Algebraic area* of lattice random walks and exclusion statistics

Joint work with Stéphane Ouvry (LPTMS) and Alexios Polychronakos (CCNY)

Li Gan

LPTMS**, Université Paris-Saclay

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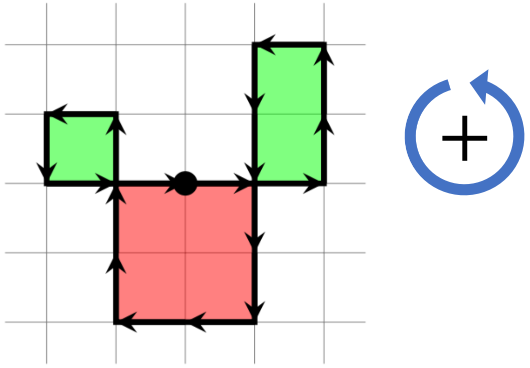
find more here

* a.k.a signed area

** Laboratoire de Physique Théorique et Modèles Statistiques

Algebraic area A of closed random walks on a square lattice

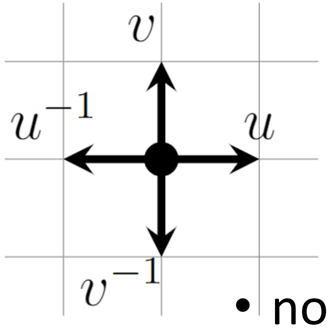


$A = 1 + 2 - 4 = -1$

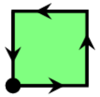
Question: a formula for the number $C_{\mathbf{n}}(A)$ of closed **n**-step lattice walks that enclose an algebraic area A?

e.g. square lattice walks:

$C_2(0) = 4, \quad C_4(0) = 28, \quad C_4(1) = C_4(-1) = 4$



square lattice walks



$v^{-1}\, u^{-1} v\, u = \mathrm{Q}^1$

- non commutative relation $v\, u = \mathrm{Q}\, u\, v$
- generating function for $\mathbf{n}$-step closed walks

$$(u + u^{-1} + v + v^{-1})^{\mathbf{n}} = \sum_A C_{\mathbf{n}}(A)\, \mathrm{Q}^A + \ldots$$

$$\mathbf{Tr}\, (v^n u^m) = \delta_{n,0}\delta_{m,0}$$

$$\Rightarrow \mathbf{Tr}\, (u + u^{-1} + v + v^{-1})^{\mathbf{n}} = \sum_A C_{\mathbf{n}}(A)\, \mathrm{Q}^A$$

e.g. $\mathbf{Tr}\, \left(u + u^{-1} + v + v^{-1}\right)^4 = 28 + 4\mathrm{Q} + 4\mathrm{Q}^{-1}$

physics

Hofstadter model: a charged particle hopping on a square lattice in a constant magnetic field

- Hamiltonian $H = u + u^{-1} + v + v^{-1}$
- $\mathrm{Q} = \mathrm{e}^{\mathrm{i}\gamma}; \gamma = 2\pi\phi/\phi_0$: magnetic flux per plaquette

  rational flux $\phi/\phi_0 = p/q$ with p, q coprime
- $\sum_A C_{\mathbf{n}}(A)\, \mathrm{Q}^A = \mathbf{Tr}\, H^{\mathbf{n}}$

aim: compute $\mathbf{Tr}\, H^{\mathbf{n}}$

rational flux, i.e., $Q = e^{2i\pi p/q}$ with p, q coprime

$v\,u = Q\,u\,v$: noncommutative 2-tori algebra, quantum torus algebra, Weyl commutation relation, Weyl braiding relation, Q-commutativity, etc.

$$u = e^{ik_x}\begin{pmatrix} Q & 0 & 0 & \cdots & 0 & 0 \\ 0 & Q^2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & Q^3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Q^{q-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad v = e^{ik_y}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

quantum trace

$$\mathbf{Tr}\, H^{\mathbf{n}} = \frac{1}{q}\int_0^{2\pi}\int_0^{2\pi}\frac{dk_x}{2\pi}\frac{dk_y}{2\pi}\,\mathrm{tr}\, H^{\mathbf{n}}$$

$u \to -uv,\ \ v \to v$

$k_x = k_y = 0,\ \mathbf{n} < q$

reduces to

usual trace

$$\mathbf{Tr}\, H^{\mathbf{n}} = \frac{1}{q}\mathrm{tr}\, H_2^{\mathbf{n}}$$

$$H_2 = \begin{pmatrix} 0 & f_1 & 0 & \cdots & 0 & 0 \\ g_1 & 0 & f_2 & \cdots & 0 & 0 \\ 0 & g_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & f_{q-1} \\ 0 & 0 & 0 & \cdots & g_{q-1} & 0 \end{pmatrix}$$

$$f_k = 1 - Q^k,\ g_k = 1 - Q^{-k}$$

aim: compute tr $H_2^{\mathbf{n}}$

- approach 1: **secular determinant** $\det\,(I - z\,H_2)$ and its relation to ***exclusion statistics***

$$\ln\det(I - zH_2) = \mathrm{tr}\,\ln(I - zH_2) = -\sum_{\mathbf{n}=1}^{\infty}\frac{z^{\mathbf{n}}}{\mathbf{n}}\mathrm{tr}\, H_2^{\mathbf{n}}$$

- approach 2: **direct computation** (combinatorics of periodic Dyck paths*)

$$\mathrm{tr}\, H_2^{\mathbf{n}} = \sum_{k_1=1}^{q}\sum_{k_2=1}^{q}\cdots\sum_{k_{\mathbf{n}}=1}^{q} h_{k_1k_2}h_{k_2k_3}\cdots h_{k_{\mathbf{n}}k_1}$$

* periodic Dyck path = Dyck bridge

Approach 1: via secular determinant $\det\left(I - z\, H_2\right)$

$$H_2 = \begin{pmatrix} 0 & f_1 & 0 & \cdots & 0 & 0 \\ g_1 & 0 & f_2 & \cdots & 0 & 0 \\ 0 & g_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & f_{q-1} \\ 0 & 0 & 0 & \cdots & g_{q-1} & 0 \end{pmatrix}$$

secular determinant $\det(I - zH_2) = \sum_{n=0}^{\lfloor q/2 \rfloor} (-1)^n Z_n z^{2n}$

