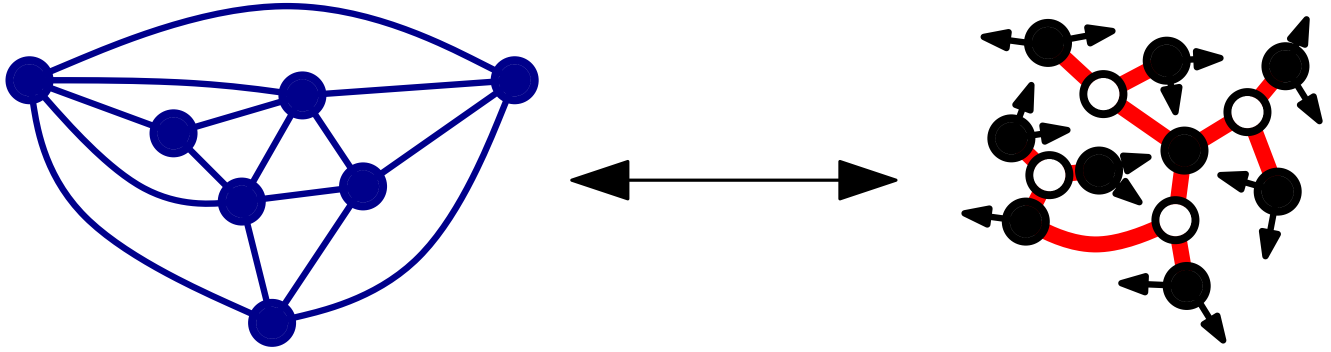
Bijections for planar maps with boundaries

Éric Fusy (CNRS/LIX)

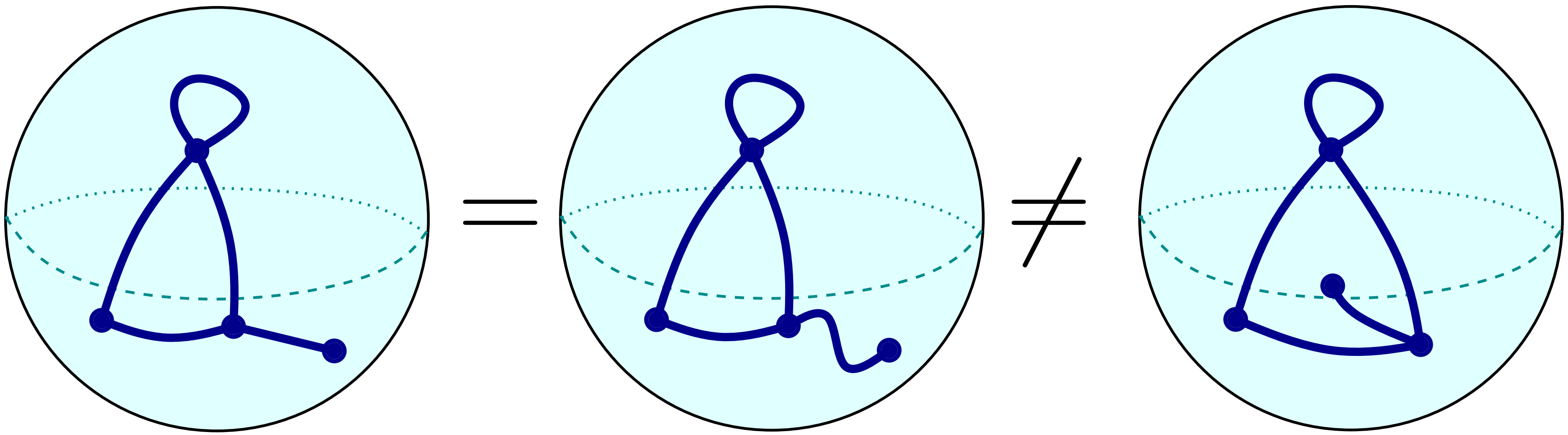
Joint work with Olivier Bernardi (Brandeis Univ.)



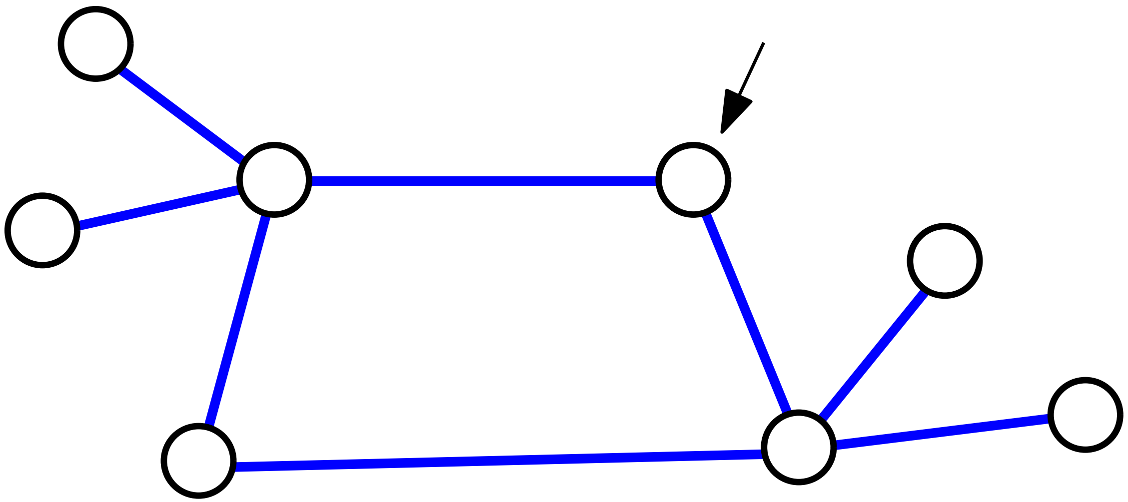
séminaire CALIN, LIPN

Planar maps

- **Planar map**= connected graph embedded on the **sphere**, considered up to continuous deformation



- **Rooted map**= map with a marked corner



A rooted map

Counting formulas for rooted maps

- **Beautiful counting formulas** discovered by Tutte

Maps with n edges

$$\frac{2 \cdot 3^n \cdot (2n)!}{n!(n+2)!}$$

Bipartite maps with n edges

$$\frac{3 \cdot 2^{n-1} \cdot (2n)!}{n!(n+2)!}$$

2-connected maps with n edges

$$\frac{4 \cdot (3n-3)!}{(n-1)!(2n)!}$$

Counting formulas for rooted maps

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- **Tutte's slicings formula (1962)**:

Let $B[n_1, n_2, \ldots, n_k]$ be the number of rooted bipartite maps with n_i faces of degree $2i$ for $i \in [1..k]$. Then

$$B[n_1, \ldots, n_k] = 2\frac{e!}{v!}\prod_{i=1}^{k}\frac{1}{n_i!}\binom{2i-1}{i-1}^{n_i}$$

where $e = \#\text{edges} = \sum_i i n_i$ and $v = \#\text{vertices} = e - k + 2$

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Maps with n edges	Bipartite maps with n edges	2-connected maps with n edges
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- **Tutte's slicings formula (1962)**:

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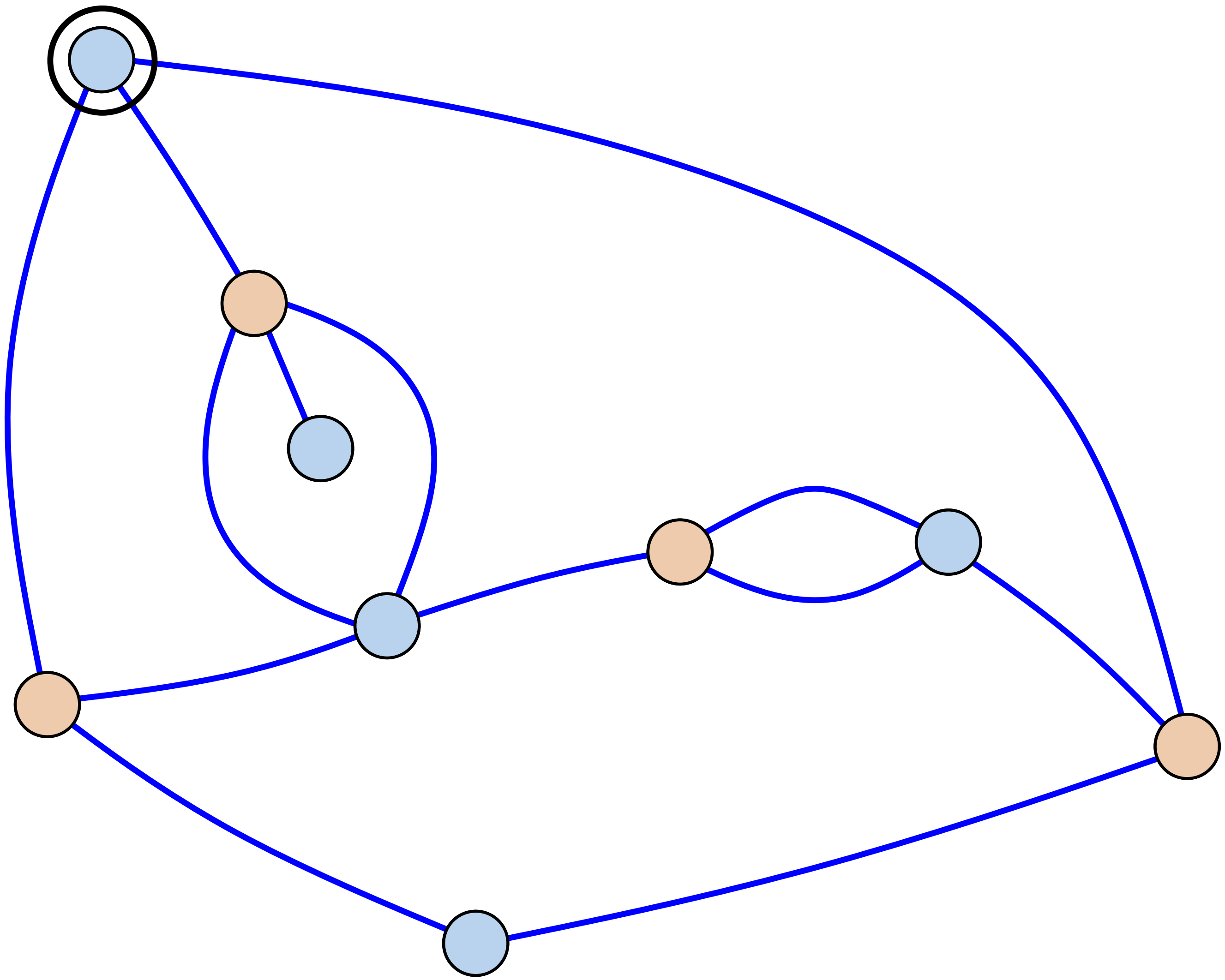
Counting methods: recursive method, matrix integrals, bijections

The BDG bijection for pointed bipartite maps

[Bouttier, Di Francesco, Guitter'04]



The BDG bijection for pointed bipartite maps

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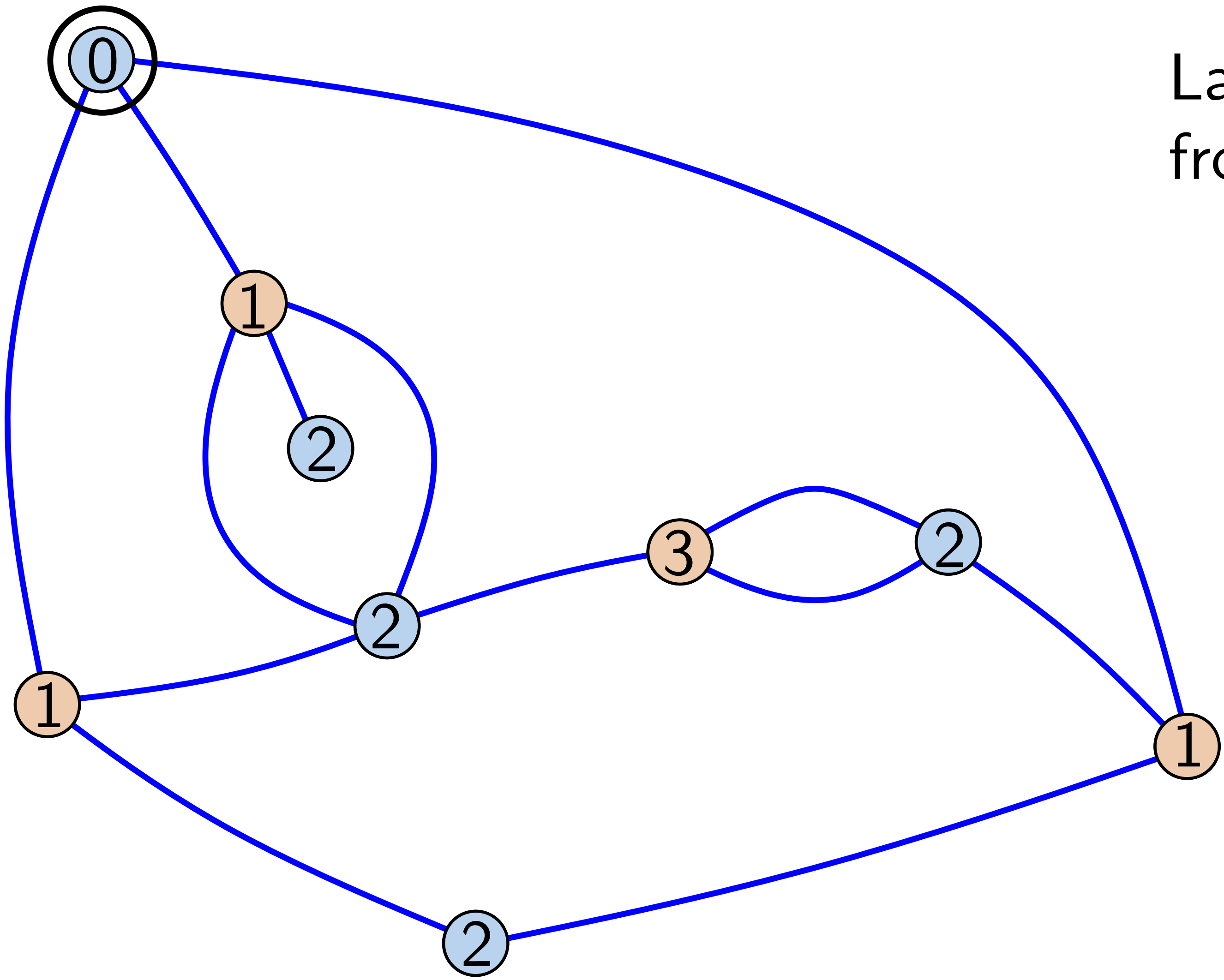

Label vertices by distance from the marked vertex

The BDG bijection for pointed bipartite maps

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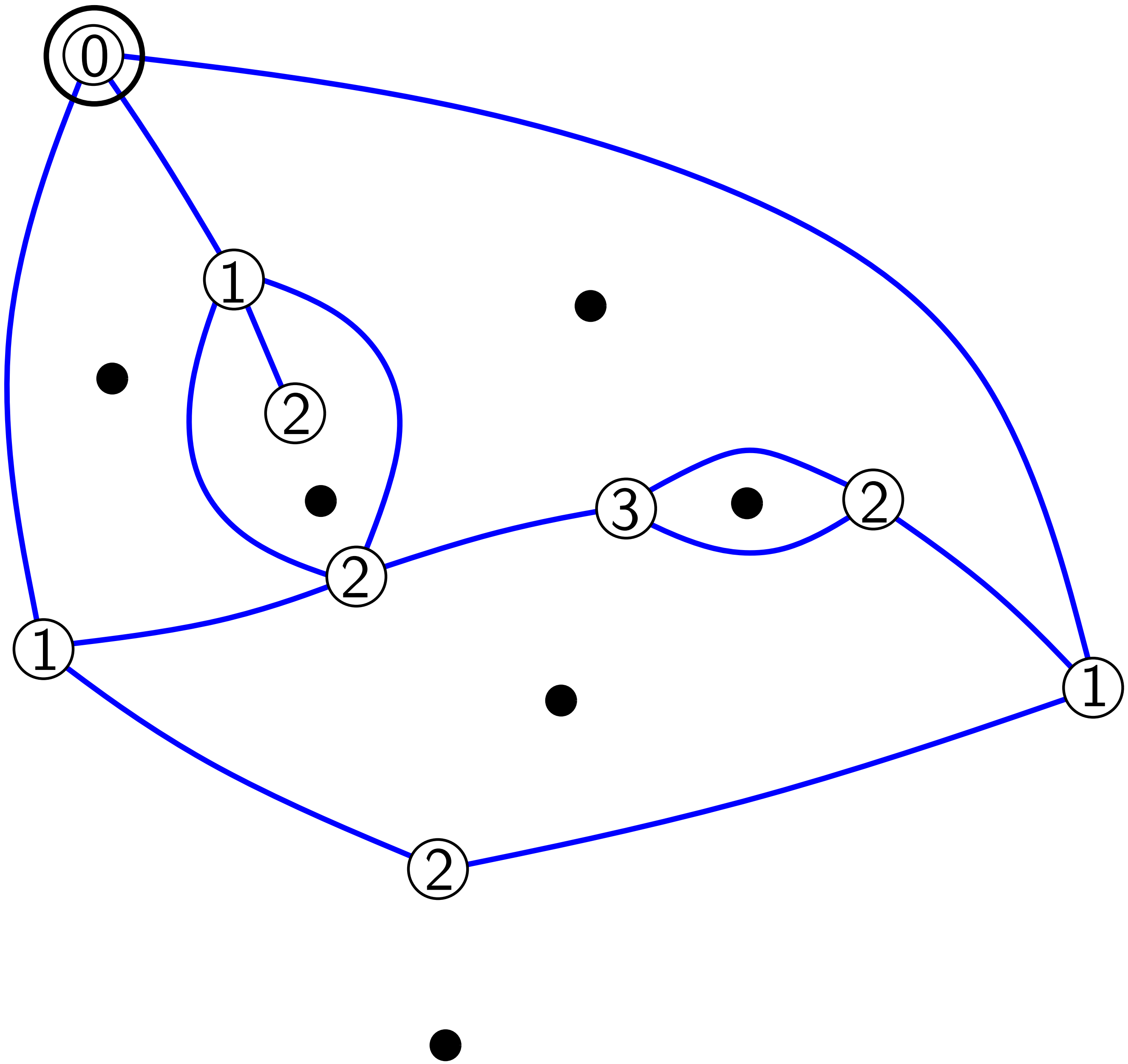

Construction of a labeled mobile

(i) Add a black vertex in each face

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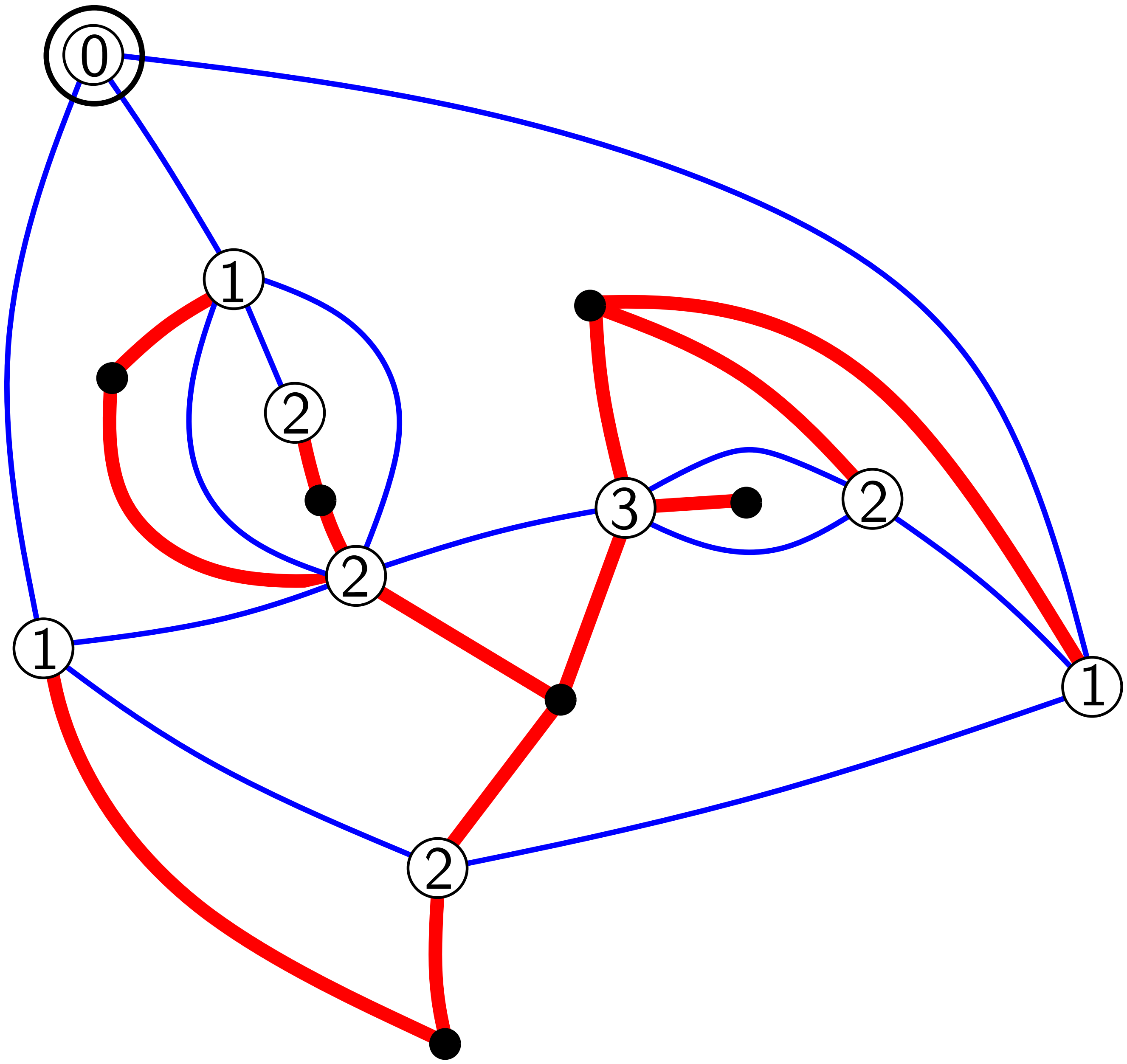
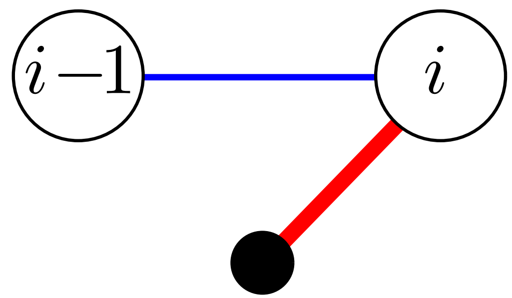
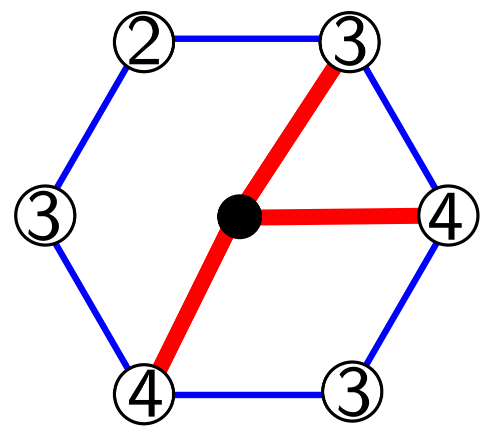


