cup K_1$ for any $n \geq 3$.
3. G is a chordless cycle, a complete graph, or a disjoint union of chordless paths.