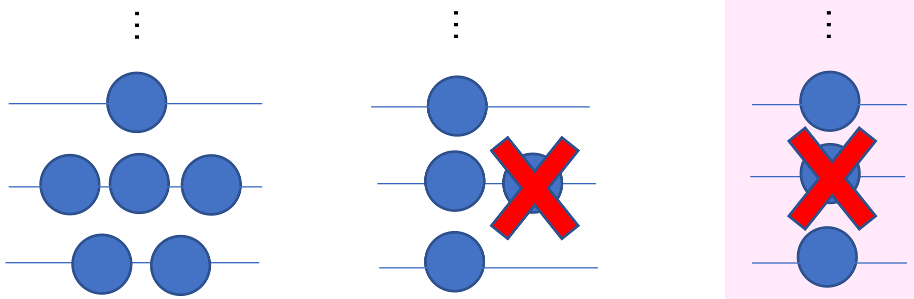
Kreft coefficient $Z_n = \sum_{k_1=1}^{q-2n+1} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} s_{k_1+2n-2} s_{k_2+2n-4} \cdots s_{k_{n-1}+2} s_{k_n}, \quad s_k := g_k f_k = 4\sin^2(k\pi p/q)$

[Kreft 1993]

"+2 shifts"

e.g. for $q = 7, \quad Z_3 = s_1\, s_3\, s_5 + s_1\, s_3\, s_6 + s_1\, s_4\, s_6 + s_2\, s_4\, s_6$

interpretation in statistical mechanics



exclusion parameter g — bosons ($g = 0$) — fermions ($g = 1$) — $g = 2$

Z_n: partition function for n particles occupying $q - 1$ quantum states. These particles obey $g = 2$ ***exclusion statistics*** (no two particles can occupy adjacent quantum states)

stronger exclusion than fermions!

closed random walks on a square lattice ⟺ ***exclusion statistics*** with exclusion parameter $g = 2$

Using techniques from statistical mechanics to compute $\mathbf{Tr}\, H_2^{\mathbf{n}}$

Approach 1: via secular determinant $\det(I - z\,H_2)$

Kreft coefficient $Z_n = \sum_{k_1=1}^{q-2n+1} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} s_{k_1+2n-2} s_{k_2+2n-4} \cdots s_{k_{n-1}+2} s_{k_n}\,,\ s_k := g_k f_k = 4\sin^2(k\pi p/q)$

introduce b_n via $\log\left(\sum_{n=0}^{\lfloor q/2 \rfloor} Z_n x^n\right) = \sum_{n=1}^{\infty} b_n x^n,\ \ \mathrm{tr}\, H_2^{\mathbf{n}=2n} = 2n(-1)^{n+1} b_n$

b_n: cluster coefficient

$$\mathrm{tr}\, H_2^{\mathbf{n}=2n} = 2n \sum_{\substack{l_1, l_2, \ldots, l_j \\ \text{composition of } n}} c_2(l_1, l_2, \ldots, l_j) \sum_{k=1}^{q-j} s_k^{l_1} s_{k+1}^{l_2} \cdots s_{k+j-1}^{l_j}$$

composition is an ordered partition

Example

Four compositions of n=3:
(3), (2,1), (1,2), (1,1,1)

l_1	l_2	l_3
3		
2	1	
1	2	
1	1	1

combinatorial factor

$$c_2(l_1, l_2, \ldots, l_j) = \frac{1}{l_1} \prod_{i=2}^{j} \binom{l_{i-1} + l_i - 1}{l_i}$$

interpretation in combinatorics?

$$C_{\mathbf{n}}(A) = 2n \sum_{\substack{l_1, l_2, \ldots, l_j \\ \text{composition of } n}} c_2(l_1, l_2, \ldots, l_j) \sum_{k_3=-l_3}^{l_3} \sum_{k_4=-l_4}^{l_4} \cdots \sum_{k_j=-l_j}^{l_j} \binom{2l_1}{l_1 + A + \sum_{i=3}^{j}(i-2)k_i} \binom{2l_2}{l_2 - A - \sum_{i=3}^{j}(i-1)k_i} \prod_{i=3}^{j} \binom{2l_i}{l_i + k_i}$$

[Ouvry, Wu 2019]

	$\mathbf{n} = 2$	4	6	8	10
$A =$ 0	4	28	232	2156	21944
± 1		8	144	2016	26320
± 2			24	616	11080
± 3				96	3120
± 4				16	840
± 5					160
± 6					40
counting	4	36	400	4900	63504

$C_{\mathbf{n}}(A)$ up to $\mathbf{n} = 10$ for square lattice walks of length $\mathbf{n}$.

$\det(I - zH_2)$ → Z_n → b_n ⇒ $\mathrm{tr}\, H_2^{\mathbf{n}=2n}$

→ generalize to the “+g shifts” (g-exclusion)

g-exclusion statistics

[Ouvry, Polychronakos 2019]

$$H_2 = \begin{pmatrix} 0 & f_1 & 0 & \cdots & 0 & 0 \\ g_1 & 0 & f_2 & \cdots & 0 & 0 \\ 0 & g_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & f_{q-1} \\ 0 & 0 & 0 & \cdots & g_{q-1} & 0 \end{pmatrix}$$

$$H_g = \begin{pmatrix} 0 & f_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & f_2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ g_1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & g_2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & f_{q-1} \\ 0 & 0 & 0 & \cdots & g_{q-g+1} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$\det(I - zH_g)$ → Z_n → $\mathrm{tr}\, H_g^{\mathbf{n}}$

$$\det(I - zH_g) = \sum_{n=0}^{\lfloor q/g \rfloor} (-1)^n Z_n z^{gn}$$

$$Z_n = \sum_{k_1=1}^{q-gn+1} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} s_{k_1+gn-g}\, s_{k_2+gn-2g} \cdots s_{k_{n-1}+g}\, s_{k_n}$$

with $s_k = g_k f_k f_{k+1} \dots f_{k+g-2}$

$$\mathrm{tr}\, H_g^{\mathbf{n}=gn} = gn \sum_{\substack{l_1, l_2, \dots, l_j \\ g\text{-composition of } n}} c_g(l_1, l_2, \dots, l_j) \sum_{k=1}^{q-j-g+2} s_k^{l_1} s_{k+1}^{l_2} \cdots s_{k+j-1}^{l_j}$$

g-composition: no more than $g-2$ zeros in succession

Example: nine $g = 3$-compositions of $n = 3$: (3), (2,1), (1,2), (1,1,1), (2,0,1), (1,0,2), (1,0,1,1), (1,1,0,1), (1,0,1,0,1)

$$c_g(l_1, l_2, \dots, l_j) = \frac{1}{l_1} \prod_{i=2}^{j} \binom{l_{i-g+1} + \cdots + l_i - 1}{l_i}$$

Approach 2: compute directly $\mathrm{tr}\, H_g^{\mathbf{n}}$ and interpreter c_g in combinatorics