Construction of a labeled mobile

(i) Add a black vertex in each face

(ii) Each map-edge gives a mobile-edge using the local rule





The BDG bijection for pointed bipartite maps

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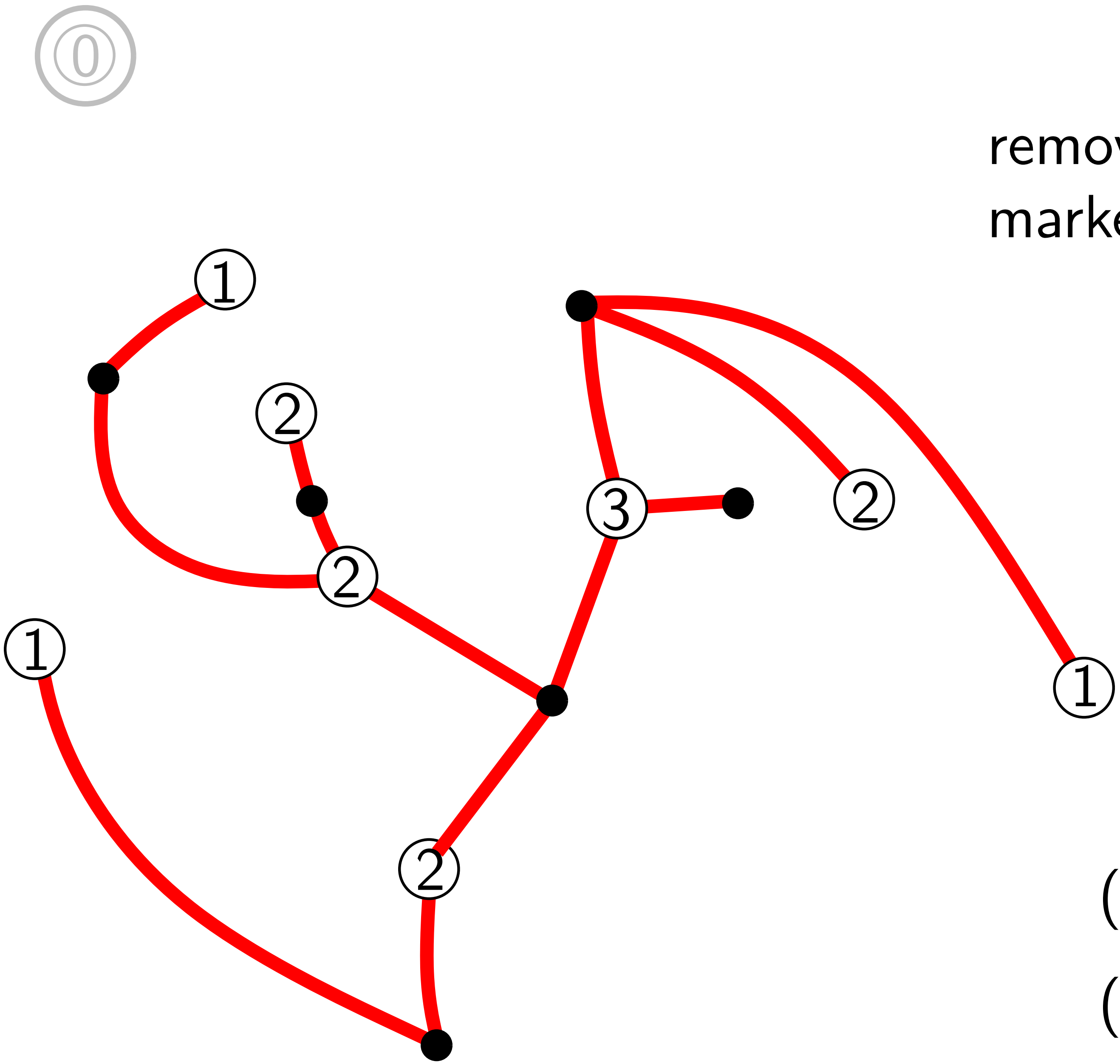
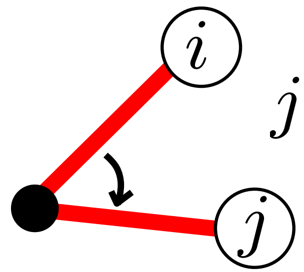


remove the map-edges and the marked vertex ⓪

Conditions:

(i) $\exists$ vertex of label 1

(ii)



$j \leq i+1$

The BDG bijection for pointed bipartite maps

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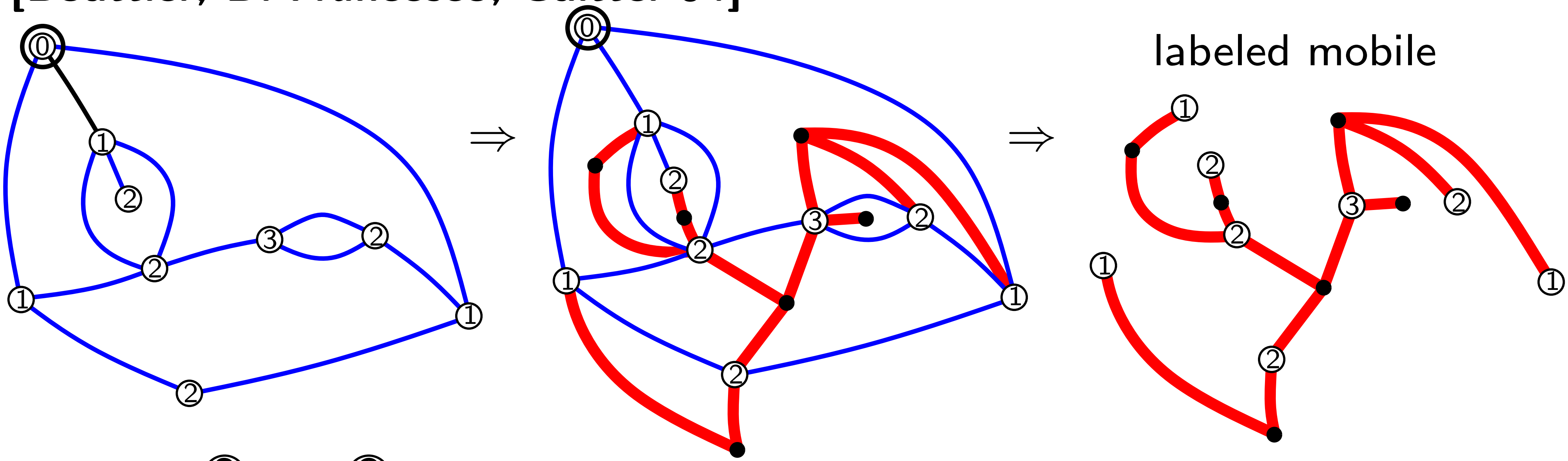

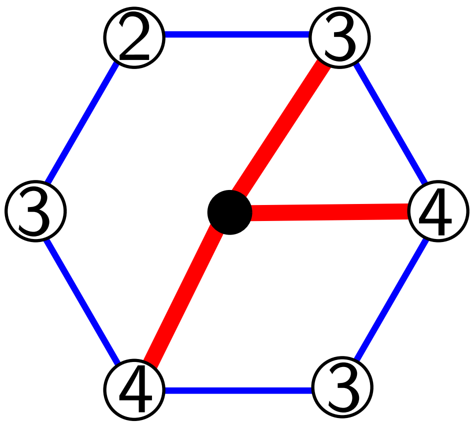

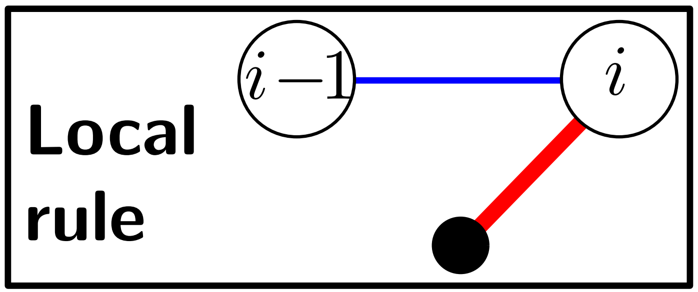

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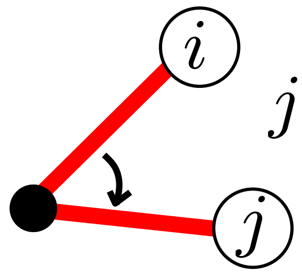
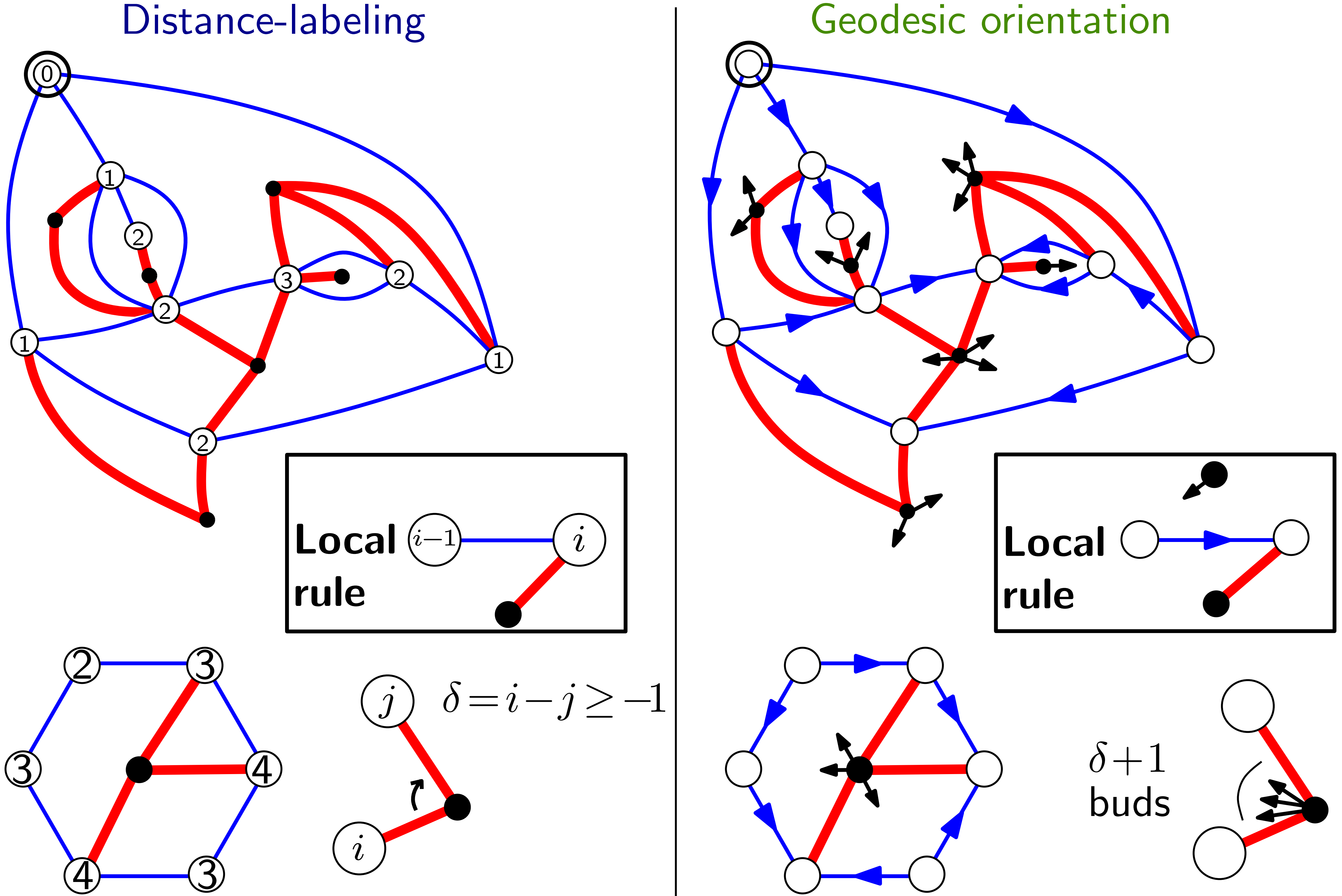


$j \leq i+1$

Theorem: The mapping is a **bijection**.

face of degree $2i$ ⟷ black vertex of degree i

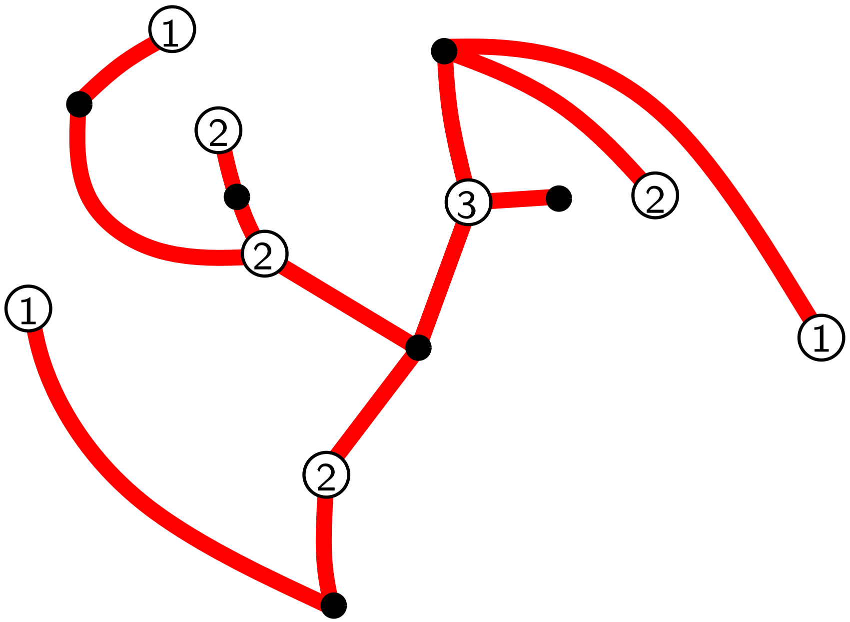
Reformulation with orientations



Mobile conditions in the two formulations

Formulation with labels

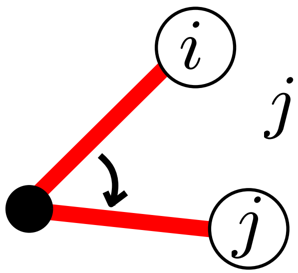
gives a labeled mobile



with the conditions:

(i) $\exists$ node of label 1

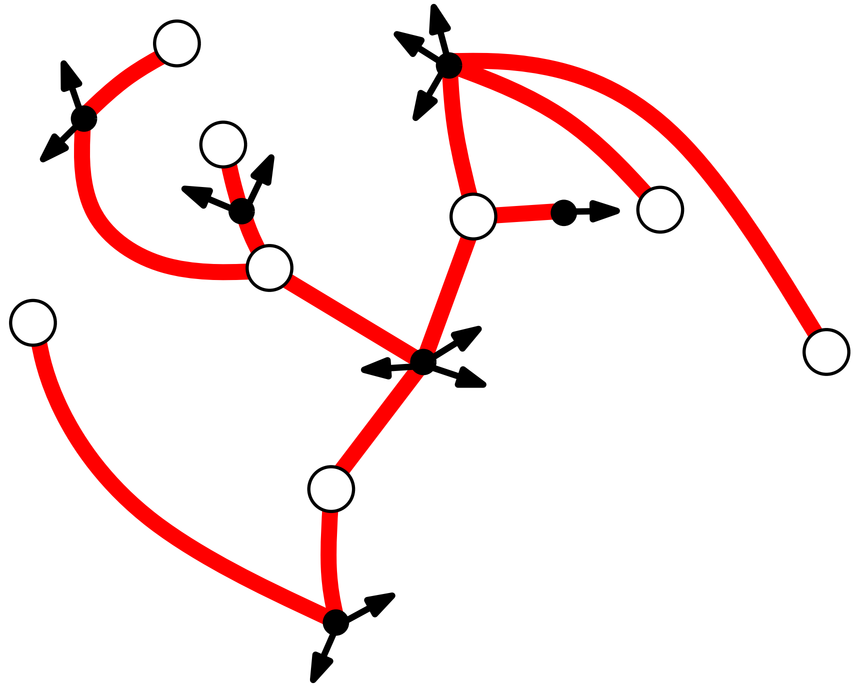
(ii)



$j \leq i+1$

Formulation with orientations

gives a "blossoming" mobile



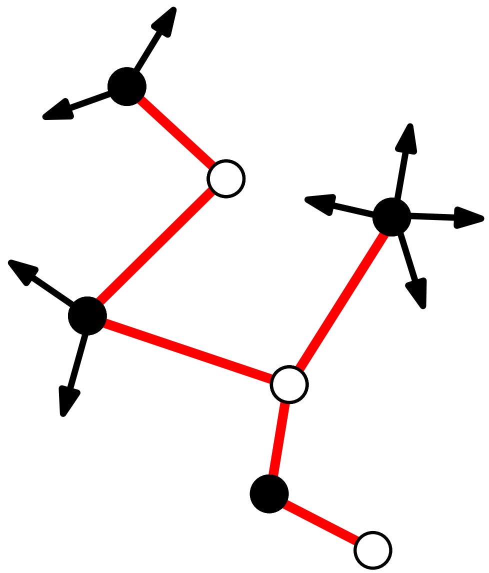
with the condition:

each black vertex has as many buds as neighbors

Definition of blossoming mobiles

- **Blossoming mobile**= bipartite tree (black/white vertices) where each corner at a black vertex carries $i \geq 0$ buds

excess = number of edges - number of buds



a blossoming mobile of excess -2

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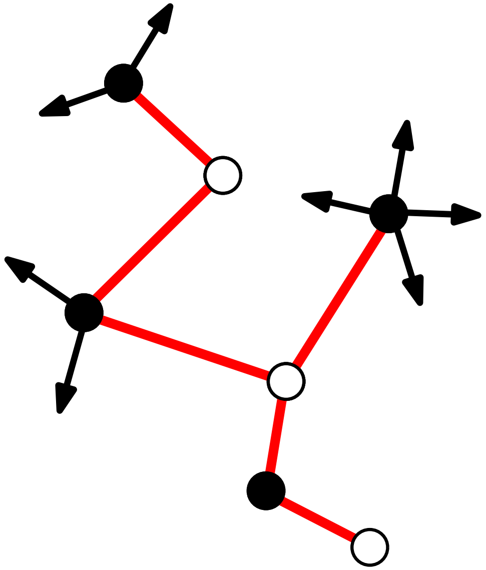
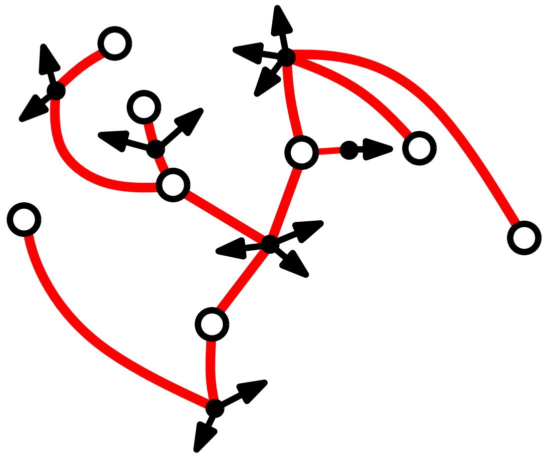


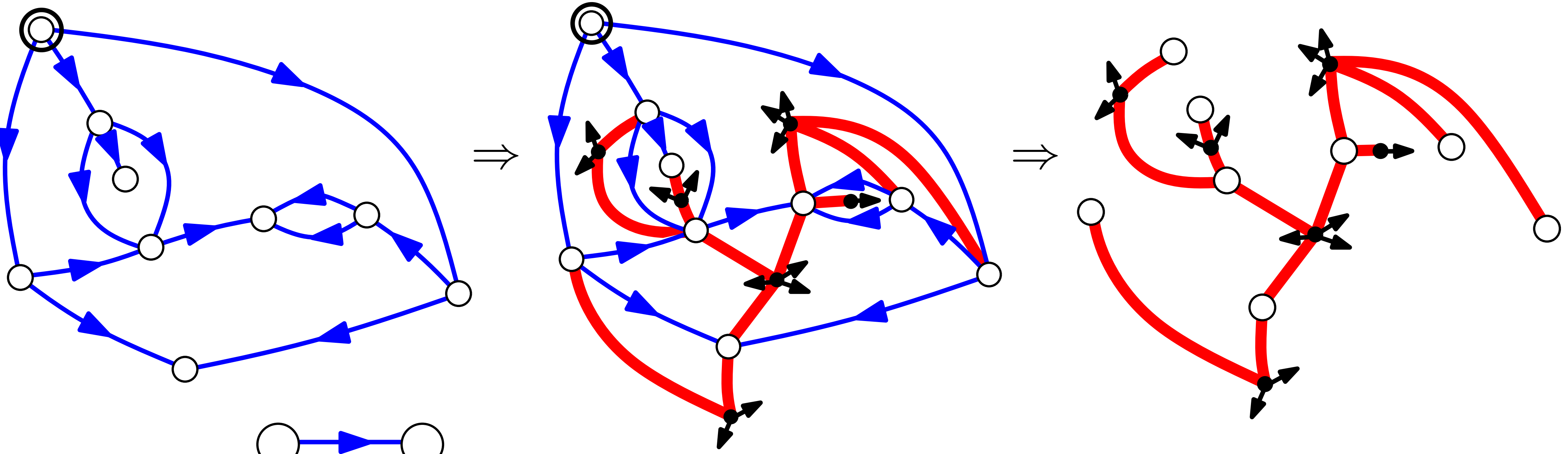
a blossoming mobile of excess -2

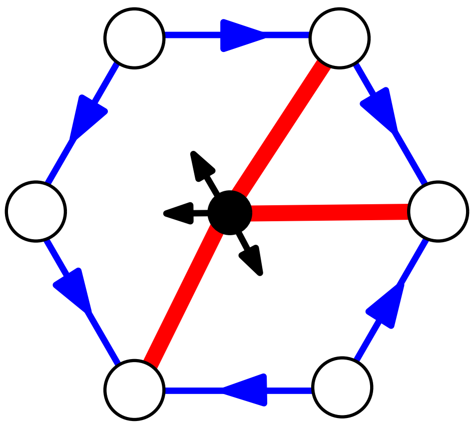
- A blossoming mobile is called **balanced** iff each black vertex has as many buds as neighbors

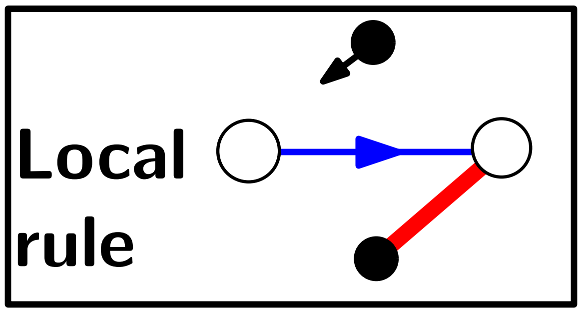
Rk: implies that the excess is 0



Summary of the reformulation







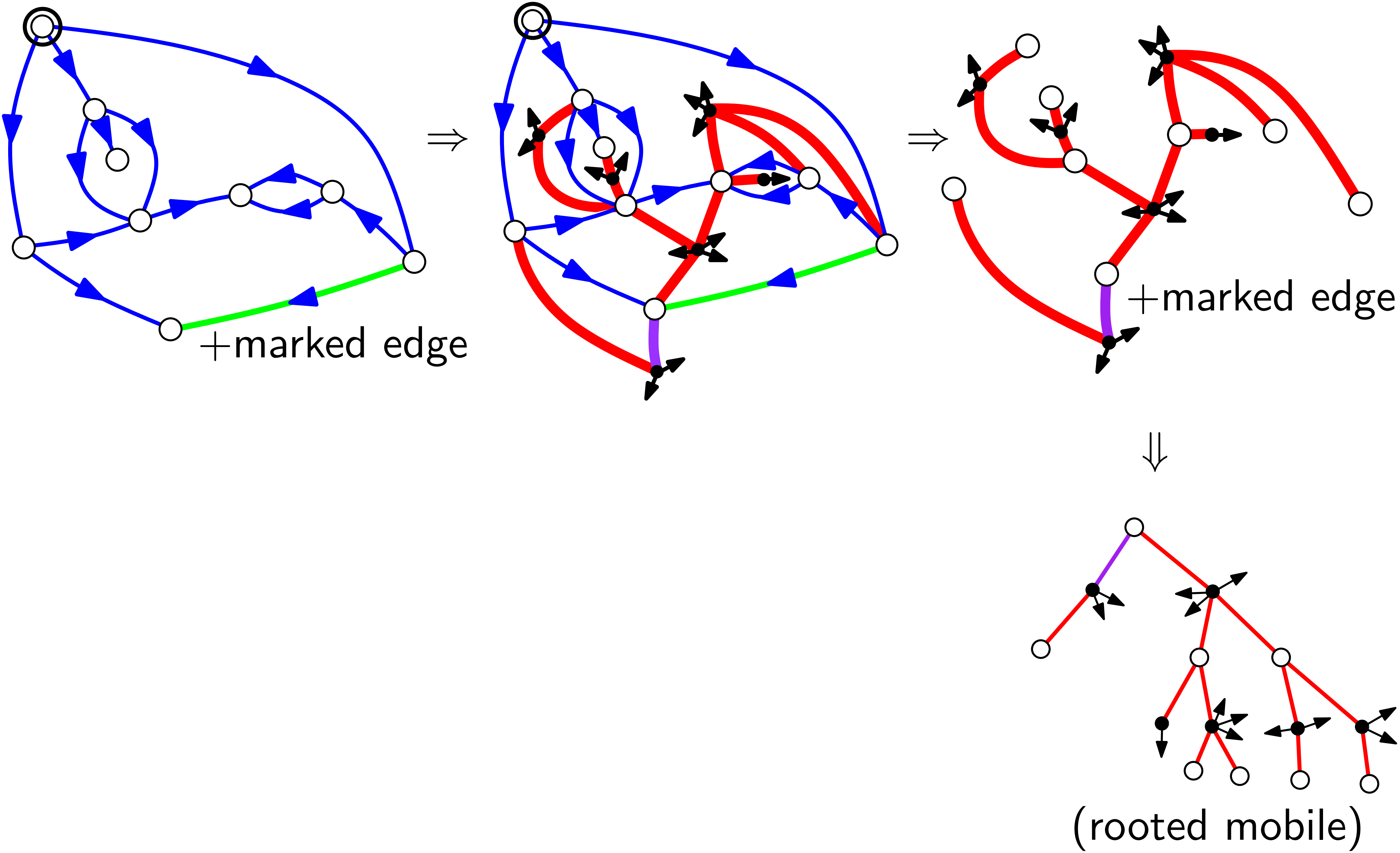
Condition:

Each black vertex has as many buds as neighbors

Theoreme: The mapping is a bijection between pointed bipartite maps and balanced blossoming mobiles

face of degree $2i$ $\longleftrightarrow$ black vertex of degree $2i$

Proof of Tutte's slicings formula



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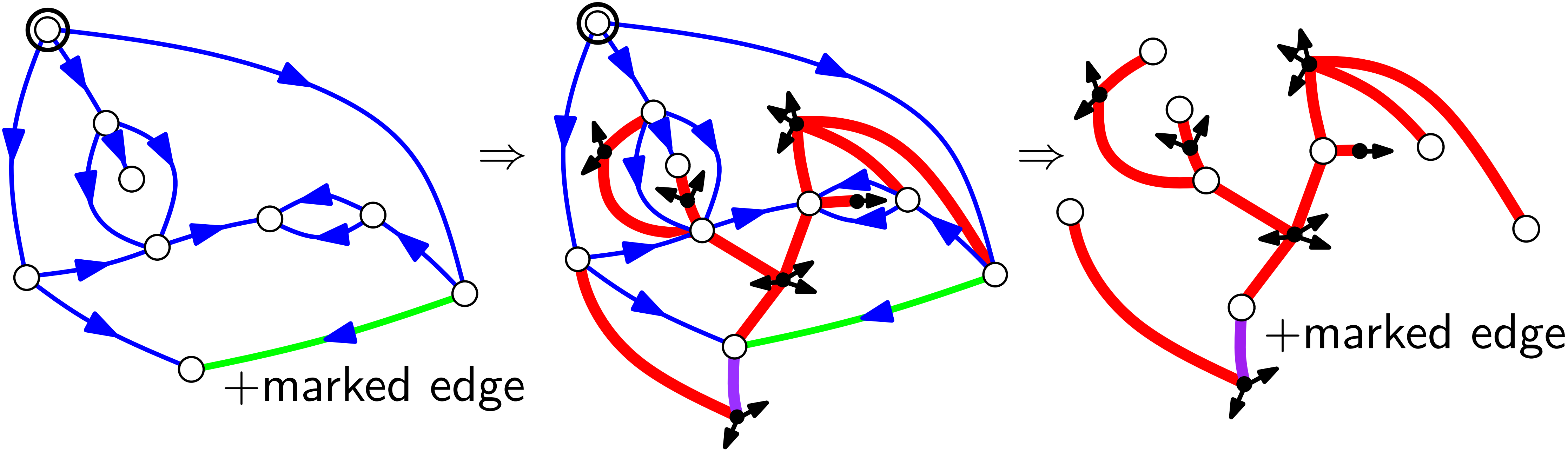


Let $B[n_1, n_2, \ldots, n_k]$ be the number of rooted bipartite maps with n_i faces of degree $2i$ for $i \in [1..k]$

- Bijection gives

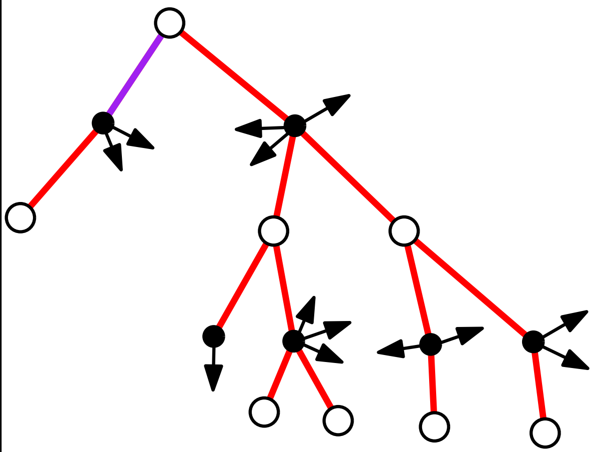
$$v \cdot B[n_1, \ldots, n_k] = 2 \cdot \text{ coeff } t_1^{n_1} \cdots t_k^{n_k} \text{ in } R(t_1, t_2, \ldots)$$

where $R \equiv R(t_1, t_2, \ldots)$ is the GF of rooted mobiles given by the equation $R = 1 + \sum_{i \geq 1} \binom{2i-1}{i-1} t_i R^i$

- Lagrange inversion formula gives:

$$[t_1^{n_1} \cdots t_k^{n_k}] R = \frac{e!}{(v-1)!} \prod_{i=1}^{k} \frac{1}{n_i!} \binom{2i-1}{i-1}^{n_i}$$

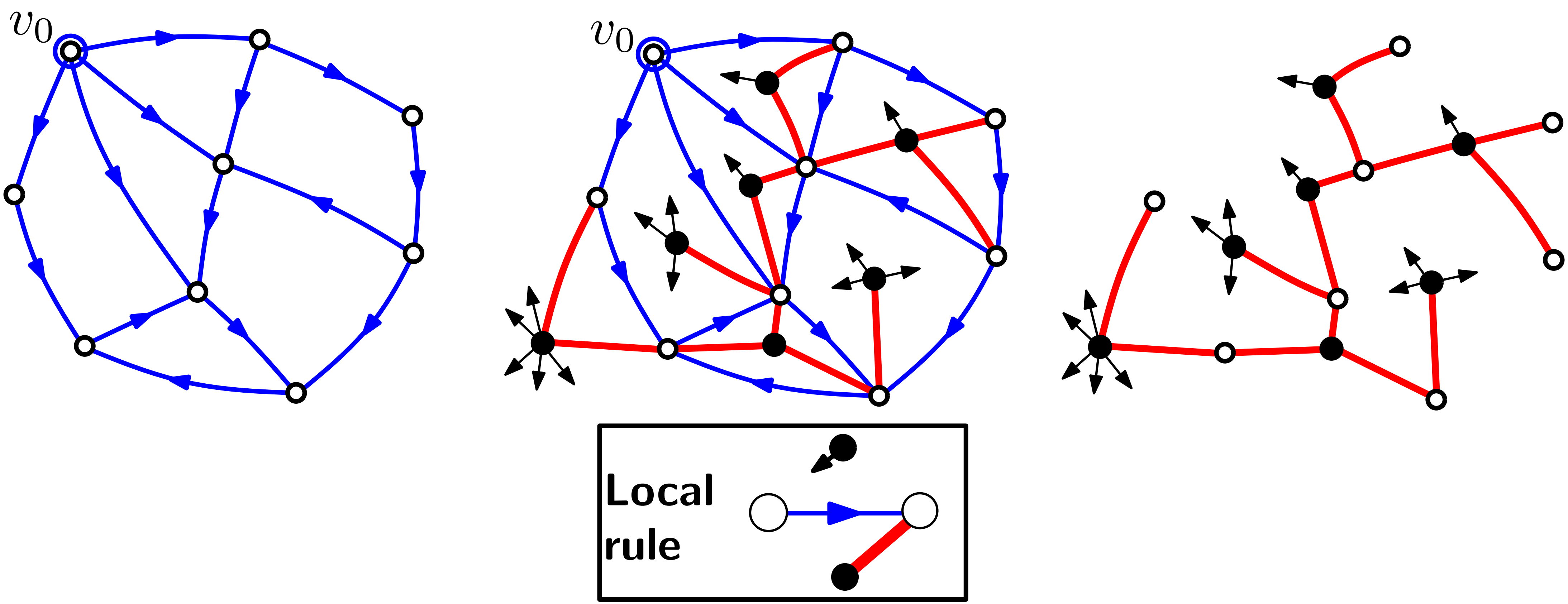
⇓



(rooted mobile)

Extension for pointed orientations with no ccw cycle

- More generally, we **obtain a blossoming mobile** (of excess 0) if we start from a vertex-pointed orientation such that :
  - the marked vertex v_0 is a **"source"** (no incoming edge)
  - every vertex is **accessible** from v_0 by a directed path
  - **there is no ccw cycle** (with $v_0 \in$ outer face)



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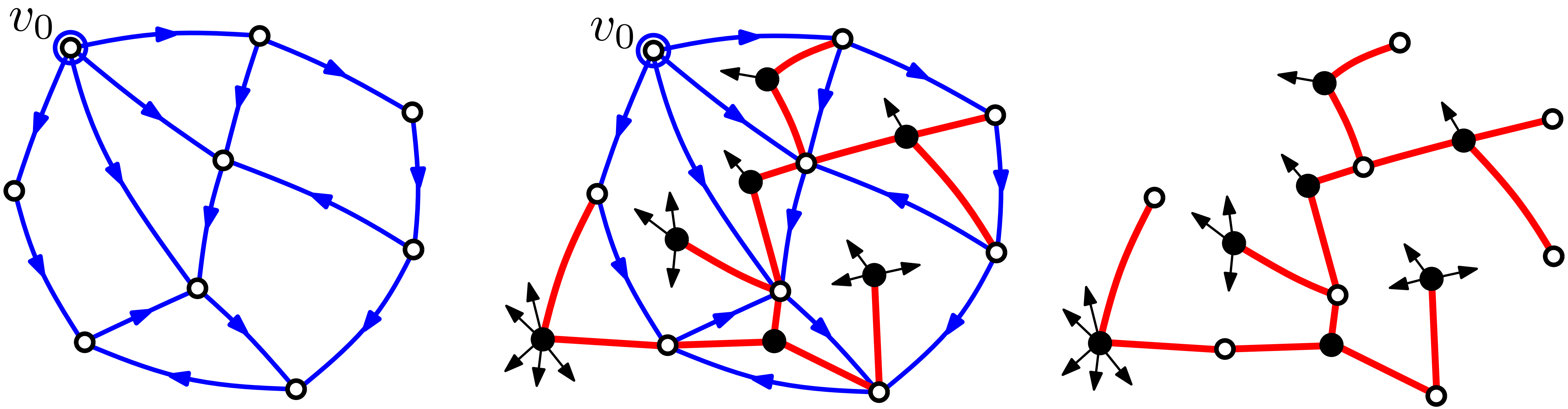
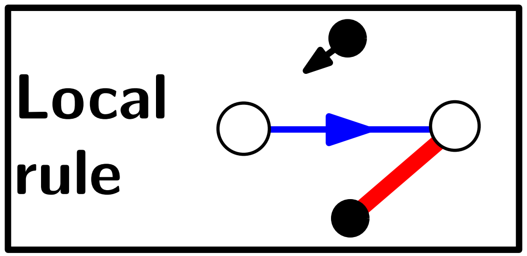
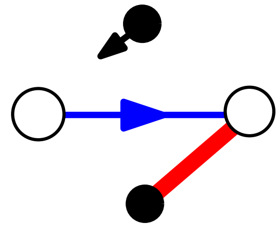




Theorem : Let $\mathcal{O}_0$ be this family of orientations, then the correspondence is a bijection with mobiles of excess 0

Proof that it gives a tree

Start from an oriented map $M \in \mathcal{O}_0$ and apply the local rule 

Let G be the graph of red edges and their incident vertices

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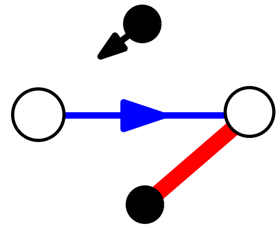

Let G be the graph of red edges and their incident vertices

G has $|V_M| - 1$, white vertices, $|F_M|$ black vertices, et $|E_M|$ edges

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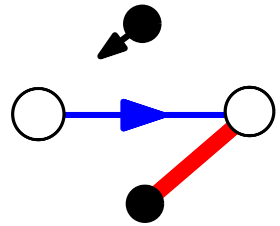
Start from an oriented map $M \in \mathcal{O}_0$ and apply the local rule 

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Euler relation: $|E_M| = |V_M| + |F_M| - 2$

$\Rightarrow G$ has one more vertices than edges

hence G is a tree iff G is acyclic

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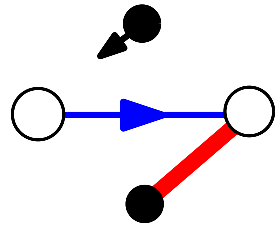
Start from an oriented map $M \in \mathcal{O}_0$ and apply the local rule 

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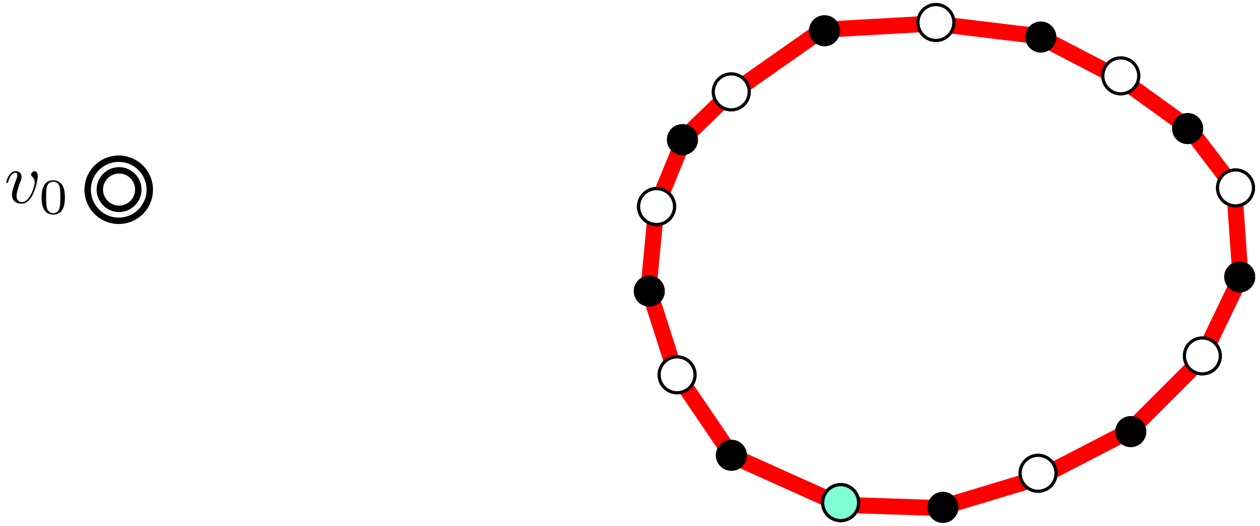

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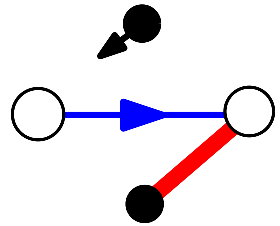

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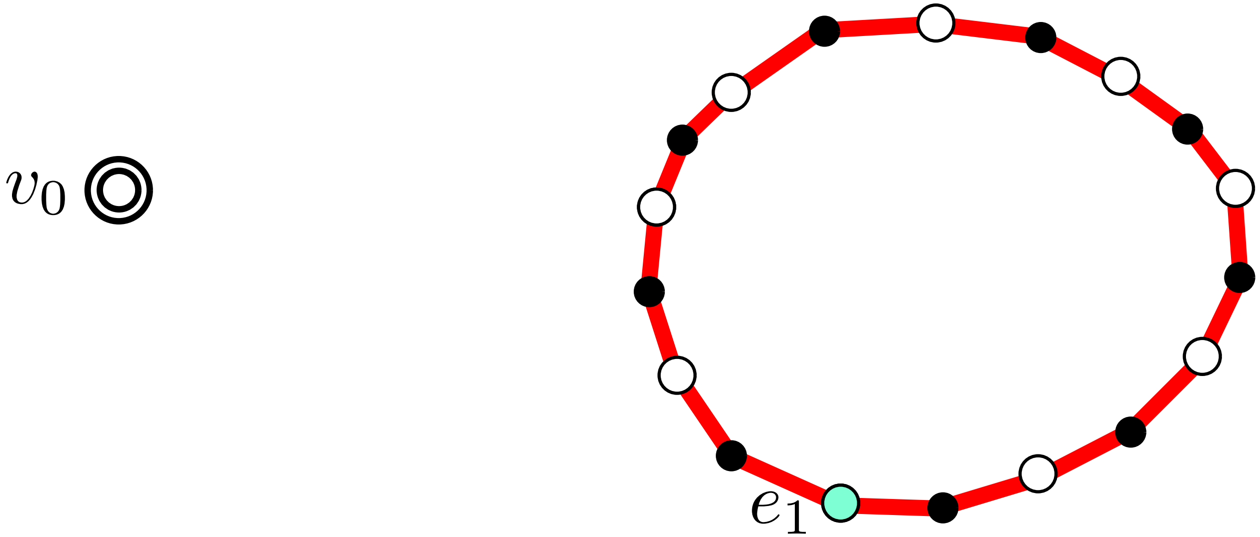

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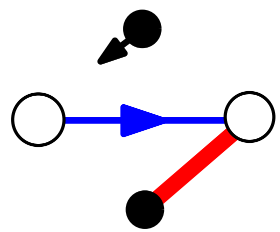
Start from an oriented map $M \in \mathcal{O}_0$ and apply the local rule 

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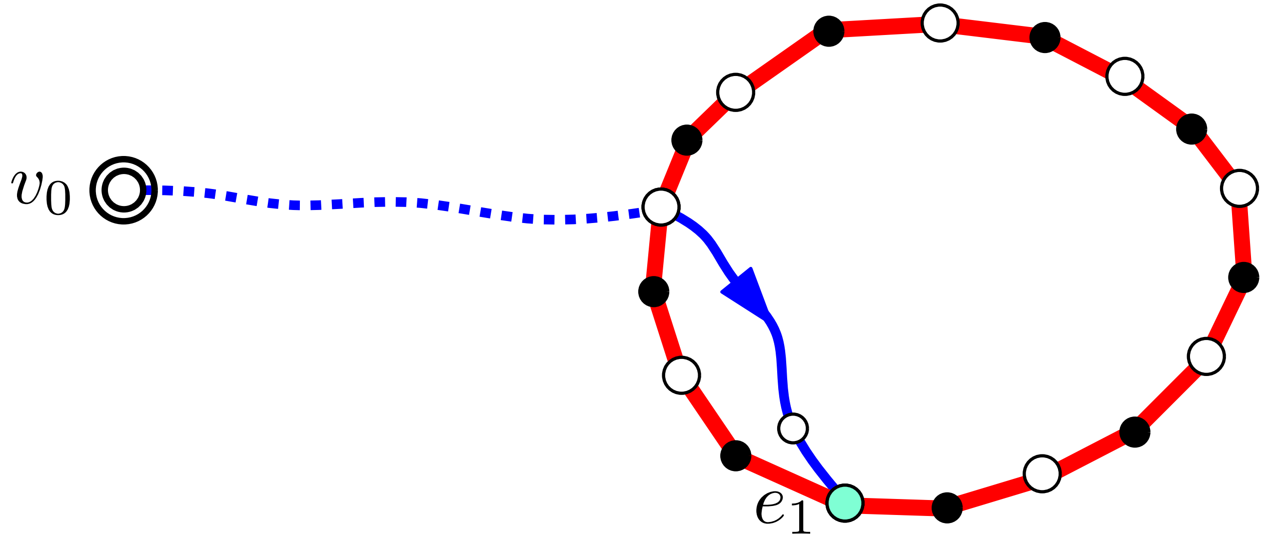

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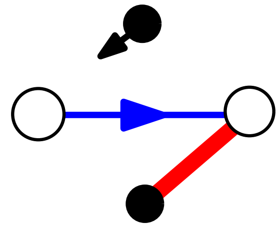
Start from an oriented map $M \in \mathcal{O}_0$ and apply the local rule 