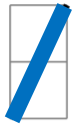
[**LG**, Ouvry, Polychronakos 2022]

$$H_g = \begin{pmatrix} 0 & f_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & f_2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ g_1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & g_2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & f_{q-1} \\ 0 & 0 & 0 & \cdots & g_{q-g+1} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

h_{ij}: matrix elements of H_g

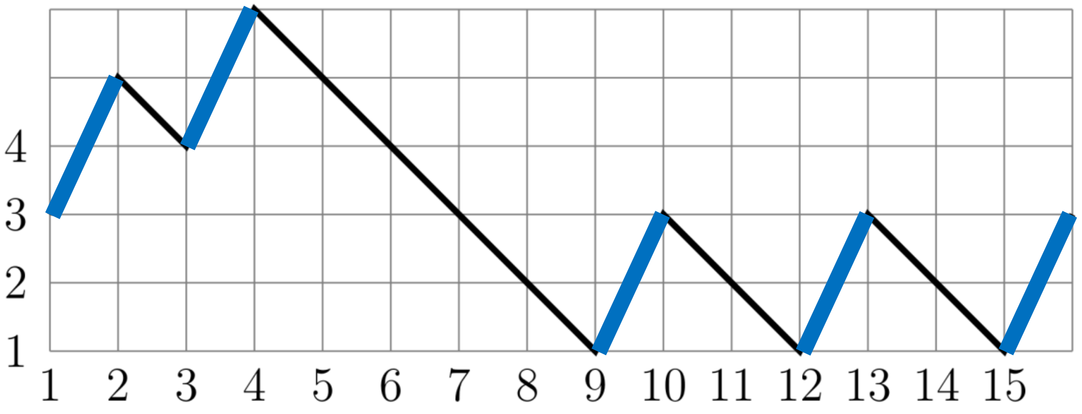
$$\mathrm{tr}\, H_g^{\mathbf{n}} = \sum_{k_1=1}^{q} \sum_{k_2=1}^{q} \cdots \sum_{k_{\mathbf{n}}=1}^{q} h_{k_1 k_2} h_{k_2 k_3} \cdots h_{k_{\mathbf{n}} k_1}$$

$$k_{i+1} - k_i = \begin{cases} -(g-1) & \rightarrow \text{up step (going up } g{-}1 \text{ floors)} \\ +1 & \rightarrow \text{down step (going down 1 floor)} \end{cases}$$

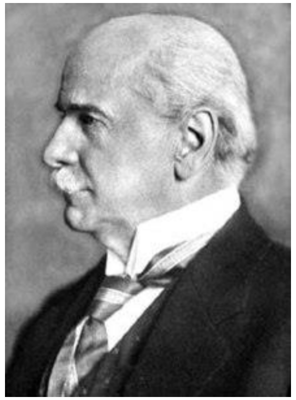
Example: $g = 3$ up step  and down step 

g-composition $l_1, \dots, l_j \rightarrow$ all possible **periodic generalized Dyck paths** with l_i up steps starting from the i-th floor

$l_4 = 1$
$l_3 = 1$
$l_2 = 0$
$l_1 = 3$



Example: a **periodic generalized Dyck path** of length 15 for the g = 3 composition 3,0,1,1 starting from the third floor with an up step.

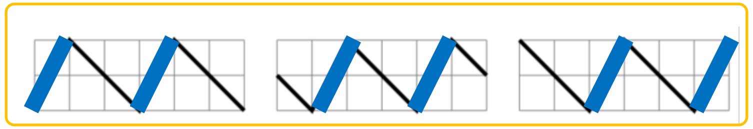
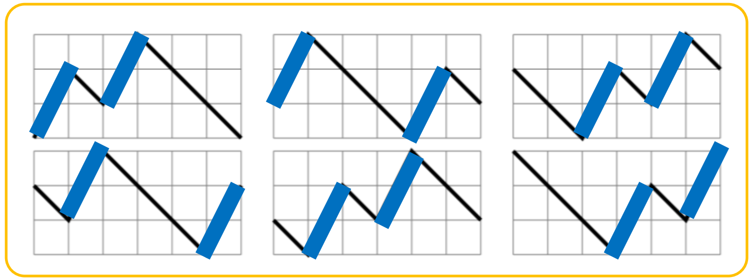
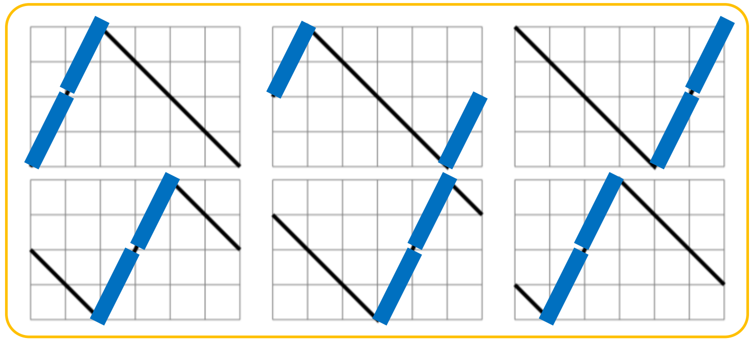


W. F. A. von Dyck
German mathematician

Approach 2: compute directly $\mathrm{tr}\, H_g^{\mathbf{n}}$ and interpreter c_g in combinatorics

g-composition $l_1, \dots, l_j \rightarrow$ all possible **periodic generalized Dyck paths** with l_i up steps starting from the i-th floor

Example: g=3-compositions of $n = 2$: (2), (1,1), (1,0,1)

	$(l_1, l_2, \dots l_j)$	c_3
	(2)	1/2
	(1,1)	1
	(1,0,1)	1

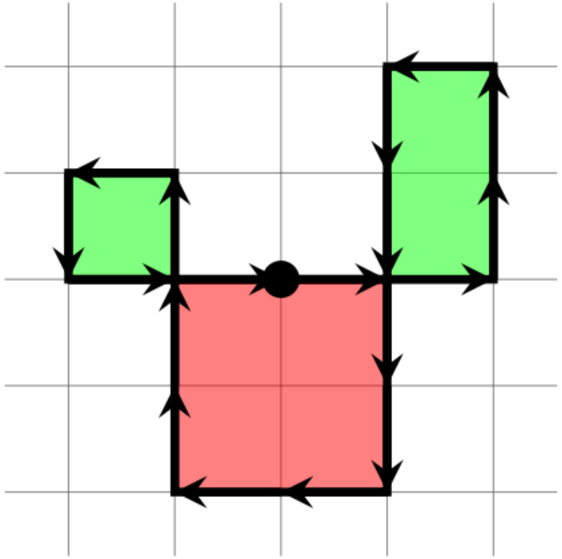
$$c_g(l_1, l_2, \dots, l_j) = \frac{1}{l_1} \prod_{i=2}^{j} \binom{l_{i-g+1} + \cdots + l_i - 1}{l_i}$$