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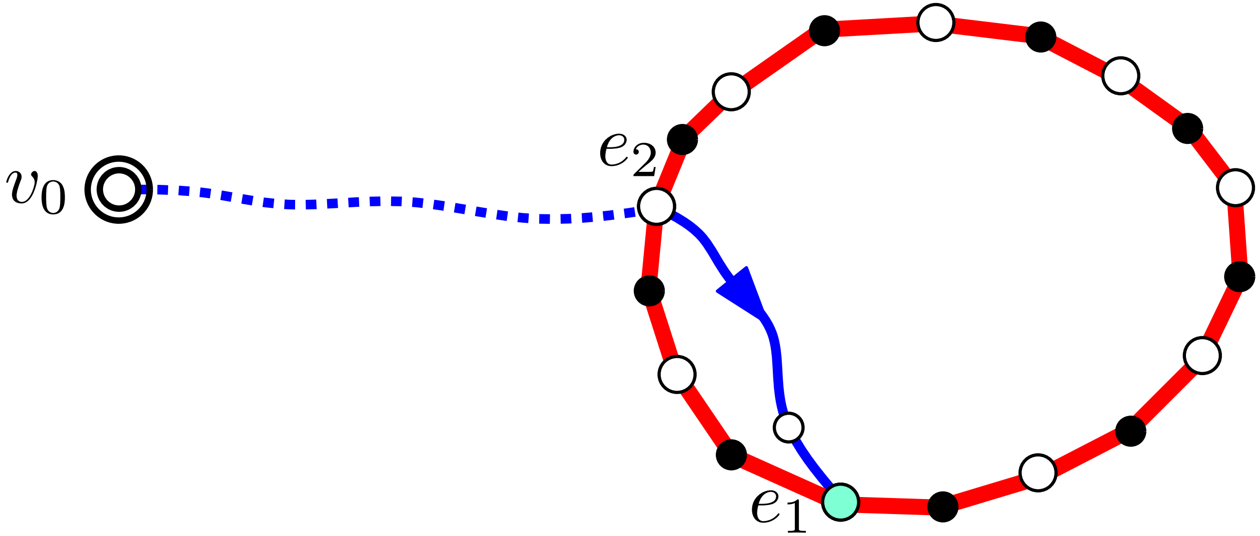

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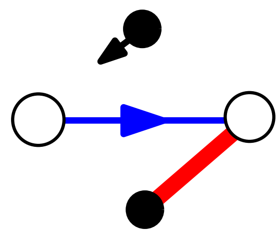
Start from an oriented map $M \in \mathcal{O}_0$ and apply the local rule 

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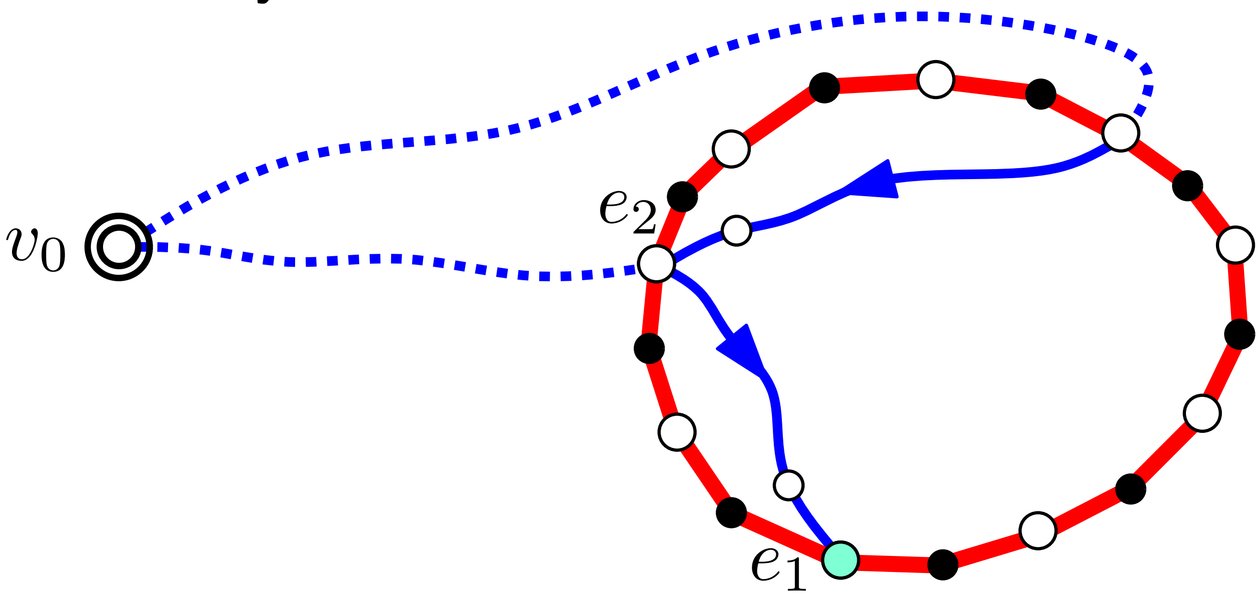

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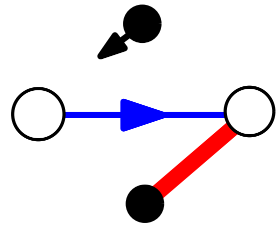

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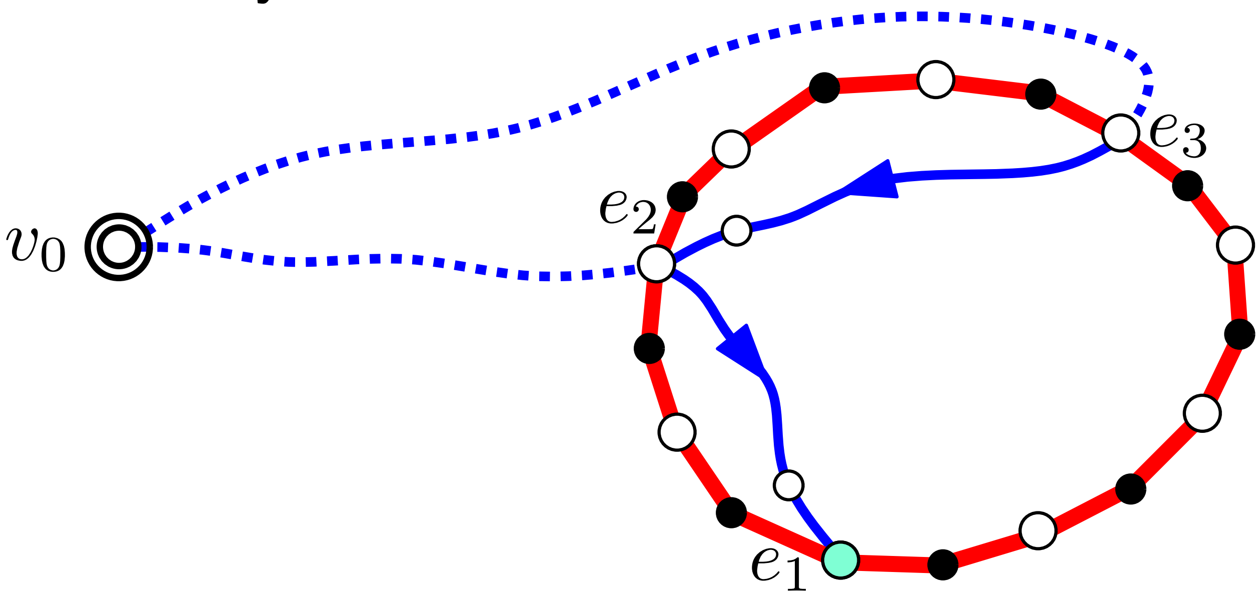

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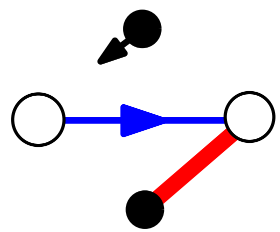
Start from an oriented map $M \in \mathcal{O}_0$ and apply the local rule 

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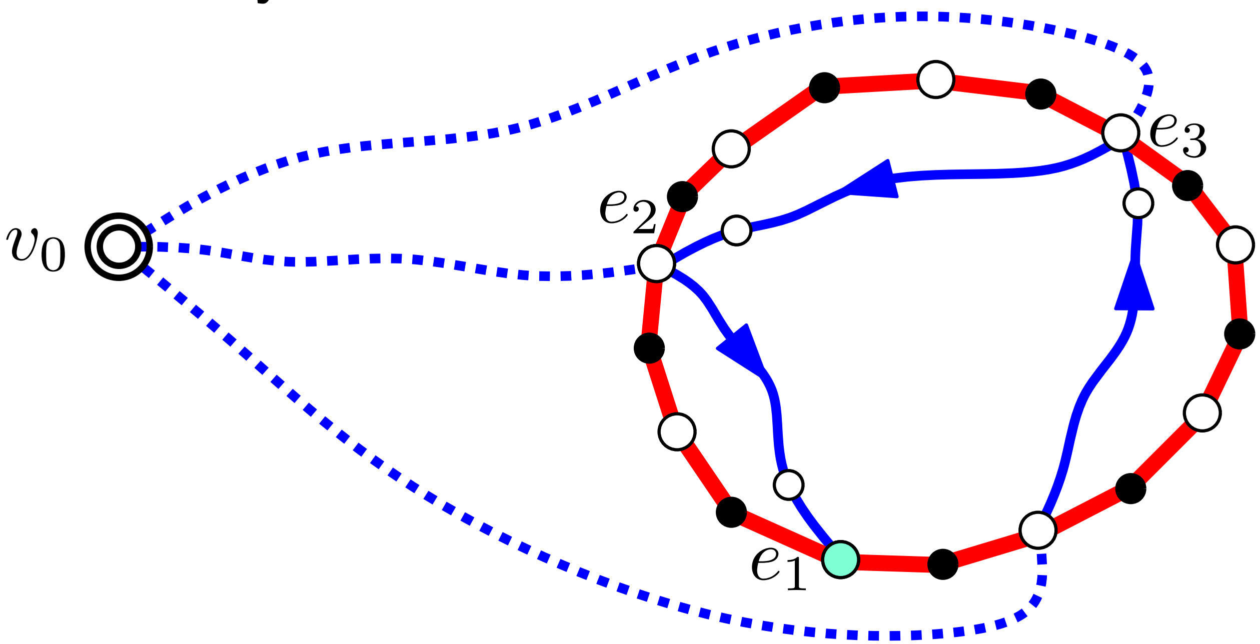

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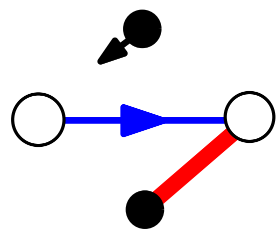
Start from an oriented map $M \in \mathcal{O}_0$ and apply the local rule 

Let G be the graph of red edges and their incident vertices

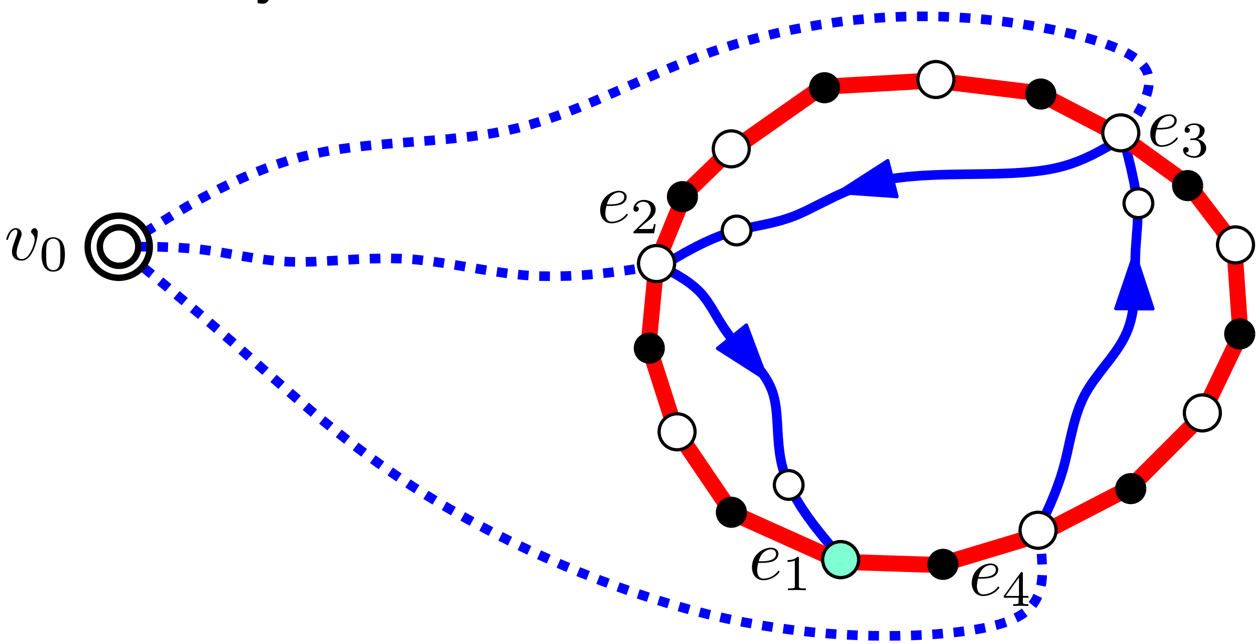
G has $|V_M| - 1$, white vertices, $|F_M|$ black vertices, et $|E_M|$ edges

Euler relation: $|E_M| = |V_M| + |F_M| - 2$

$\Rightarrow G$ has one more vertices than edges

hence G is a tree iff G is acyclic

Assume G has a cycle :



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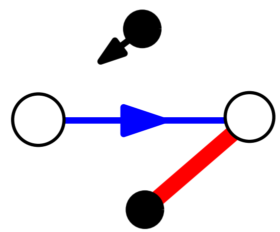

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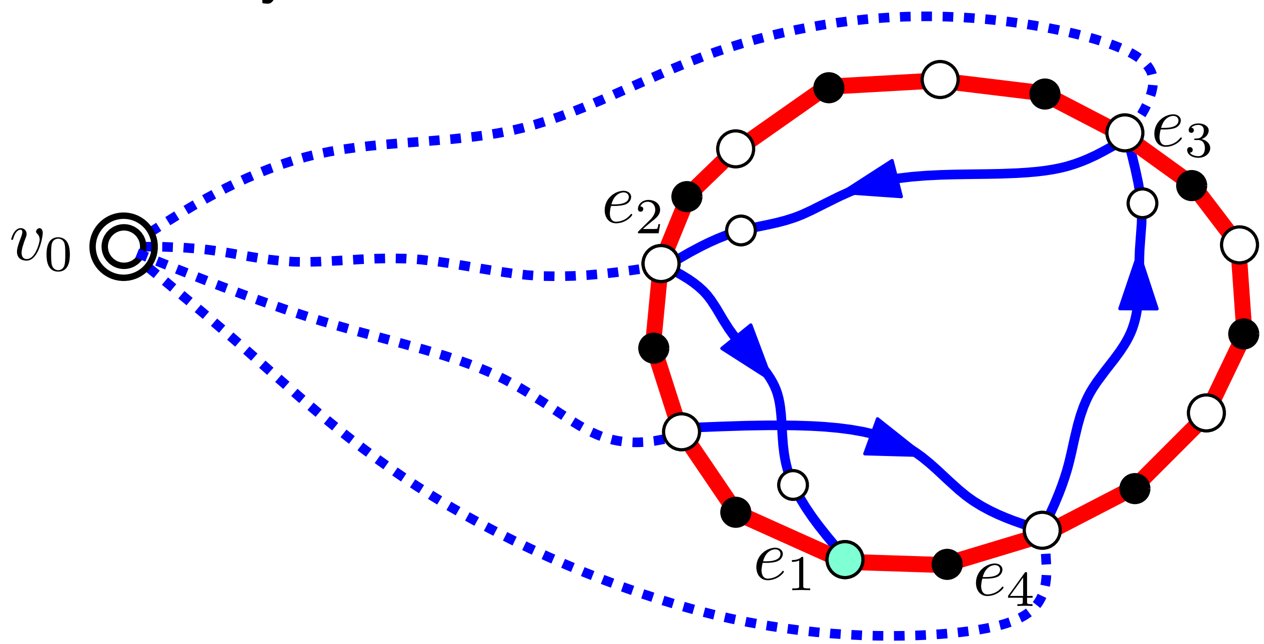

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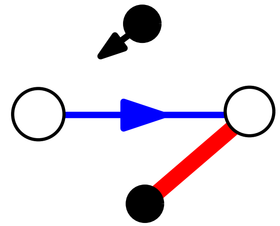
Start from an oriented map $M \in \mathcal{O}_0$ and apply the local rule 

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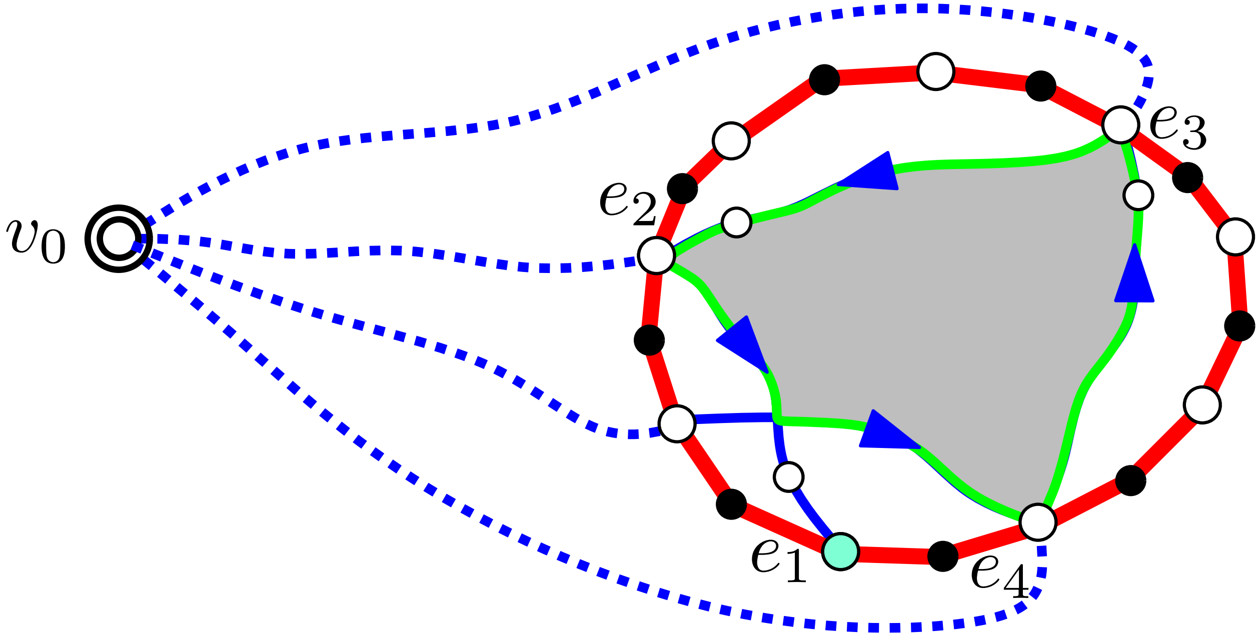


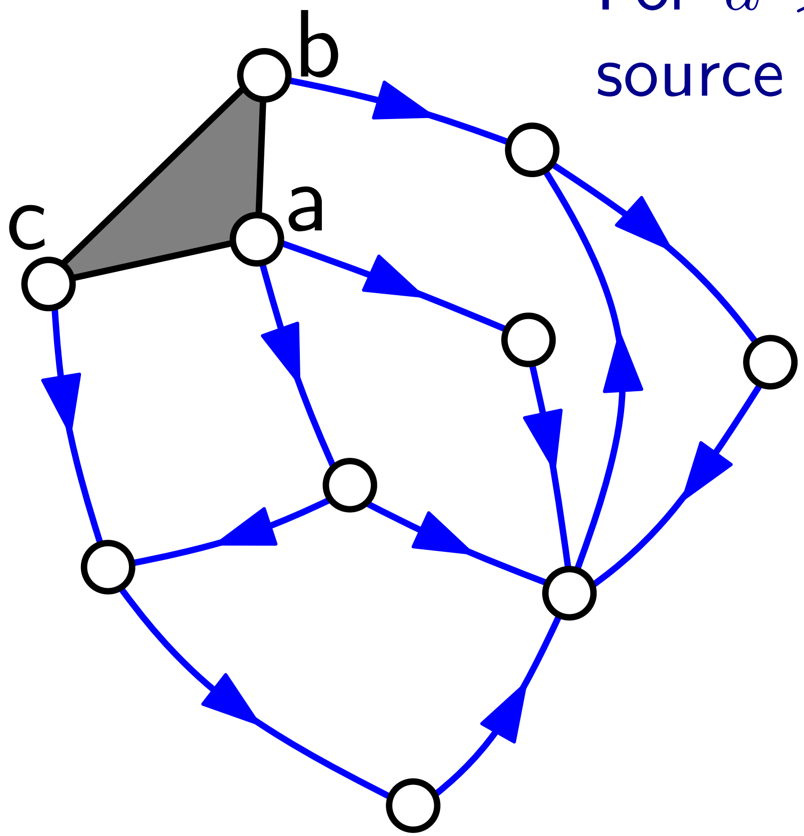
prisoner ccw cycle

$\Rightarrow$ contradiction

Extension for mobiles of negative excess

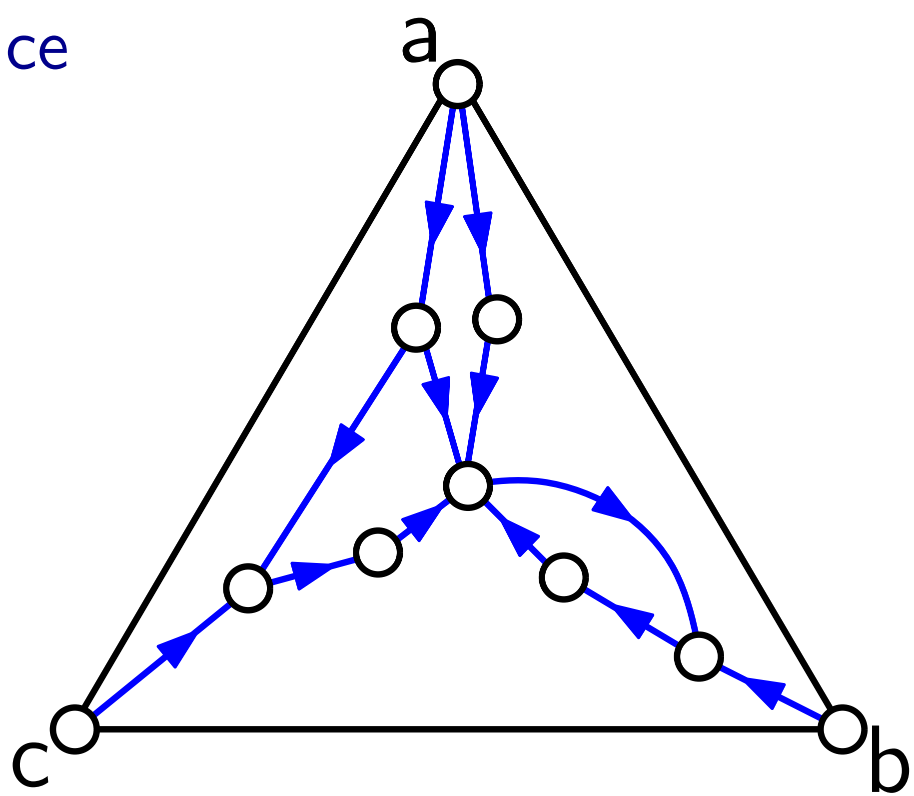
More generally the "source" can be a d-gon, for any $d \geq 0$

Example for $d = 3$



For $d > 0$, we take the d-gonal source as the outer face

$\Rightarrow$



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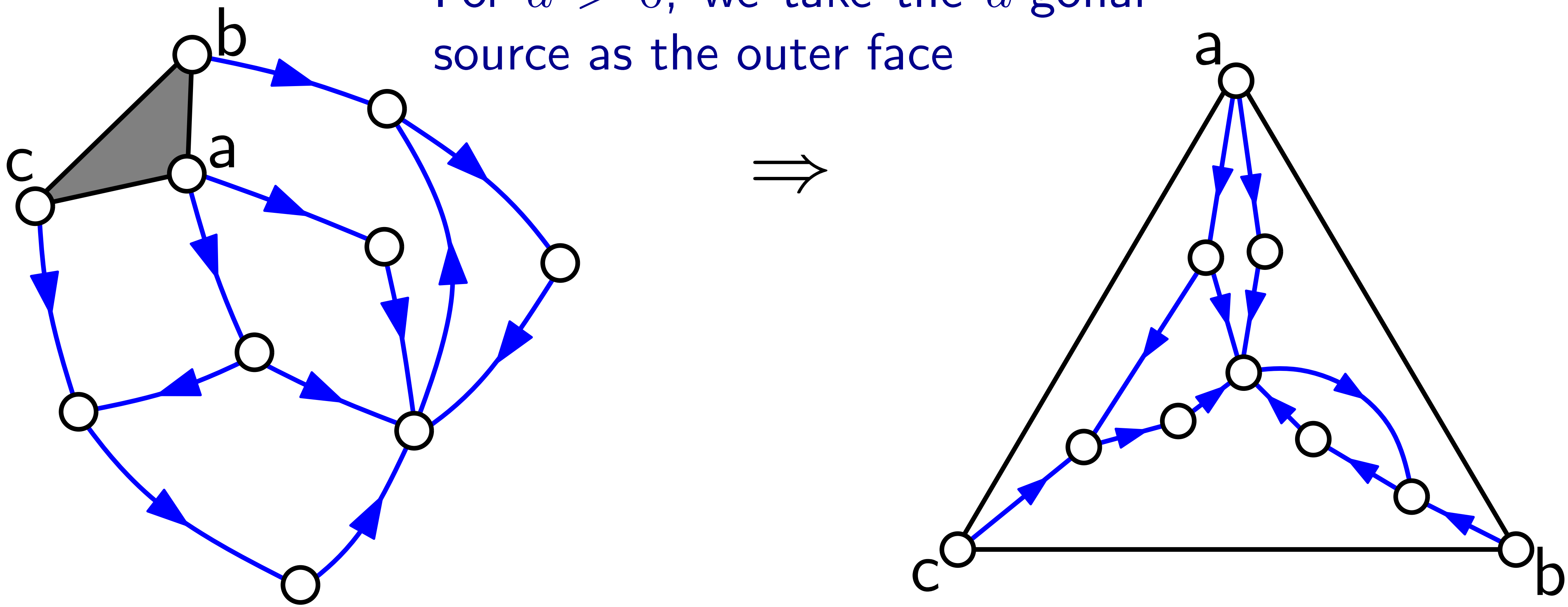
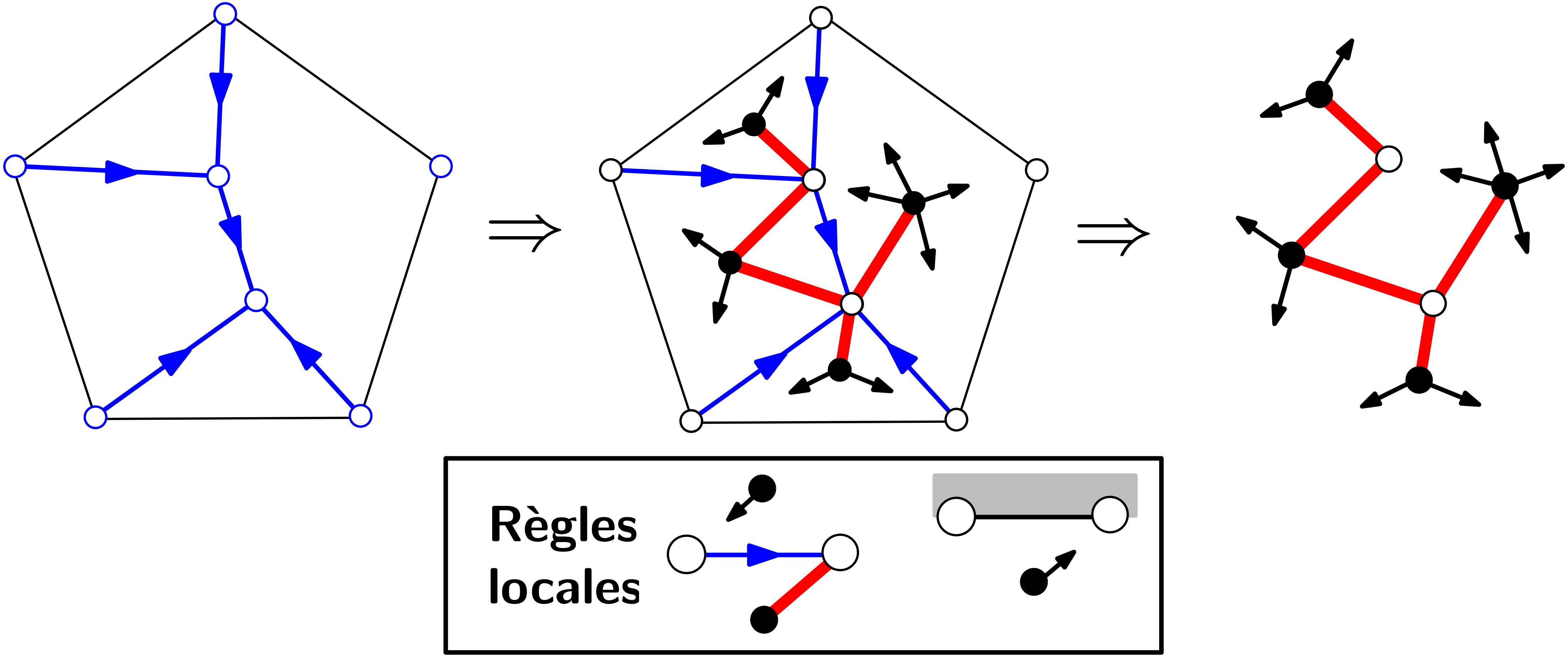


Let $\mathcal{O}_{-d}$ be the family of these orientations, still with the conditions

- the d-gonal **source** has no ingoing edge
- **accessibility** of every vertex from the source
- **no ccw cycle**

Extension for mobiles of negative excess



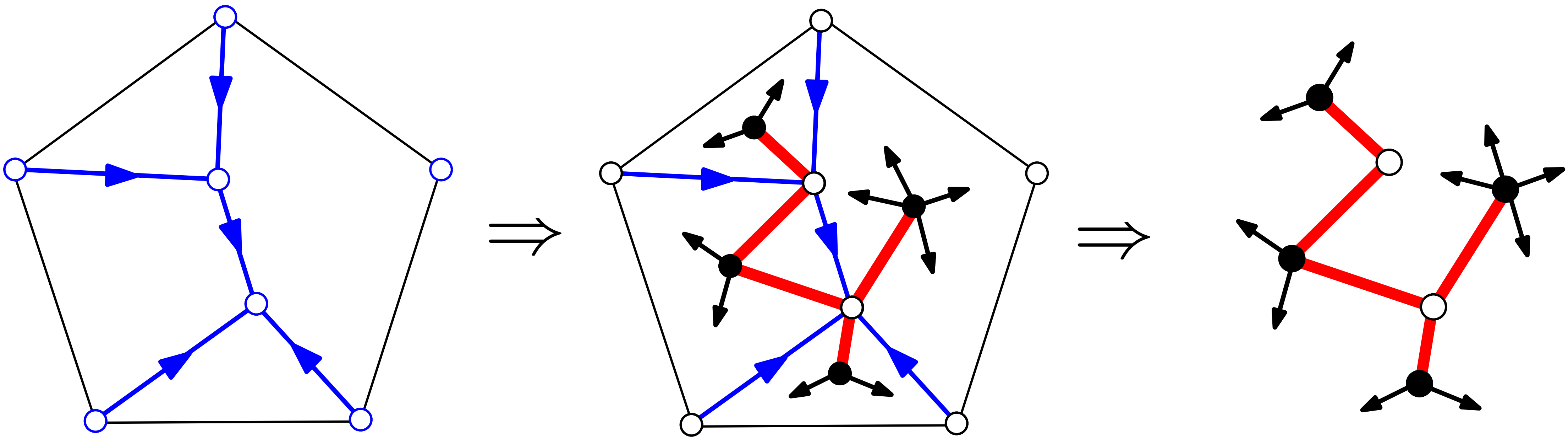
Theorem [Bernardi-F'10]: For $\delta \leq 0$, the correspondence Φ is a **bijection** between $\mathcal{O}_\delta$ and mobiles of excess δ.

degrees of the inner faces $\longleftrightarrow$ degrees of the black ver

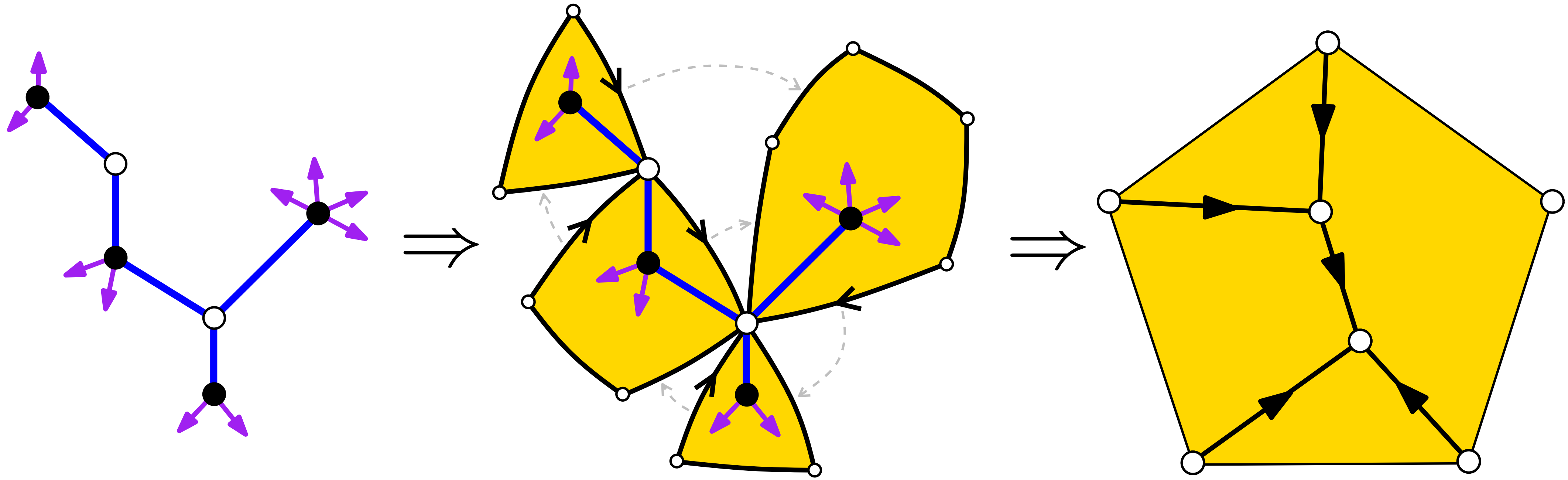
indegrees of internal vertices $\longleftrightarrow$ degrees of white vertice

cf [Bernardi'07], [Bernardi-Chapuy'10]

Extension for mobiles of negative excess



- Inverse mapping (tree $\rightarrow$ cactus $\rightarrow$ closure operations)



Specializing the correspondence

The correspondence Φ is a bijection between the family $\mathcal{O} = \cup_{d \geq 0} \mathcal{O}_{-d}$ of oriented maps and mobiles of nonpositive excess

Idea: Let $\mathcal{F}$ be the family of planar maps we consider (e.g. bipartite maps, simple triangulations, etc.)

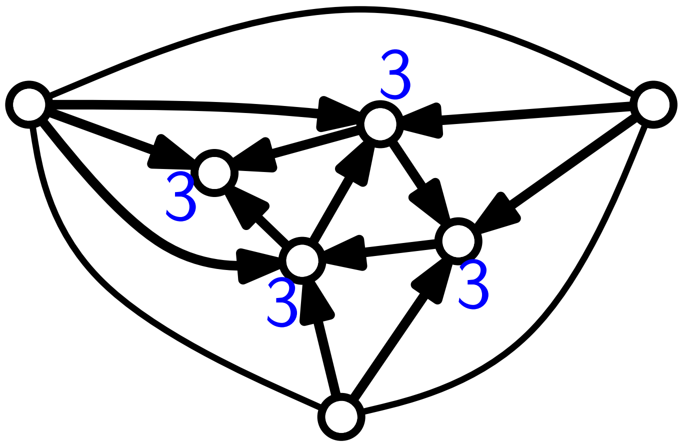
Prove that a map is in $\mathcal{F}$ iff it admits a **canonical orientation** in $\mathcal{O}$ specified by face-degrees and vertex-indegrees conditions

Specialize Φ to the corresponding subfamily $\mathcal{O}_{\mathcal{F}} \subseteq \mathcal{O}$

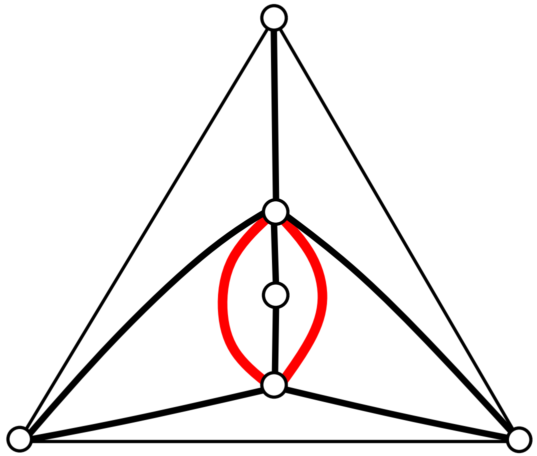
Gives a bijection between $\mathcal{F}$ and a well characterized family of mobiles

Application to simple triangulations

For a triangulation T, a **3-orientation** of T is an orientation of the inner edges of T such that every inner vertex has **indegree 3**



Rk: If a triangulation T admits a 3-orientation, then T is simple



Assume there is a 2-cycle C

If there are k vertices inside C then there are $3k-1$ edges inside C

$\Rightarrow$ total indegree is too large compared to the number of edges

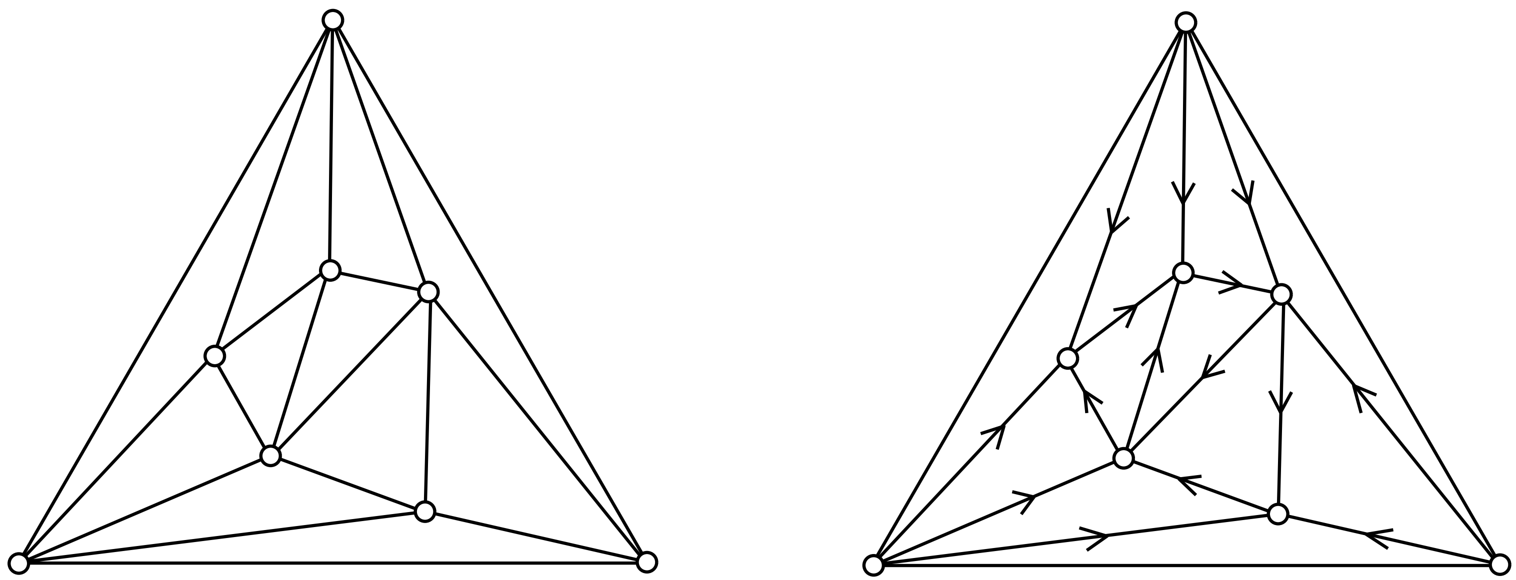
Existence of a canonical 3-orientation

Theorem (Schnyder'89): Any simple triangulation admits a 3-orientation

Theorem: Let T be a simple triangulation. Then T has a unique 3-orientation with no ccw cycle, the **minimal 3-orientation**

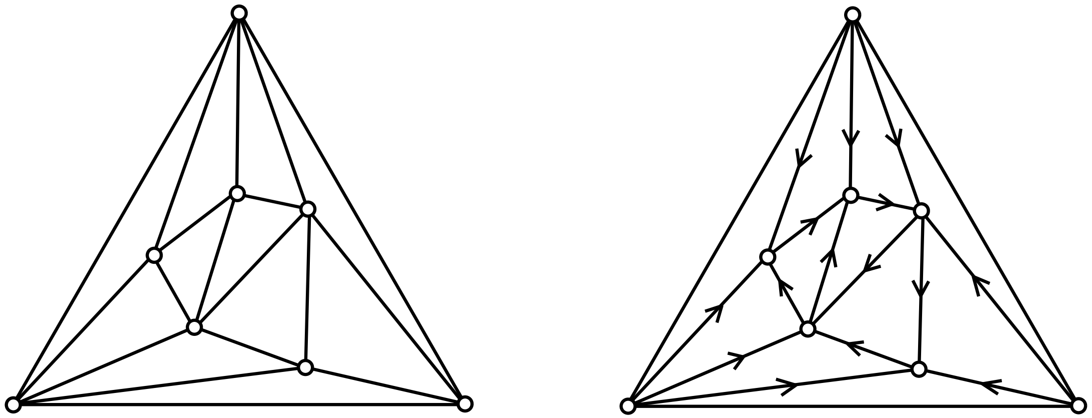
(set of 3-orientations is a lattice, flip $=$ reverse cw to ccw)

[Ossona de Mendez'94], [Brehm'03], [[Felsner'03]]

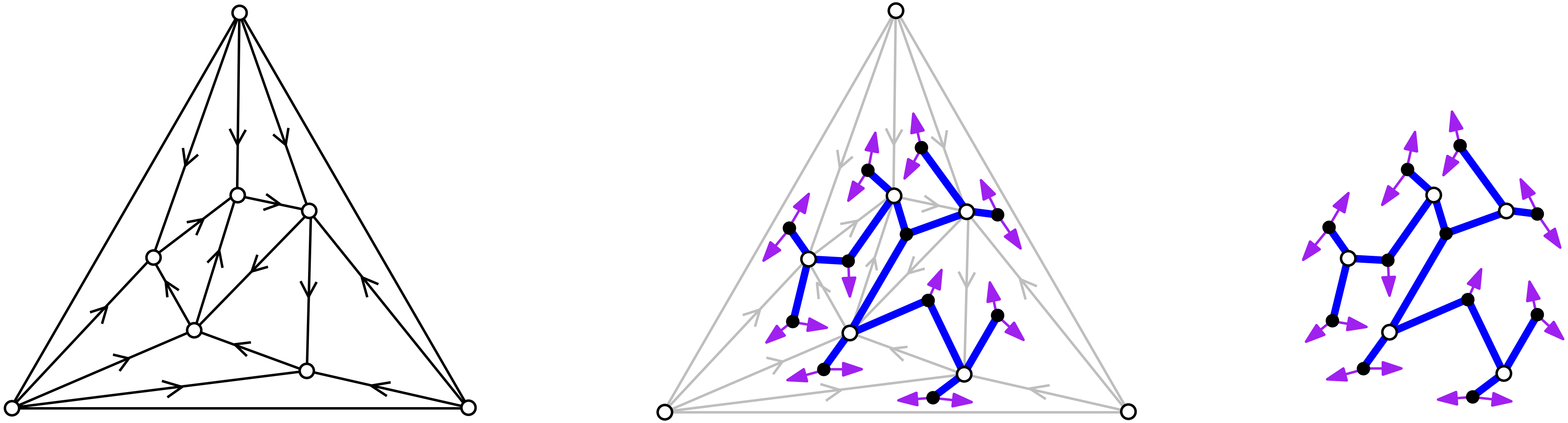


Bijection for simple triangulations

- From the lattice property (**taking the min**) we have
  family $\mathcal{T}$ of simple triangulations $\leftrightarrow$ **subfamily $\mathcal{O}_{\mathcal{T}}$ of $\mathcal{O}$** where:
  - faces have degree 3
  - inner vertices have indegree 3



- From the **bijection Φ specialized to $\mathcal{F}$**, we have
  $\mathcal{F}$ $\leftrightarrow$ **mobiles** where all vertices have **degree 3**

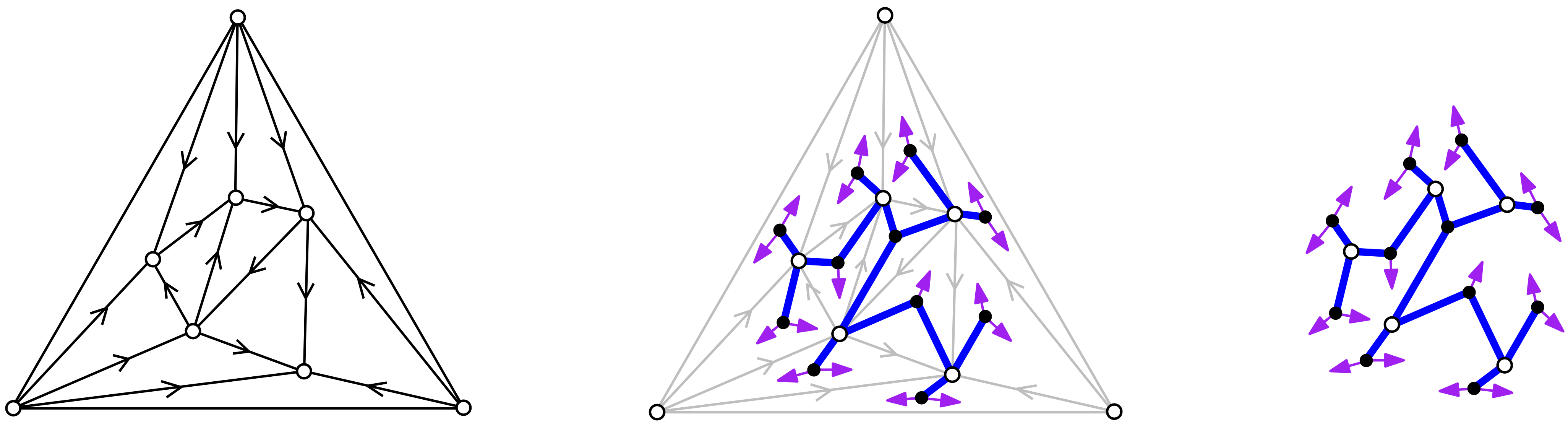


[Bernardi, F'10], other bijection in [Poulalhon, Schaeffer'03]

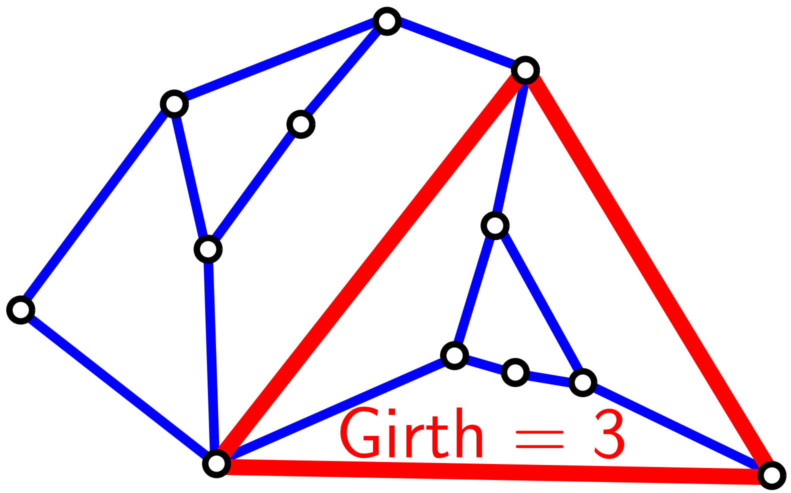
Counting simple triangulations

Counting: The generating function of mobiles with vertices of degree 3 rooted on a white corner is $T(x) = U(x)^3$, where $U(x) = 1 + xU(x)^4$.