- $l_i\, c_g$ is the number of such generalized Dyck paths **starting from the i-th floor with an up step**
- $(l_1 + l_2 + \cdots + l_j)\; c_g = n\, c_g$ is the total number of such generalized Dyck paths **starting with an up step**
- $gn\, c_g$ is the total number of such generalized Dyck paths

$$\mathrm{tr}\, H_g^{\mathbf{n}=gn} = gn \sum_{\substack{l_1, l_2, \dots, l_j \\ g\text{-composition of } n}} c_g(l_1, l_2, \dots, l_j) \sum^{q-j-g+2} s_k^{l_1} s_{k+1}^{l_2} \cdots s_{k+j-1}^{l_j}$$

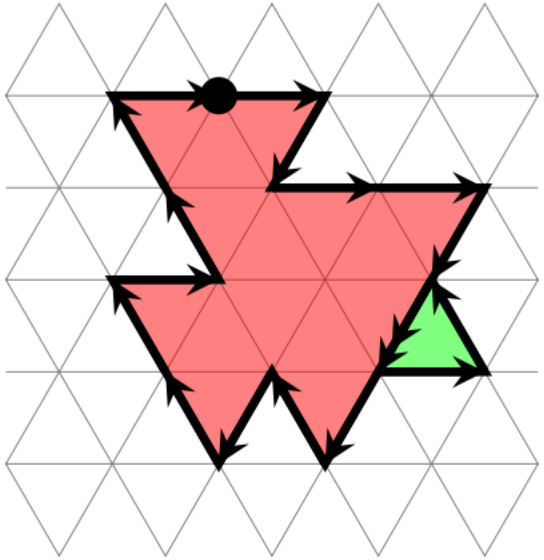
with $s_k = g_k f_k f_{k+1} \cdots f_{k+g-2}$

Closed random walks on various lattices



square

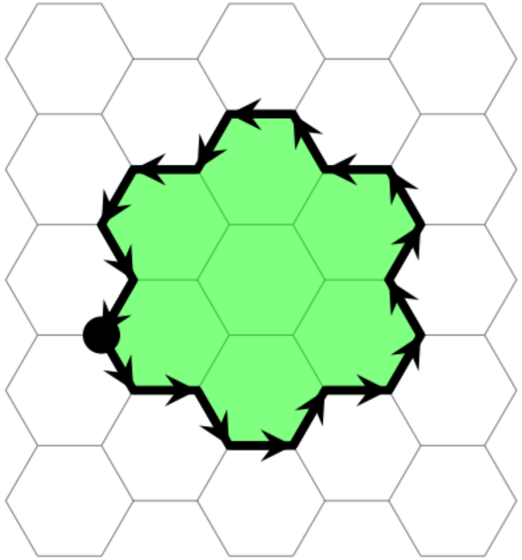
$g = 2$



chiral* triangular

$g = 3$

[Ouvry, Polychronakos 2019]

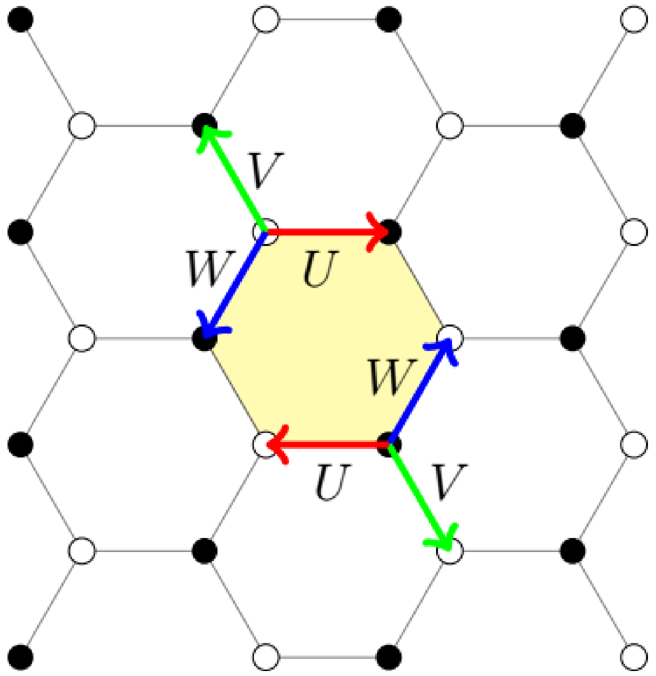


honeycomb

mixture of $g = 1$ and $g = 2$

[**LG**, Ouvry, Polychronakos 2022]

* Only three out of six directions at each step are allowed.



Honeycomb lattice walks

Hamiltonian $H = U + V + W$

honeycomb algebra $U^2 = V^2 = W^2 = I, \quad (UVW)^2 = \mathrm{Q}$

$$\Rightarrow U = \begin{pmatrix} 0 & u \\ u^{-1} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & v \\ v^{-1} & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & \mathrm{Q}^{1/2} v u^{-1} \\ \mathrm{Q}^{-1/2} u v^{-1} & 0 \end{pmatrix}$$

$$H = \begin{pmatrix} 0 & u + v + \mathrm{Q}^{1/2} v u^{-1} \\ u^{-1} + v^{-1} + \mathrm{Q}^{-1/2} u v^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}, \qquad H_{1,2} = AA^\dagger$$

$$\det(I - zH) = \det(I - z^2 H_{1,2}) = \sum_{n=0}^{q} (-1)^n Z_n z^{2n}$$

$\det(I - zH)$ → Z_n → b_n ⇒ $\mathrm{tr}\, H^{\mathbf{n}}$

Honeycomb lattice walks: (1,2)-exclusion

$$H_{1,2} = \begin{pmatrix} \tilde{s}_1 & f_1 & 0 & \cdots & 0 & 0 \\ g_1 & \tilde{s}_2 & f_2 & \cdots & 0 & 0 \\ 0 & g_2 & \tilde{s}_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \tilde{s}_{q-1} & f_{q-1} \\ 0 & 0 & 0 & \cdots & g_{q-1} & \tilde{s}_q \end{pmatrix}$$