Consequently, the number of (rooted) simple triangulations with $2n$ faces is $\dfrac{1}{n(2n-1)}\dbinom{4n-2}{n-1}$.



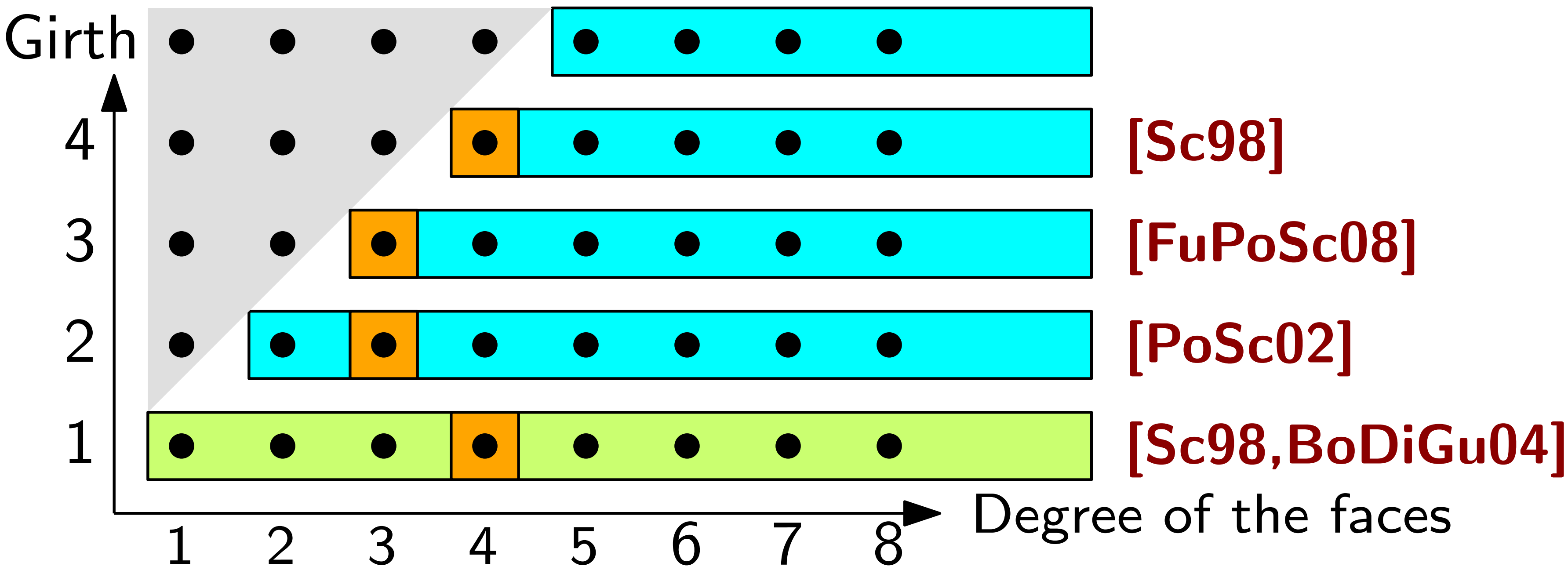
Extension to any girth and face-degrees



girth=length shortest cycle

Rk: girth ≤minimal face-degree

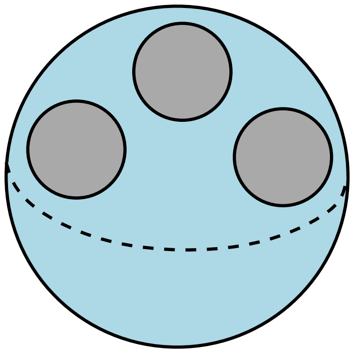
Our approach works in any girth d, with control on the face-degrees



Other approach using slice decompositions **[Bouttier,Guitter'15]**

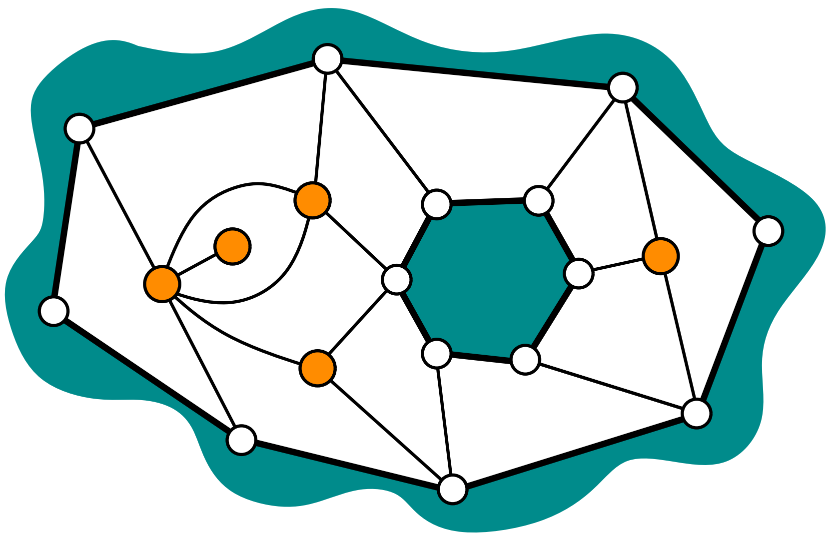
Maps with boundaries

- Sphere with k holes $=$ sphere where k disks have been removed



sphere with 3 holes

- Map with k boundaries $=$ graph embedded on the sphere with k holes
  the boundaries are occupied by cycles of edges



A quadrangulations with 2 boundaries
of lengths 8 and 6, and 5 internal vertices

Maps with boundaries

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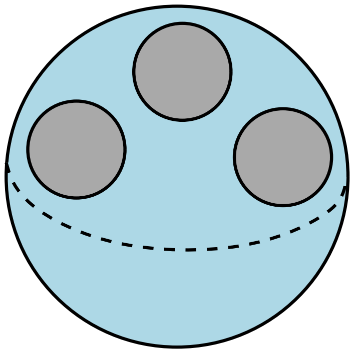

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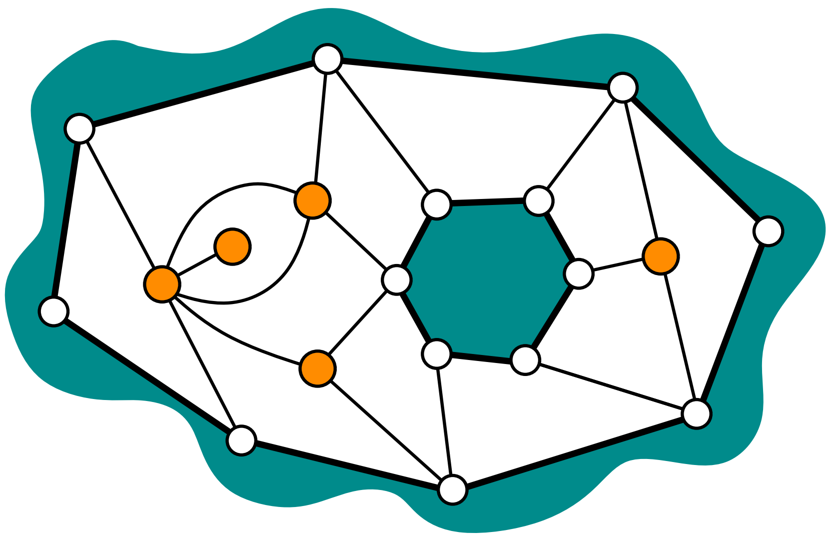


A quadrangulations with 2 boundaries
of lengths 8 and 6, and 5 internal vertices

(also $=$ planar map with k distinguished faces whose contours
are vertex-disjoint simple cycles)

Counting triangulations with boundaries

b boundaries of lengths $k_1, \ldots, k_b$
n internal vertices

- Without loops and multiple edges, formula only for $b = 1$

No loops (girth=2)

$$t_n^{(k)} = \frac{2^{n+1}(2k-3)!}{(k-2)!^2} \frac{(3n+2k-3)!}{n!(2n+2k-2)!}$$

[Mullin'65] (recursive method)
bijective proof in **[Poulalhon,Schaeffer'02]**

No loops & multiple edges (girth=3)

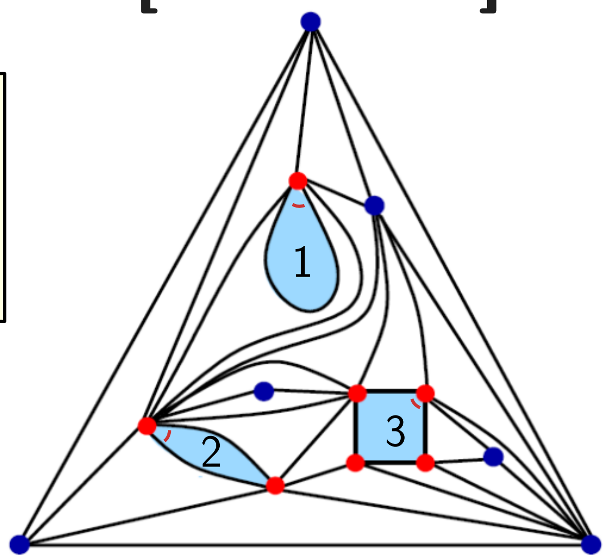
$$s_n^{(k)} = \frac{2(2k-3)!}{(k-1)!(k-3)!} \frac{(4n+2k-5)!}{n!(3n+2k-3)!}$$

[Brown'64] (recursive method)
bijective proofs in **[Poulalhon,Schaeffer'06]** **[Bernardi,F'10]**

- With loops and multiple edges, nice factorized formula **[Krikun'07]**

$$a_n^{(k_1,\ldots,k_b)} = \frac{4^{n-1}(2k+3n-5)!!}{(n-b+1)!(2k+n-1)!!} \prod_{j=1}^{b} k_j \binom{2k_j}{k_j}$$

bijective proof in **[Bernardi, F'15]**

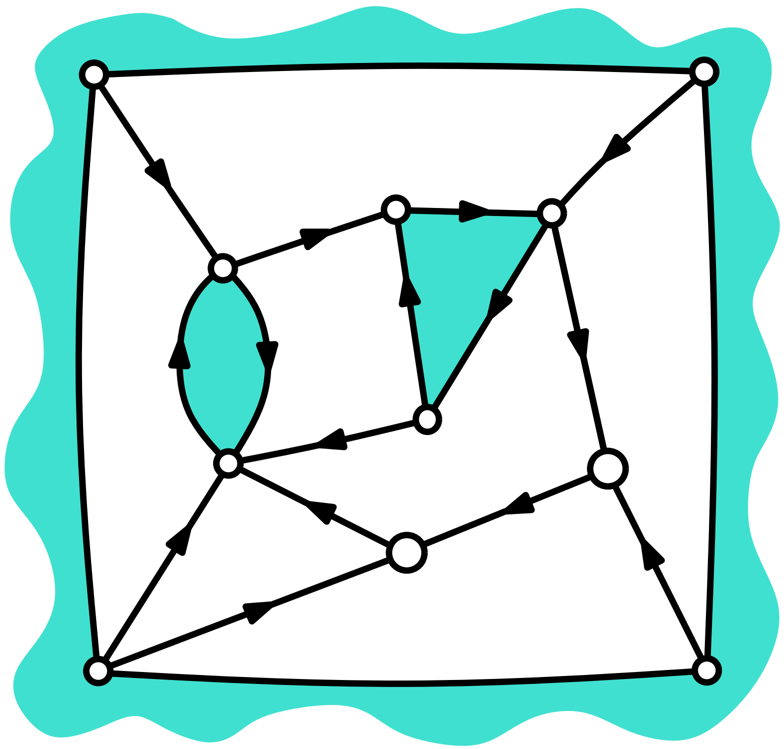


Orientations for maps with boundaries

For maps with boundaries we consider orientations such that **every inner boundary is a cw cycle** and the outer cycle is a boundary. These are called **boundary-orientations**

To apply the mobile construction we still require the orientations to satisfy:

- the outer d-gon is a **source** (no ingoing edge)
- every vertex can be **reached** by a directed path starting from the source
- **there is no ccw cycle**



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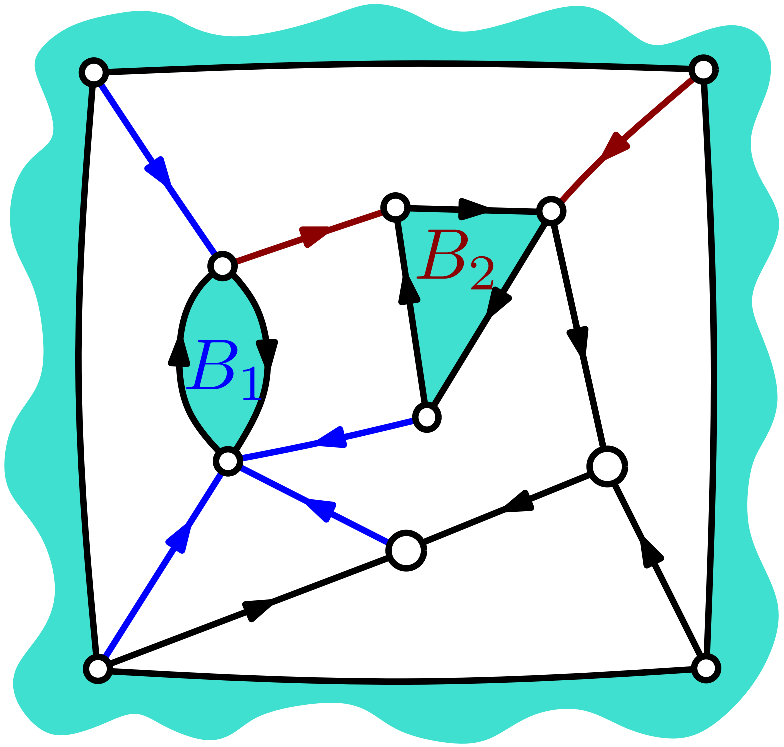
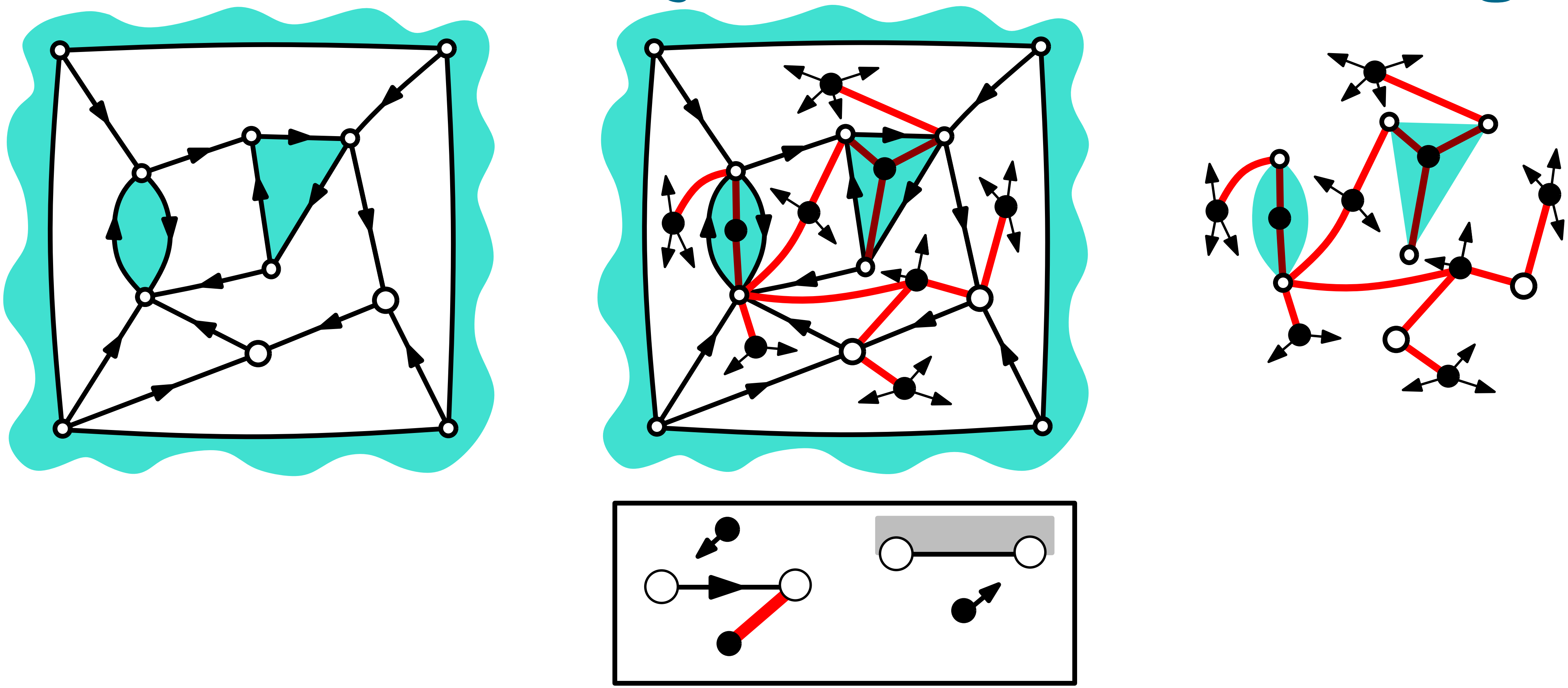


indegree of a boundary B : total number of edges toward B

B_1 has indegree 4
B_2 has indegree 2

Extension of the bijection Φ to this setting



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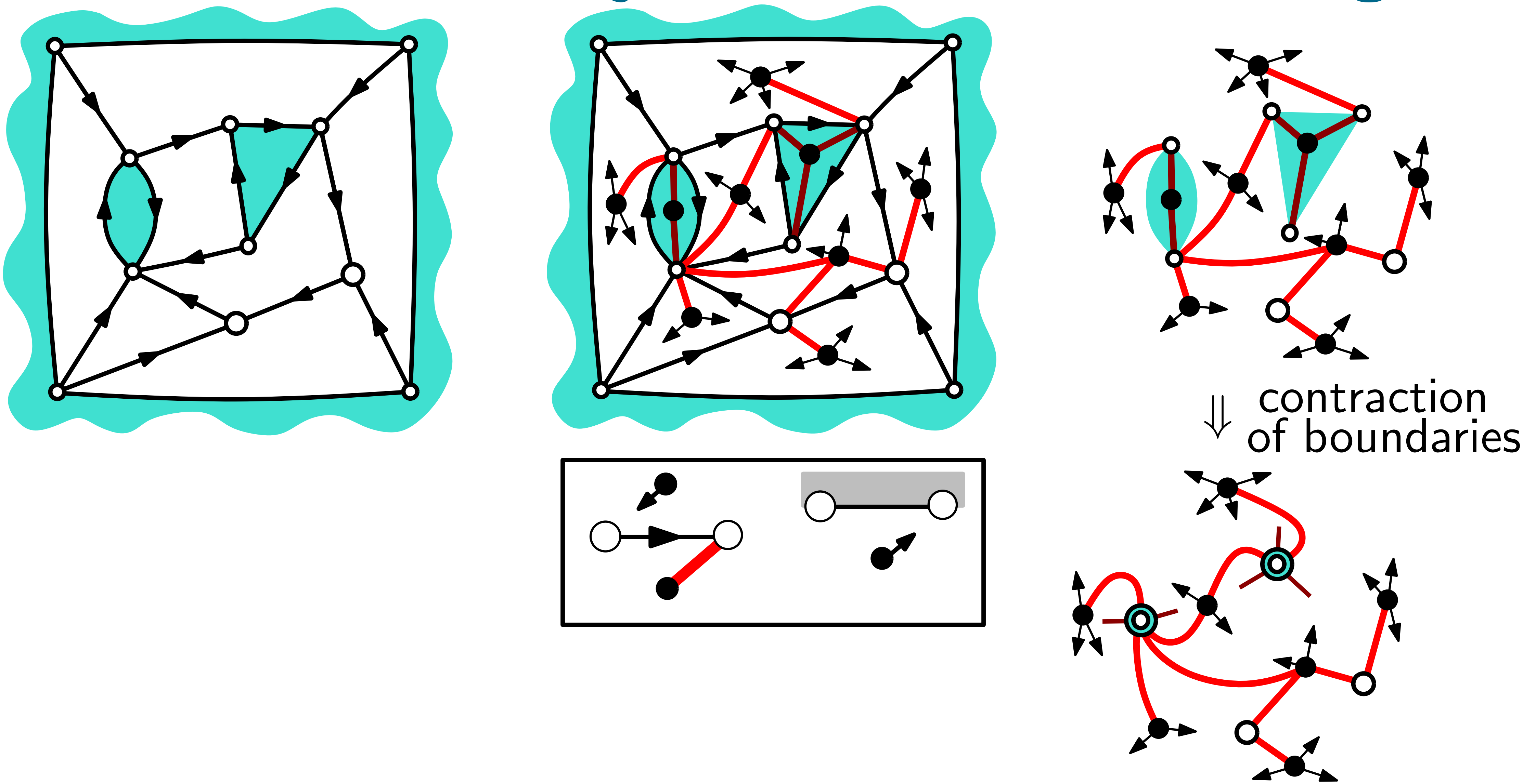

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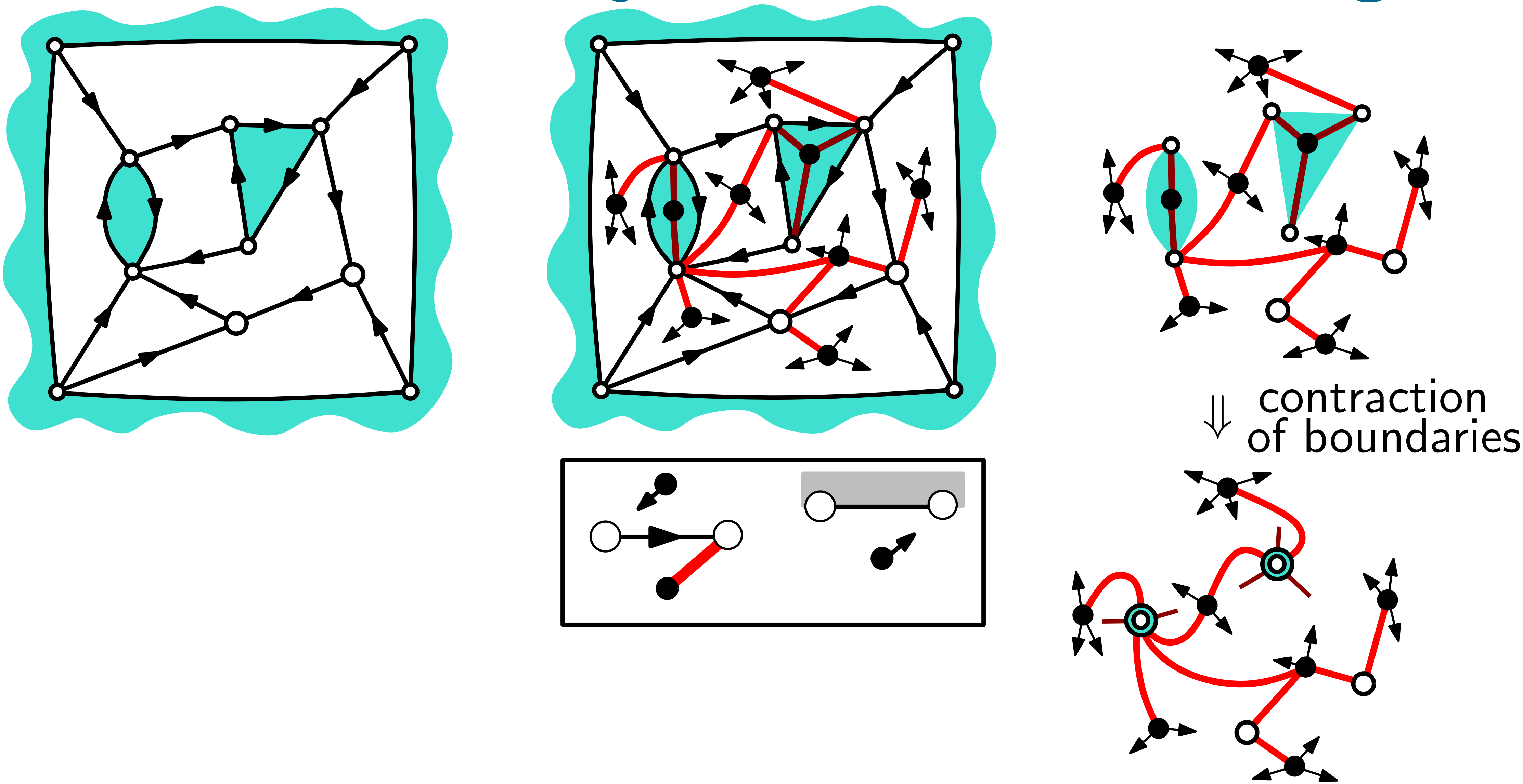
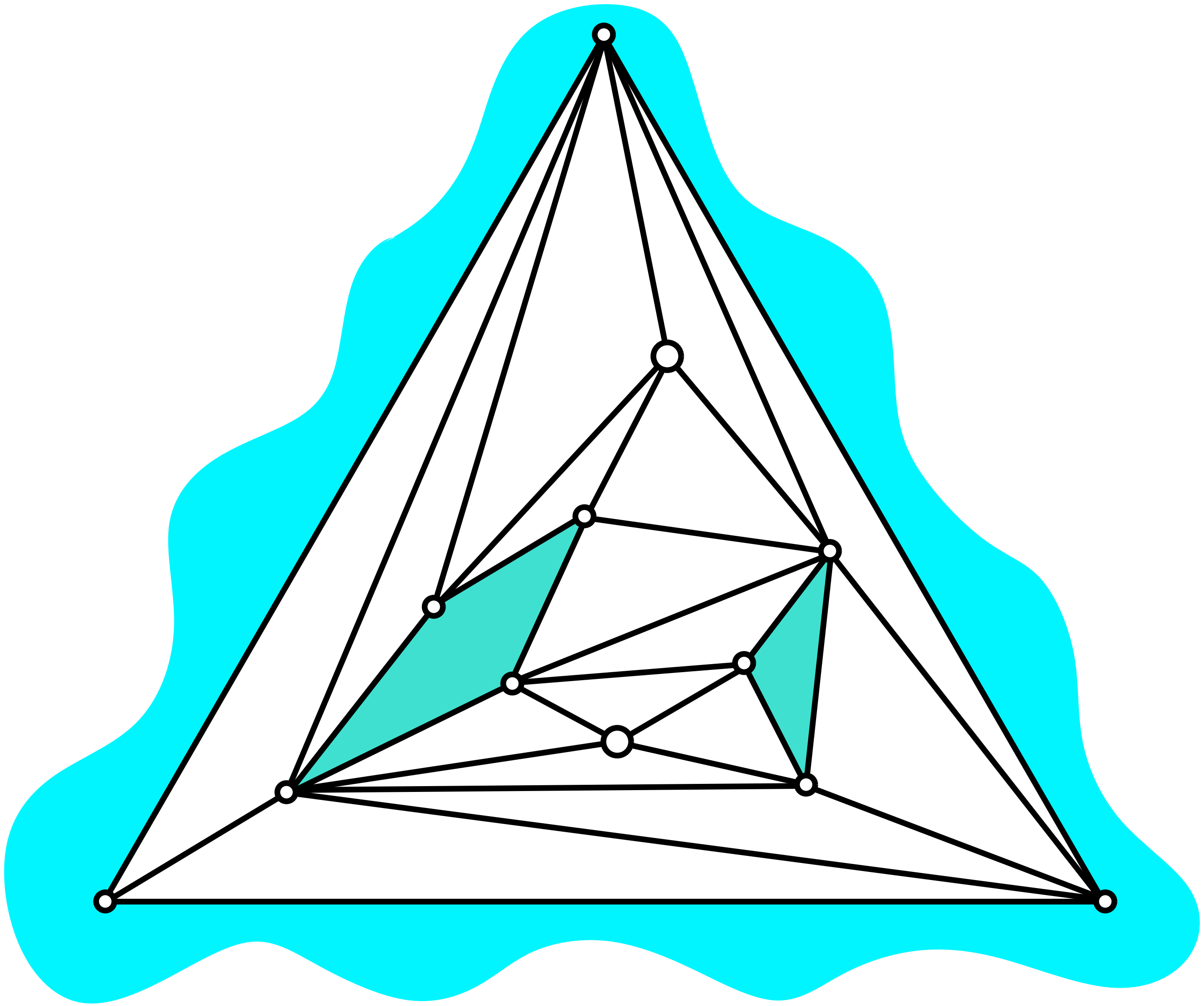


vertex ○ of indegree k ⟷ vertex ○ of degree k

internal boundary { degré r / b entrantes ⟷ vertex ◎ { r legs / b neighbours

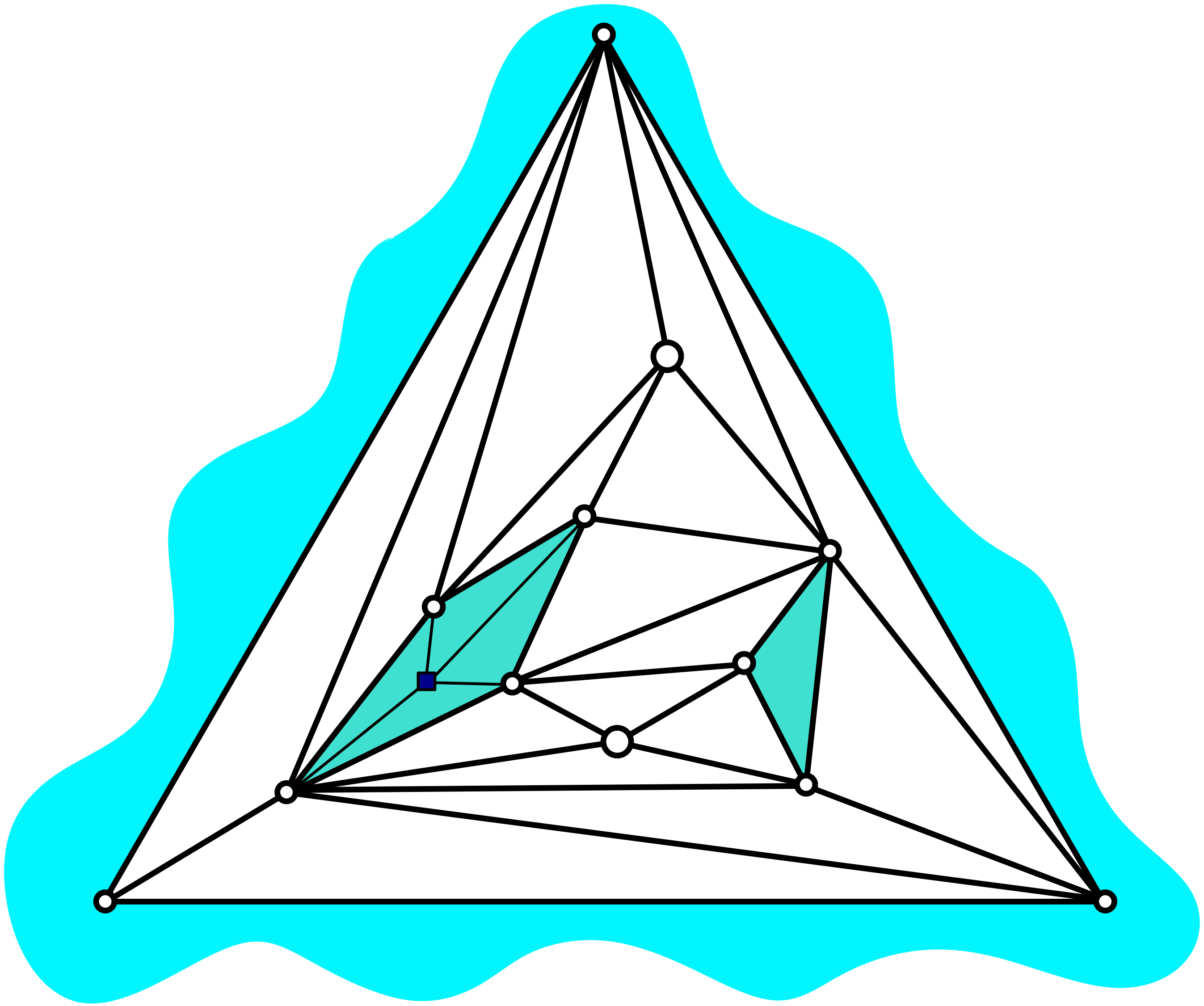
internal face of degree p ⟷ vertex ● of degree p

Orientations for simple triangulations with boundaries



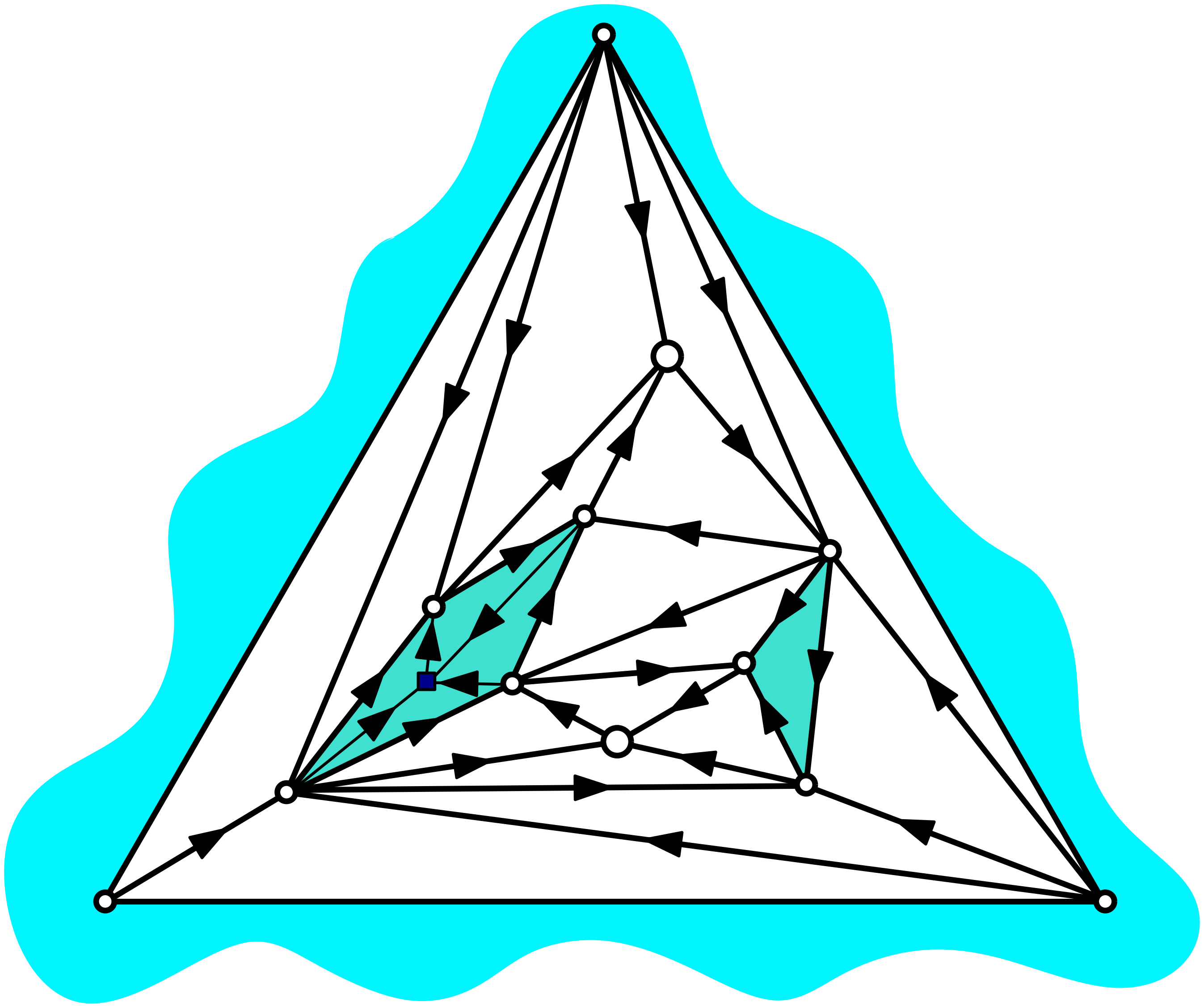
Orientations for simple triangulations with boundaries

Triangulate each inner boundary of length > 3



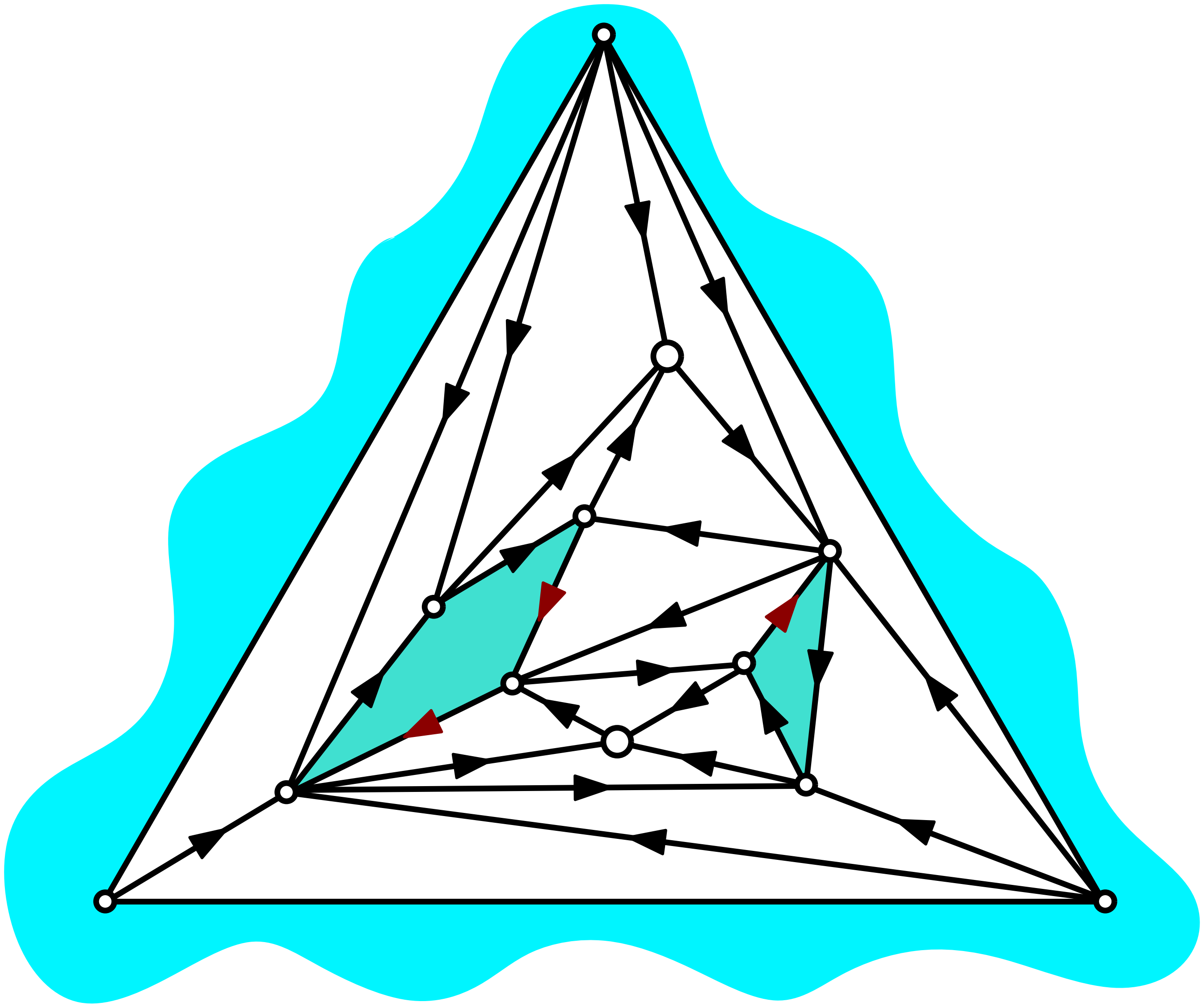
Orientations for simple triangulations with boundaries

Triangulate each inner boundary of length > 3 and compute the minimal 3-orientation



Orientations for simple triangulations with boundaries

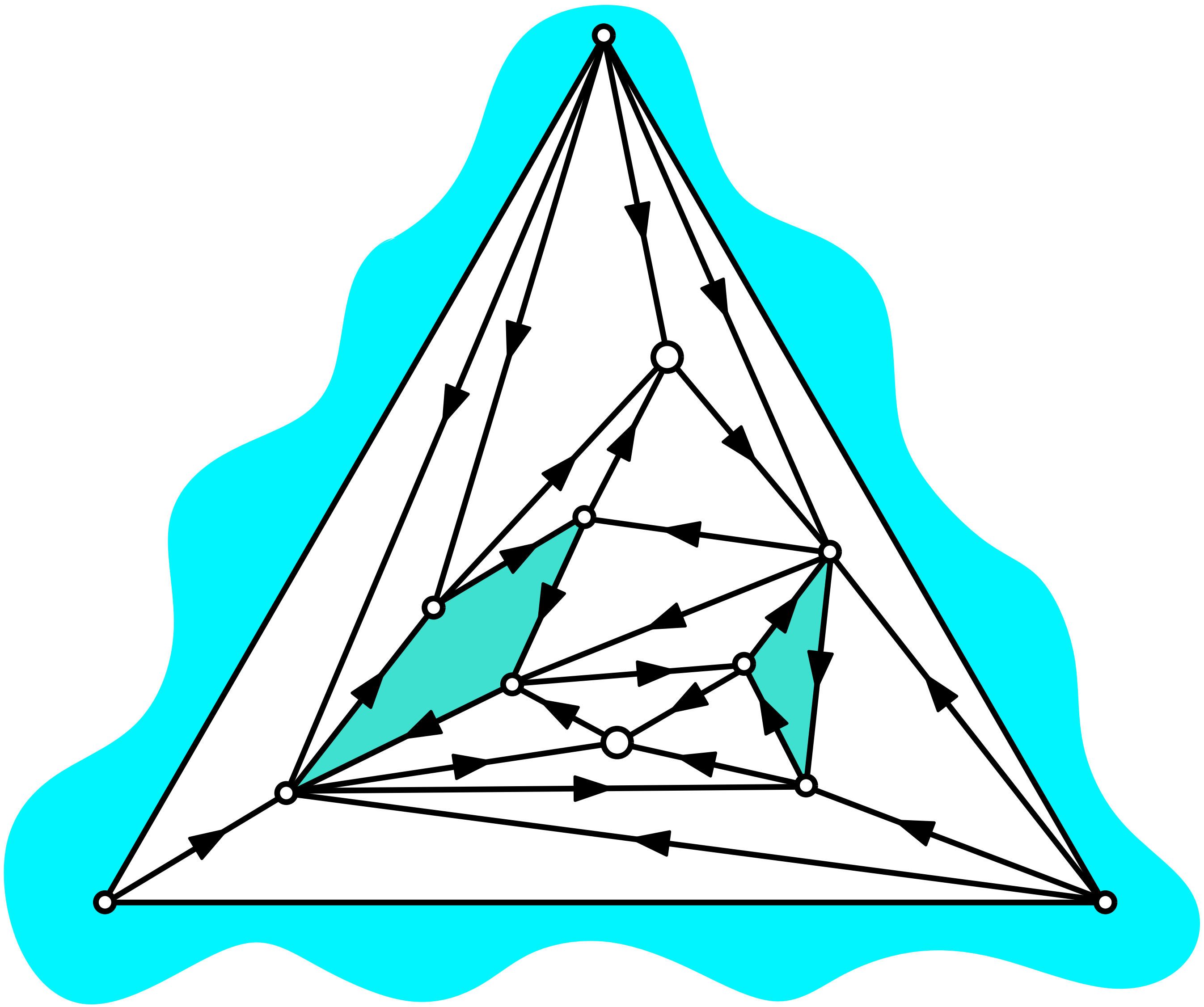
delete the added edges inside boundaries
and reorient the inner boundaries as cw cycles



Orientations for simple triangulations with boundaries

Each inner boundary of length i has indegree $i+3$
Each internal vertex has indegree 3