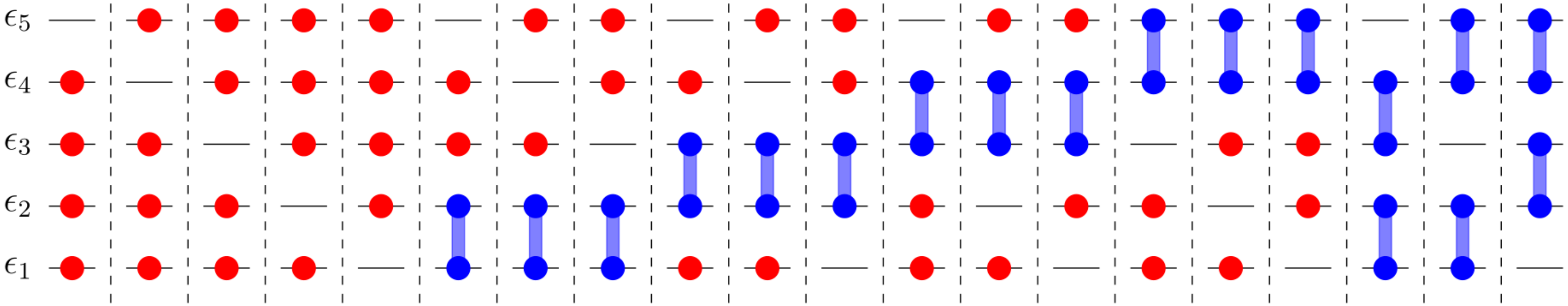
$$\det(I - zH_{1,2}) = \sum_{n=0}^{q} (-1)^n Z_n z^n$$

$$s_k = g_k f_k, \quad \tilde{s}_k = 1 + s_k$$

e.g. $q = 5$

$$Z_4 = \tilde{s}_4\tilde{s}_3\tilde{s}_2\tilde{s}_1 + \tilde{s}_5\tilde{s}_3\tilde{s}_2\tilde{s}_1 + \tilde{s}_5\tilde{s}_4\tilde{s}_2\tilde{s}_1 + \tilde{s}_5\tilde{s}_4\tilde{s}_3\tilde{s}_1 + \tilde{s}_5\tilde{s}_4\tilde{s}_3\tilde{s}_2 + \tilde{s}_4\tilde{s}_3(-s_1) + \tilde{s}_5\tilde{s}_3(-s_1)$$
$$+ \tilde{s}_5\tilde{s}_4(-s_1) + \tilde{s}_4\tilde{s}_1(-s_2) + \tilde{s}_5\tilde{s}_1(-s_2) + \tilde{s}_5\tilde{s}_4(-s_2) + \tilde{s}_2\tilde{s}_1(-s_3) + \tilde{s}_5\tilde{s}_1(-s_3) + \tilde{s}_5\tilde{s}_2(-s_3)$$
$$+ \tilde{s}_2\tilde{s}_1(-s_4) + \tilde{s}_3\tilde{s}_1(-s_4) + \tilde{s}_3\tilde{s}_2(-s_4) + (-s_3)(-s_1) + (-s_4)(-s_1) + (-s_4)(-s_2)$$

mixture of $g = 1$ (fermion) and $g = 2$ exclusion



Z_4 for $q = 5$: all possible occupancies of 5 levels by 4 particles with either **fermions** or **two-fermion bound states**

Honeycomb lattice walks: (1,2)-exclusion

$\det(I - zH)$ ⇒ Z_n ⇒ b_n ⇒ $\mathrm{tr}\, H^{\mathbf{n}}$

$$\mathrm{tr}\, H_{1,2}^{\mathbf{n}} = \mathbf{n} \sum_{\substack{\tilde l_1,\dots,\tilde l_{j+1}; l_1,\dots,l_j \\ (1,2)-\text{composition of } \mathbf{n}}} c_{1,2}(\tilde l_1,\dots,\tilde l_{j+1}; l_1,\dots,l_j) \sum_{k=1}^{q-j} \tilde s_k^{\tilde l_1} s_k^{l_1} \tilde s_{k+1}^{\tilde l_2} s_{k+1}^{l_2}\cdots$$

(1,2)-composition of $\mathbf{n}$ if

$$\mathbf{n} = (\tilde l_1 + \cdots + \tilde l_{j+1}) + 2(l_1 + \cdots + l_j),\ \tilde l_i \geq 0,\ l_i > 0$$

combinatorial factor

$$c_{1,2}(\tilde l_1,\dots,\tilde l_{j+1}; l_1,\dots,l_j) = \frac{(\tilde l_1 + l_1 - 1)!}{\tilde l_1!\, l_1!} \prod_{k=2}^{j+1} \binom{l_{k-1} + \tilde l_k + l_k - 1}{l_{k-1} - 1,\ \tilde l_k,\ l_k}$$

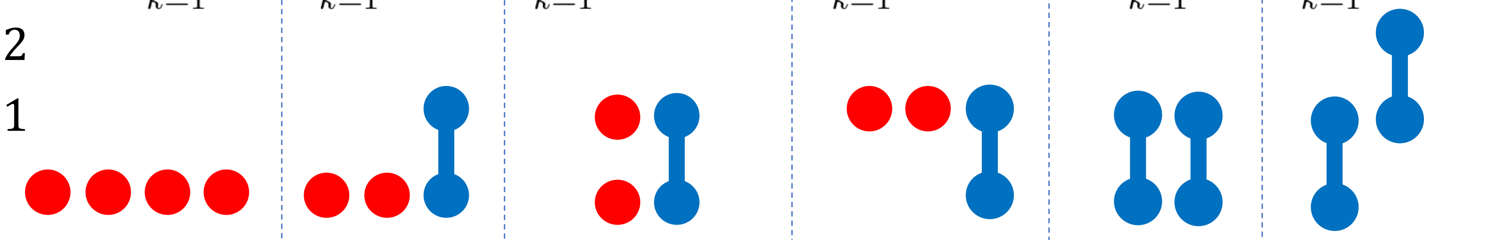
combinatorial interpretation from cluster coefficients

$$-b_4 = \frac{1}{4}\sum_{k=1}^{q} \tilde s_k^4 + \sum_{k=1}^{q-1} \tilde s_k^2 s_k + \sum_{k=1}^{q-1} \tilde s_k s_k \tilde s_{k+1} + \sum_{k=1}^{q-1} s_k \tilde s_{k+1}^2 + \frac{1}{2}\sum_{k=1}^{q-1} s_k^2 + \sum_{k=1}^{q-2} s_k s_{k+1}$$

$k+2$

$k+1$

k



e.g. (1,2)-composition of 4: (4,0; 0) (2,0; 1) (1,1; 1) (0,2; 1) (0,0; 2) (0,0,0; 1,1)

Honeycomb lattice walks: (1,2)-exclusion

$$\mathrm{tr}\, H_{1,2}^{\mathbf{n}} = \mathbf{n} \sum_{\substack{\tilde{l}_1,\ldots,\tilde{l}_{j+1};l_1,\ldots,l_j \\ (1,2)-\text{composition of } \mathbf{n}}} c_{1,2}(\tilde{l}_1,\ldots,\tilde{l}_{j+1};l_1,\ldots,l_j) \sum_{k=1}^{q-j} \tilde{s}_k^{\tilde{l}_1} s_k^{l_1} \tilde{s}_{k+1}^{\tilde{l}_2} s_{k+1}^{l_2} \cdots$$

$$s_k = 4\sin^2(k\pi p/q)\ , \quad \tilde{s}_k = 1 + s_k$$

$$C_{\mathbf{n}}(A) = n \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{composition of } n'=0,1,2,\ldots,n \\ j\le\min(n',n-n'+1)}} c_n(l_1,l_2,\ldots,l_j) \sum_{k_3=-l_3}^{l_3} \sum_{k_4=-l_4}^{l_4} \cdots \sum_{k_j=-l_j}^{l_j} \binom{2l_1}{l_1 + A + \sum_{i=3}^{j}(i-2)k_i} \binom{2l_2}{l_2 - A - \sum_{i=3}^{j}(i-1)k_i} \prod_{i=3}^{j} \binom{2l_i}{l_i + k_i}$$

$$c_n(l_1,l_2,\ldots,l_j) = \frac{1}{l_1 l_2 \ldots l_j} \sum_{m_1=0}^{\min(l_1,l_2)} \sum_{m_2=0}^{\min(l_2,l_3)} \cdots \sum_{m_{j-1}=0}^{\min(l_{j-1},l_j)} \left(\prod_{i=1}^{j-1} m_i \binom{l_i}{m_i} \binom{l_{i+1}}{m_i} \right) \binom{n + \sum_{i=1}^{j} l_i - \sum_{i=1}^{j-1} m_i - 1}{2\sum_{i=1}^{j} l_i - 1}$$

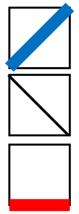
[**LG**, Ouvry, Polychronakos 2022]

(1,2)-exclusion and Motzkin path

$$H_{1,2} = \begin{pmatrix} \tilde{s}_1 & f_1 & 0 & \cdots & 0 & 0 \\ g_1 & \tilde{s}_2 & f_2 & \cdots & 0 & 0 \\ 0 & g_2 & \tilde{s}_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \tilde{s}_{q-1} & f_{q-1} \\ 0 & 0 & 0 & \cdots & g_{q-1} & \tilde{s}_q \end{pmatrix}$$

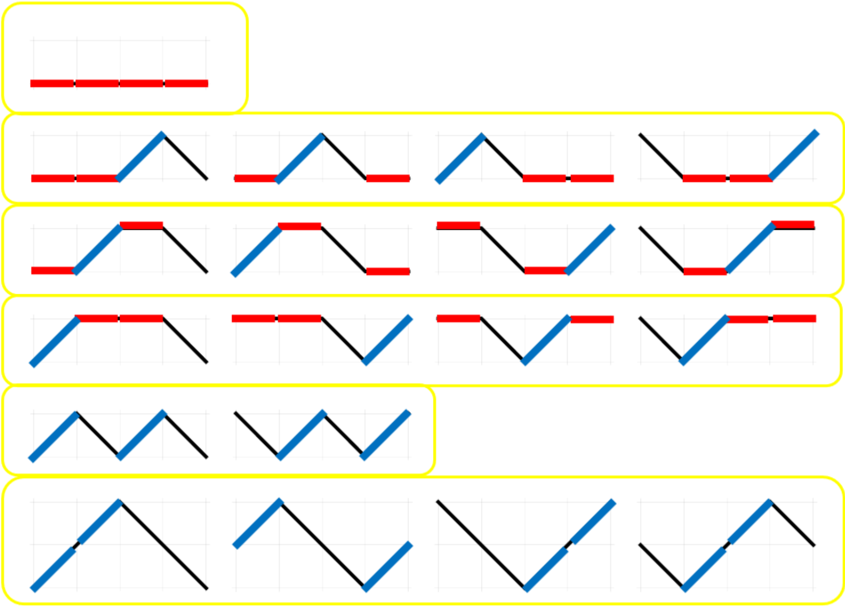
$$\mathrm{tr}\, H_{1,2}^{\mathbf{n}} = \sum_{k_1=1}^{q} \sum_{k_2=1}^{q} \cdots \sum_{k_{\mathbf{n}}=1}^{q} h_{k_1 k_2} h_{k_2 k_3} \cdots h_{k_{\mathbf{n}} k_1}$$

$$k_{i+1} - k_i = \begin{cases} -1 & \rightarrow \text{up step (going up 1 floor)} \\ +1 & \rightarrow \text{down step (going down 1 floor)} \\ 0 & \rightarrow \text{horizontal step} \end{cases}$$



T. S. Motzkin
Israeli-American mathematician

length $\mathbf{n} = 4$



$(\tilde{l}_1, \tilde{l}_2, \ldots, \tilde{l}_{j+1}; l_1, l_2, \ldots, l_j)$	$c_{1,2}$
$(4, 0; 0)$	$1/4$
$(2, 0; 1)$	1
$(1, 1; 1)$	1
$(0, 2; 1)$	1
$(0, 0; 2)$	$1/2$
$(0,0,0; 1,1)$	1

$$c_{1,2}(\tilde{l}_1, \ldots, \tilde{l}_{j+1}; l_1, \ldots, l_j) = \frac{(\tilde{l}_1 + l_1 - 1)!}{\tilde{l}_1!\, l_1!} \prod_{k=2}^{j+1} \binom{l_{k-1} + \tilde{l}_k + l_k - 1}{l_{k-1} - 1,\ \tilde{l}_k,\ l_k}$$

- $l_i\, c_{1,2}$ is the number of such Motzkin paths **starting from the *i*-th floor with an up step**
- $(l_1 + l_2 + \cdots + l_j)\, c_{1,2}$ is the total number of such Motzkin paths **starting with an up step**
- $\mathbf{n}\, c_{1,2}$ is the total number of such Motzkin paths

→ generalization: $(1, g)$-exclusion, $(g_1, g_2, \ldots)$-exclusion

(1,g)-exclusion

$$\operatorname{tr} H_{1,g}^{\mathbf{n}} = \mathbf{n} \sum_{\substack{\tilde{l}_1,\ldots,\tilde{l}_{j+g-1};l_1,\ldots,l_j \\ (1,g)\text{-composition of } \mathbf{n}}} c_{1,g}(\tilde{l}_1, \ldots, \tilde{l}_{j+g-1}; l_1, \ldots, l_j) \sum_{k=1}^{q-j-g+2} \tilde{s}_k^{\tilde{l}_1} s_k^{l_1} \tilde{s}_{k+1}^{\tilde{l}_2} s_{k+1}^{l_2} \cdots$$

We define the sequence of integers $\tilde{l}_1, \tilde{l}_2, \ldots, \tilde{l}_{j+g-1}; l_1, l_2, \ldots, l_j$, $j \geq 1$, as a $1, g$-composition of $\mathbf{n}$ if they satisfy the conditions

$$\mathbf{n} = (\tilde{l}_1 + \tilde{l}_2 + \cdots + \tilde{l}_{j+g-1}) + g(l_1 + l_2 + \cdots + l_j)$$
$$\tilde{l}_i \geq 0 \ ; \ l_i \geq 0, \ l_1, l_j > 0, \ \text{at most } g-2 \text{ successive vanishing } l_i$$

That is, the l_j's are the usual g-compositions of integers $1, 2, \ldots, \lfloor \mathbf{n}/g \rfloor$ and the $\tilde{l}_i$'s are additional nonnegative integers (we also include the trivial composition $\tilde{l}_1 = \mathbf{n}$.) For example, there are seven $(1,3)$ compositions of 5

- $j = 0$: (5); $j = 1$: $(2,0,0;1)$, $(1,1,0;1)$, $(1,0,1;1)$, $(0,2,0;1)$, $(0,1,1;1)$, $(0,0,2;1)$

and five $(1,4)$ compositions of 5

- $j = 0$: (5); $j = 1$: $(1,0,0,0;1)$, $(0,1,0,0;1)$, $(0,0,1,0;1)$, $(0,0,0,1;1)$

$$c_{1,g}(\tilde{l}_1, \tilde{l}_2, \ldots, \tilde{l}_{j+g-1}; l_1, l_2, \ldots, l_j) = \frac{(\tilde{l}_1 + l_1 - 1)!}{\tilde{l}_1! \, l_1!} \prod_{k=2}^{j+g-1} \binom{\tilde{l}_k + \sum_{i=k-g+1}^{k} l_i - 1}{\sum_{i=k-g+1}^{k-1} l_i - 1, \ \tilde{l}_k, \ l_k}$$

(1,g)-exclusion

The number of $(1,g)$-compositions of a given integer $\mathbf{n}$ is

$$N_{1,g}(\mathbf{n}) = 1 + \sum_{k=0}^{\lfloor \mathbf{n}/g \rfloor - 1} \sum_{m=0}^{(g-1)k} \binom{k}{m}_g \binom{\mathbf{n} + m - gk - 1}{m + g - 1},$$

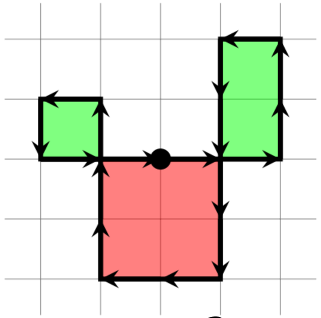
where the g-nomial coefficient is defined as

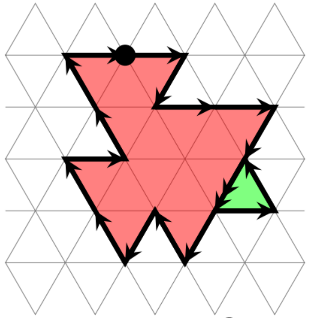
$$\binom{k}{m}_g = [x^m](1 + x + x^2 + \cdots + x^{g-1})^k = \sum_{j=0}^{\lfloor m/g \rfloor} (-1)^j \binom{k}{j} \binom{k + m - gj - 1}{k - 1}.$$

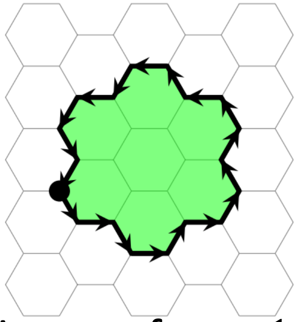
Equivalently, the generating function of the $N_{1,g}(\mathbf{n})$'s is

$$\sum_{\mathbf{n}=0}^{\infty} x^n N_{1,g}(\mathbf{n}) = \frac{(1-x)^{g-2}(1+x^{g-1}-x^g) - x^{g-1}}{(1-x)^{g-1}(1+x^{g-1}-x^g) - x^{g-1}}.$$

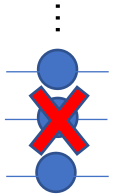
Take-home message


$g = 2$

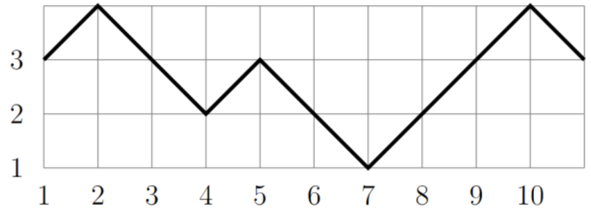

$g = 3$


mixture of $g = 1$ and $g = 2$

algebraic area enumeration of lattice random walks


quantum exclusion statistics

combinatorics of Dyck/Motzkin paths



Future plans

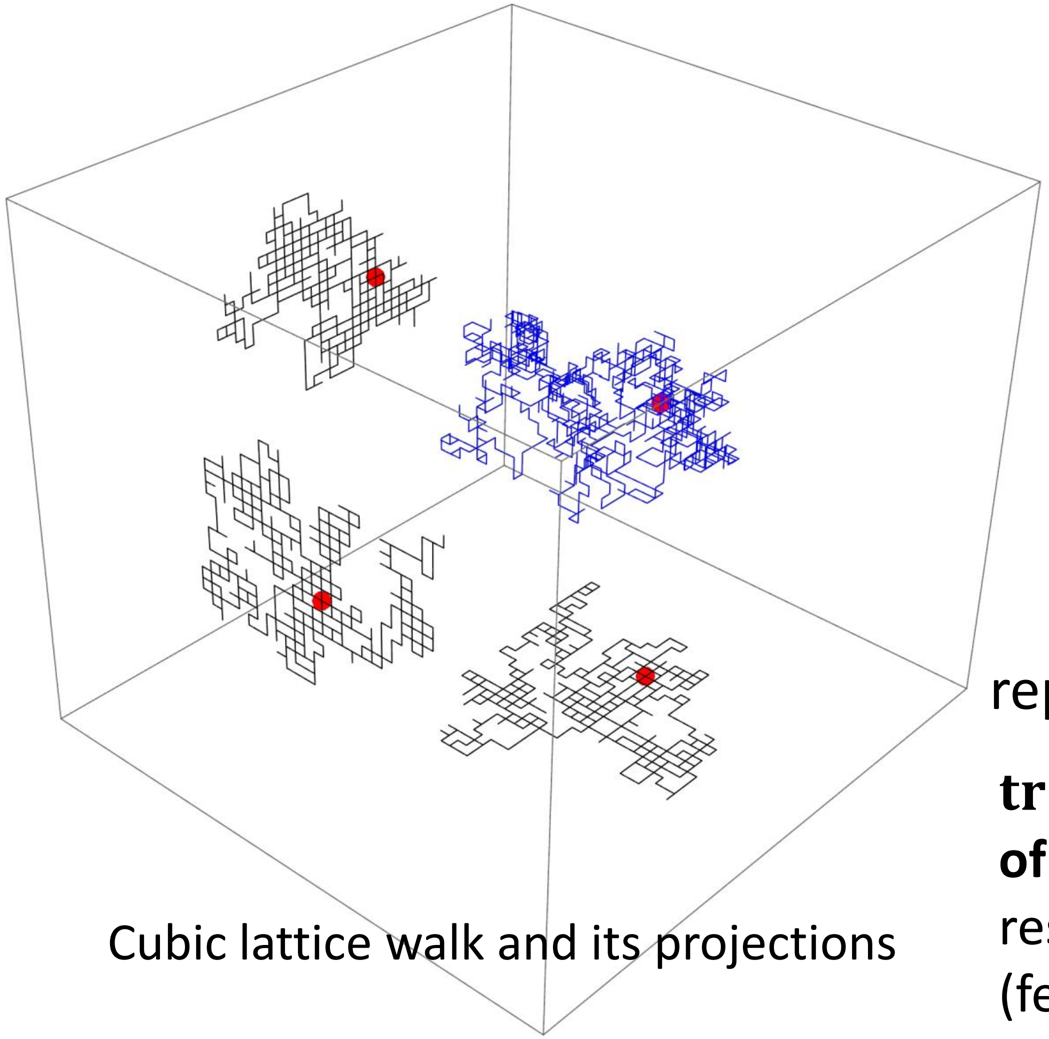
- Higher dimensional walks?

  3D cubic lattice walks: mixture of $g = 1$, $g = 1$, and $g = 2$ exclusion [**LG** 2023]

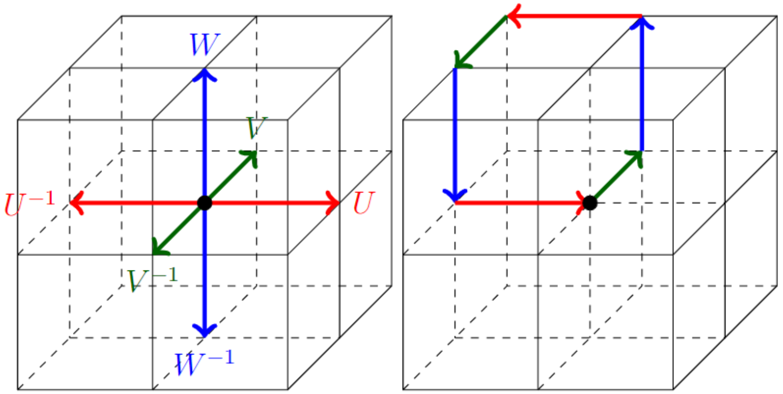
  arbitrary dimension (ongoing work)

Cubic lattice walks: (1,1,2)-exclusion with constraints [LG 2023]

algebraic area of 3D walks: sum of algebraic areas obtained from the walk's projection onto the three Cartesian planes



Cubic lattice walk and its projections



$$UW^{-1}V^{-1}U^{-1}WV = \mathrm{Q}$$

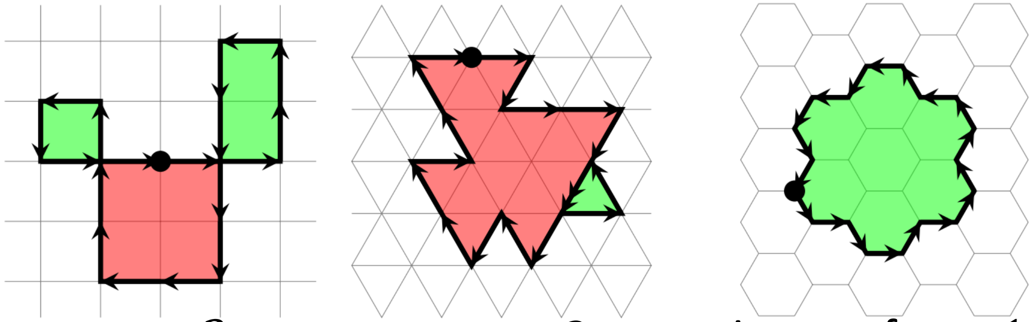
Hamiltonian $H = U + V + W + U^{-1} + V^{-1} + W^{-1}$

algebra $V\,U = \mathrm{Q}\,U\,V,\ W\,V = \mathrm{Q}\,V\,W,\ U\,W = \mathrm{Q}\,W\,U$

representation $U = u \otimes I,\ \ V = v \otimes I,\ \ W = (v^{-1}u^{-1}) \otimes u$

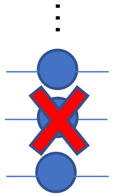
tr $H^{\mathbf{n}}$ can be mapped onto the cluster coefficients of **three types of particles** that obey exclusion statistics with g=1, g=1, and g=2, respectively, subject to the constraint that the numbers of g=1 (fermions) exclusion particles of two types are equal.

Take-home message



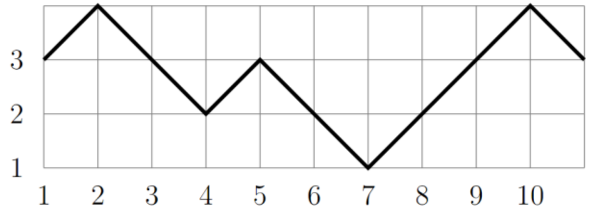
$g = 2$ $g = 3$ mixture of $g = 1$ and $g = 2$

algebraic area enumeration of lattice random walks



quantum exclusion statistics

combinatorics of Dyck/Motzkin paths



Future plans

- Higher dimensional walks?
  3D cubic lattice walks: mixture of $g = 1$, $g = 1$, and $g = 2$ exclusion [LG 2023]
  arbitrary dimension (ongoing work)
- Connection to exactly solvable models?
  e.g. open Ising spin-1/2 chain: ***free-fermionic*** spectrum $\pm\epsilon_1 \pm \epsilon_2 \pm \cdots$ with ϵ_k obtained from $g = 2$ exclusion matrix H_2 [Baxter 1989], closed chain (in preparation), SU(N) or mixed spin chain (ongoing work)
- Other applications? Polymer physics, particle physics, quantum information, etc.

Thank You! One more thing: seek a postdoc position starting from Jan. 2024 :)