Such a boundary-orientation is called a **pseudo-3-orientation**



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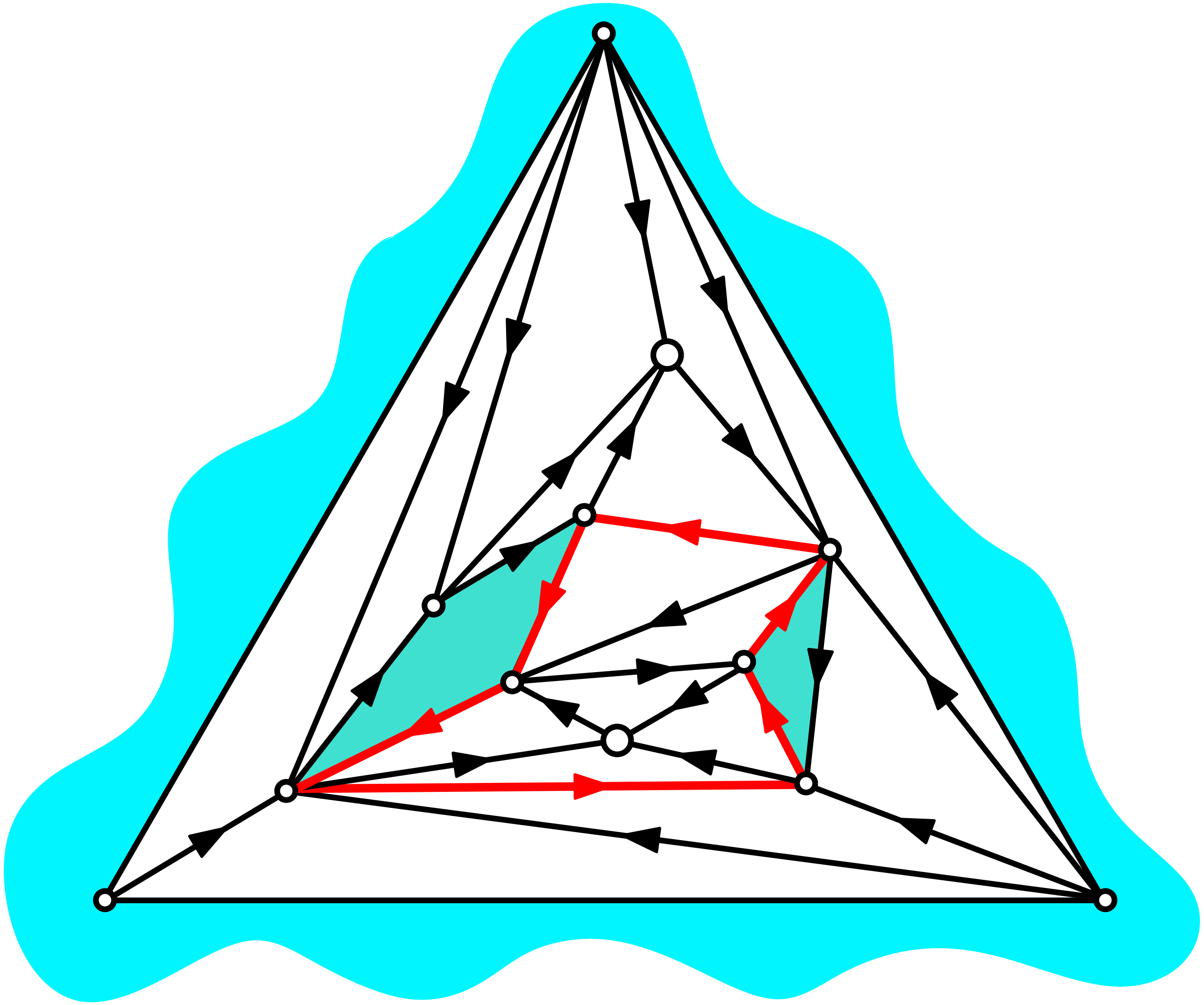
Take the **minimal** such orientation (no ccw cycle)



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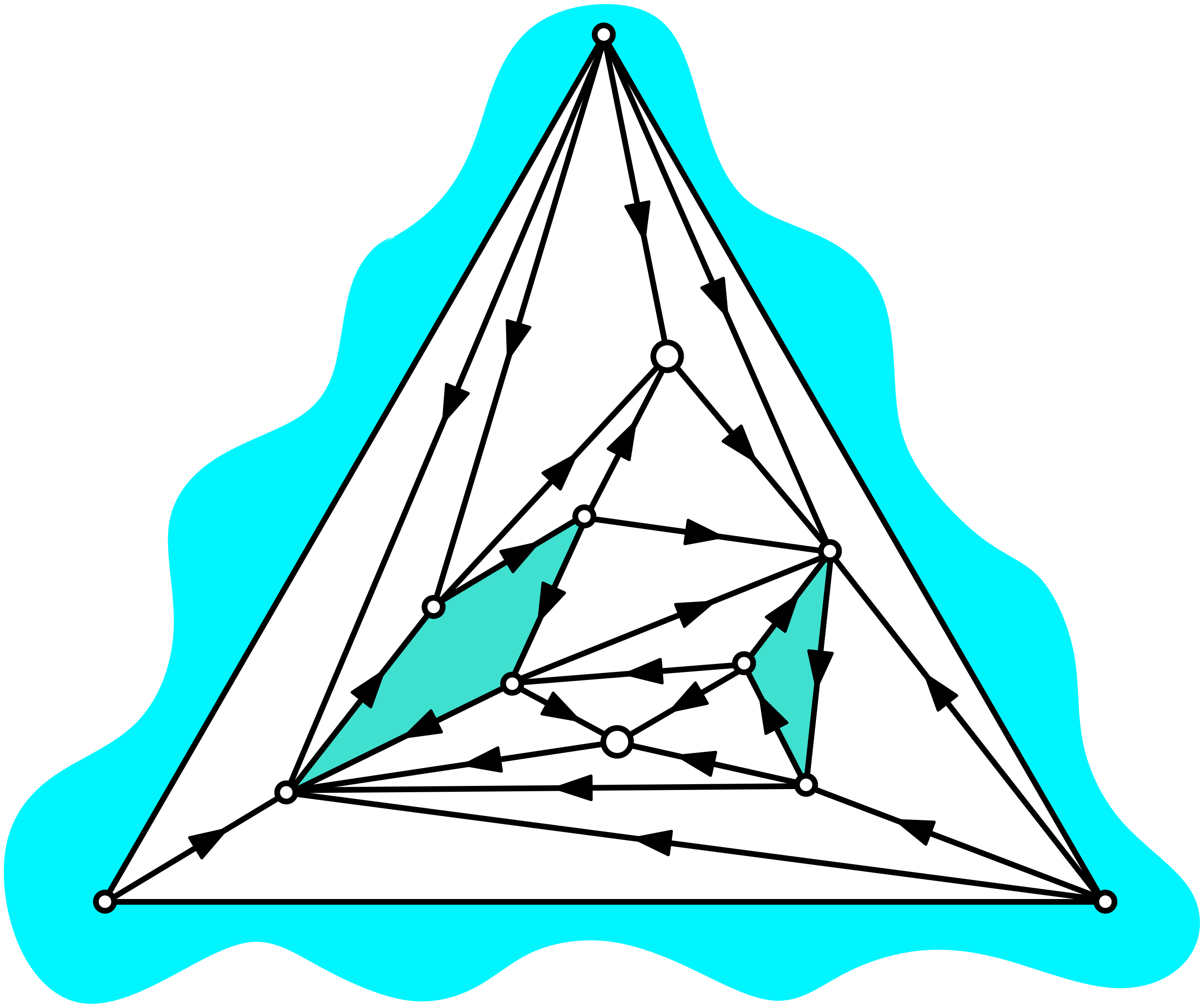
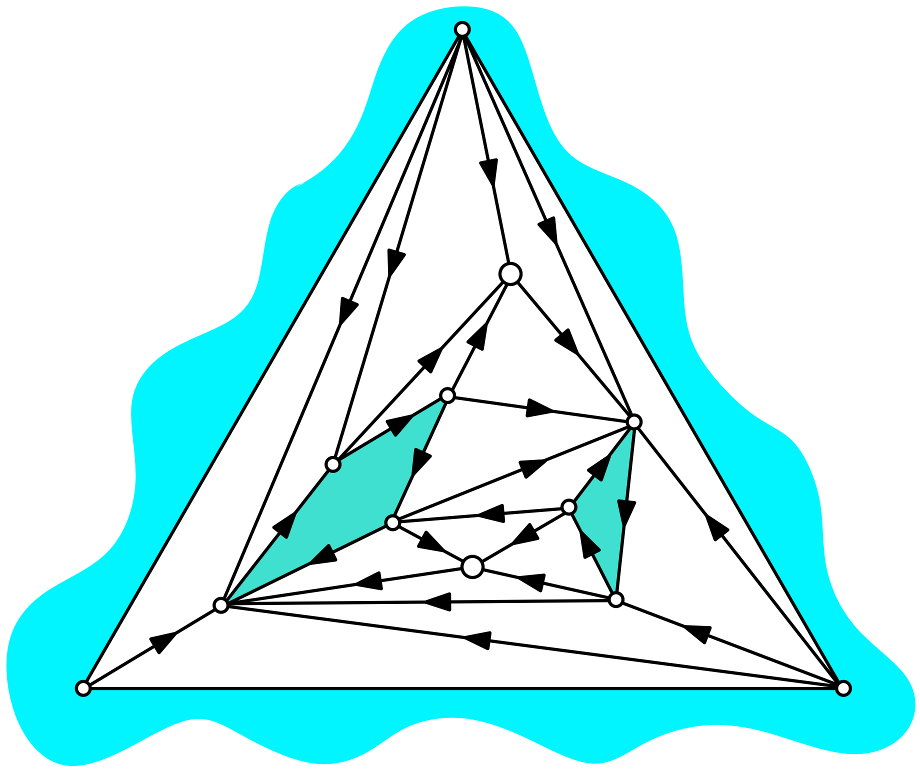
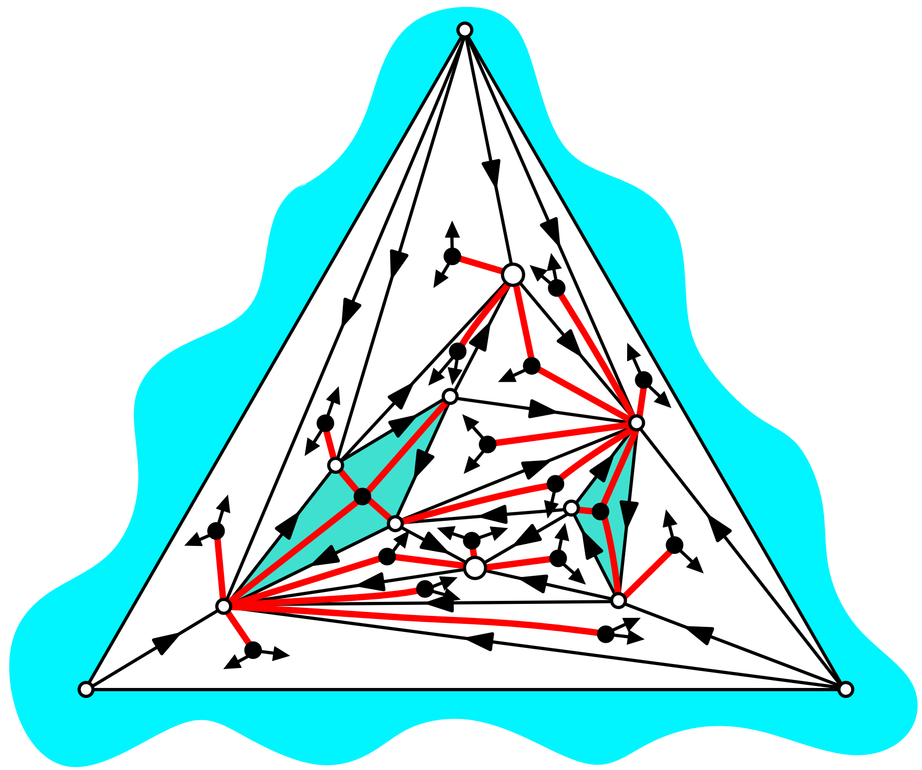


Mobiles for simple triangulations with boundaries

Apply the bijection Φ to the minimal pseudo-3-orientation



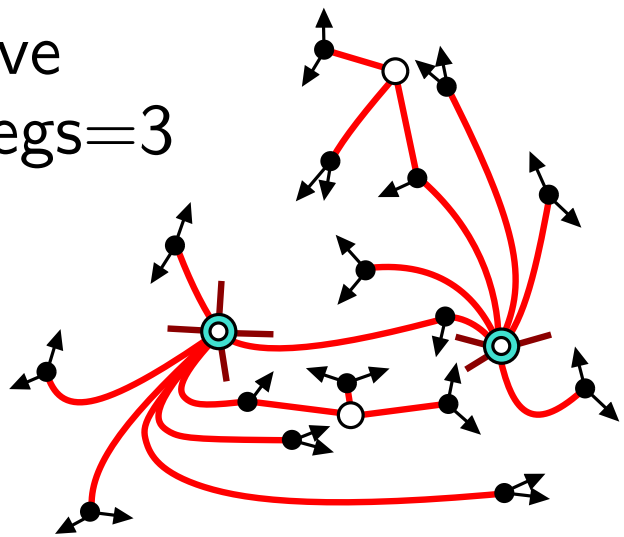
$\Rightarrow$



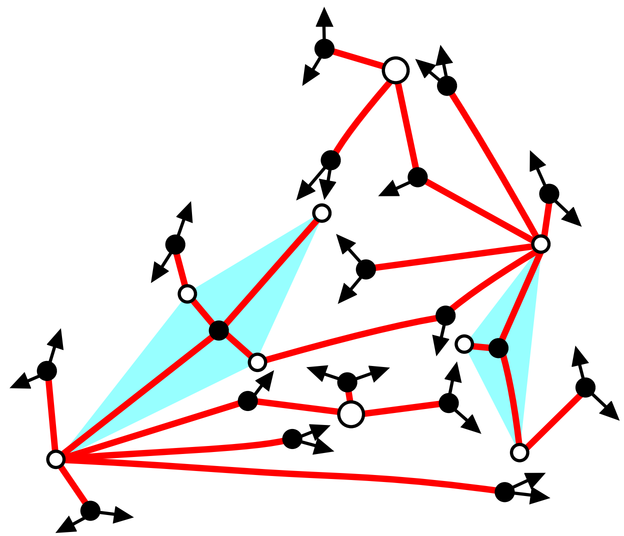
$\Downarrow$

white vertices have
#neighbours−#legs=3

black vertices
have degree 3



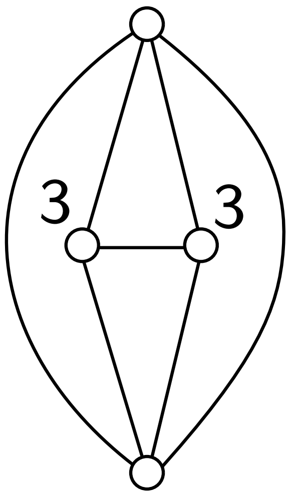
$\Leftarrow$



Obstacles for the existence of pseudo-3- orientations

Not all 2-cycles are forbidden!

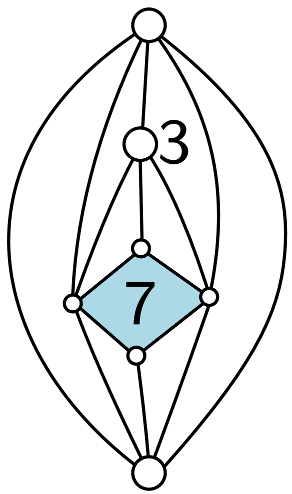
conctractible 2-cycle



5 edges inside
total indegree 6 inside

Forbidden

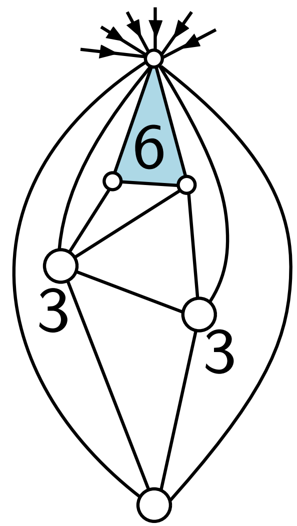
non-conctractible 2-cycle
not touching any boundary
from the inside



9 non-boundary edges inside
total indegree 10 inside

Forbidden

non-conctractible 2-cycle
touching a boundary
from the inside



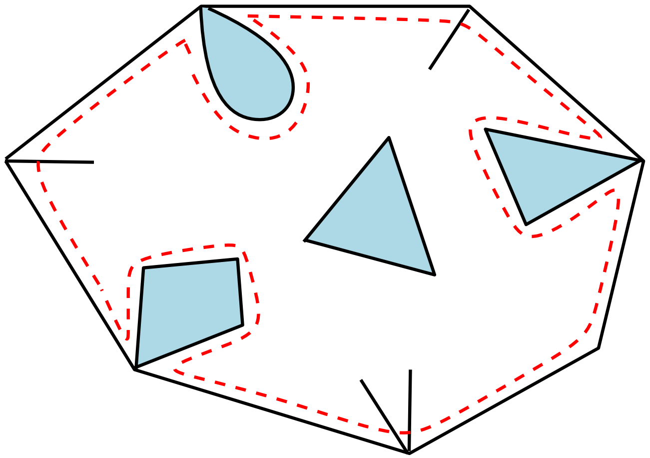
8 non-boundary edges inside
total indegree 6 inside

Not forbidden

Pseudo-girth parameter

For a map with boundaries that is planarly embedded

pseudo-girth = length of a shortest curve of the form



curve of length 15

(curve that is the outer border of a region consisting of non-boundary faces)

Rk: girth $\leq$ pseudo-girth $\leq$ contractible girth

Pseudo-girth parameter

For a map with boundaries that is planarly embedded

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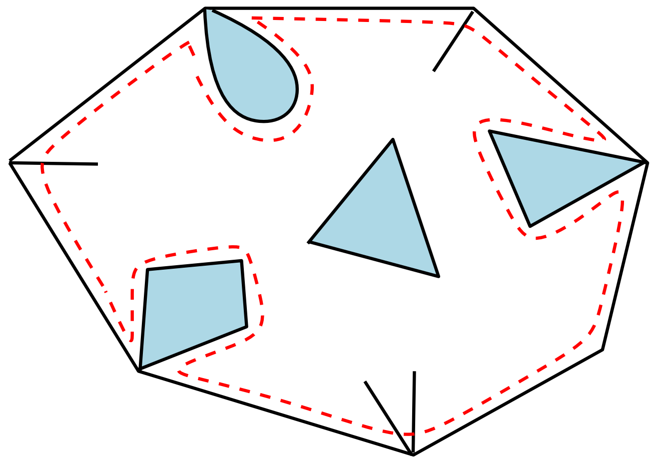


curve of length 15

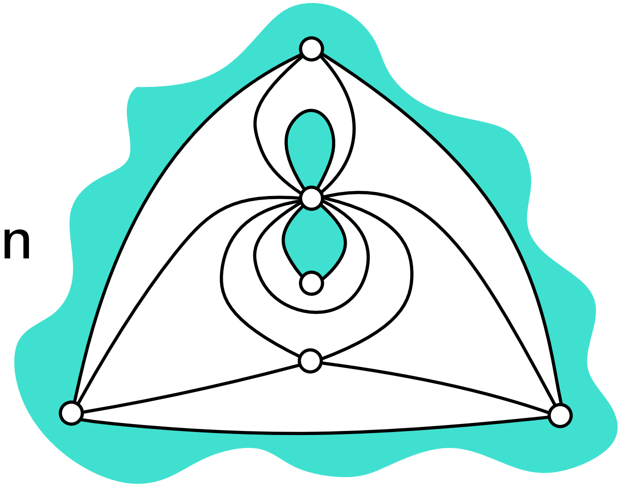
(curve that is the outer border of a region consisting of non-boundary faces)

Rk: girth $\leq$ pseudo-girth $\leq$ contractible girth

The map is called **pseudo-simple** if the pseudo-girth is ≥ 3

Results for pseudo-simple triangulations with boundaries

A triangulation with boundaries
(outer face being a triangular boundary-face)
is pseudo simple iff admits a pseudo 3–orientation



bijection with explicit mobiles

internal face (degree 3) ⟷ black vertex of degree 3

inner boundary of length i ⟷ white vertex with i legs and $i+3$ neighbours

Counting formula:

Let $N[n; a, k_1, \ldots, k_r]$ be the number of pseudo-simple triangulations where:

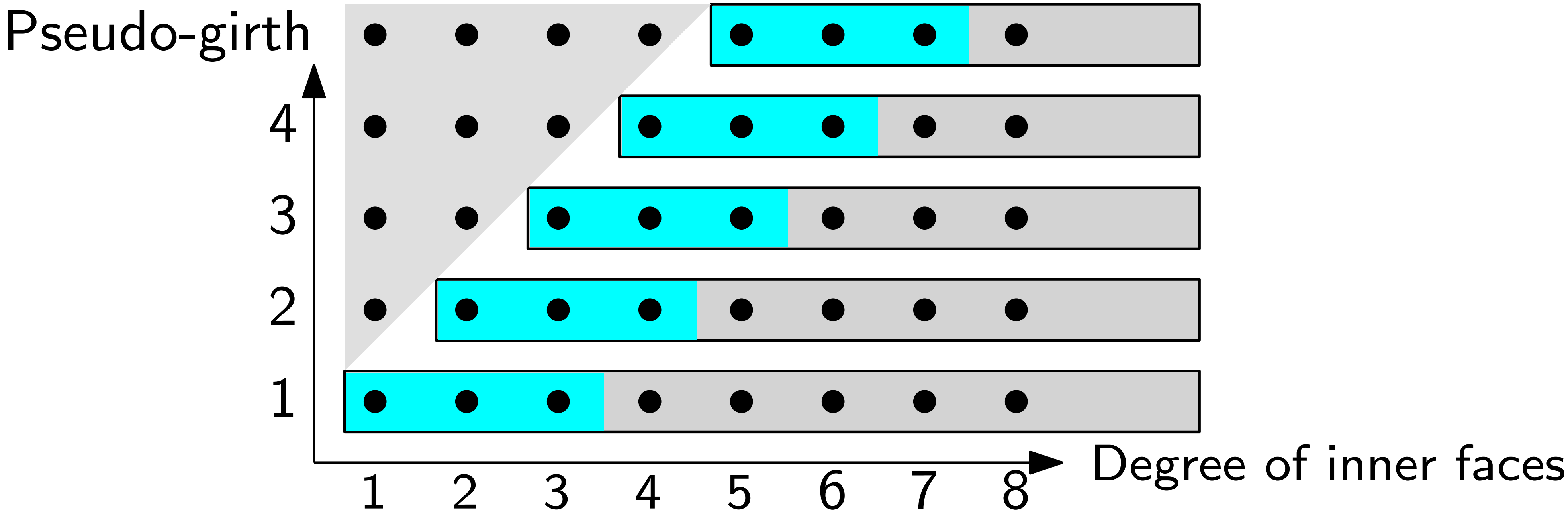
- the outer boundary has length a
- the inner boundaries $B_1, \ldots, B_r$ have lengths $k_1, \ldots, k_r$
- there are n internal vertices
- in every boundary, a vertex is distinguished

$$N[n; a; k_1, \ldots, k_r] = \frac{2(2a-3)!}{(a-3)!(a-1)!} \frac{(4n+4r+2L-5)!}{(n-1)!(3m+4r+2L-3)!} \prod_{i=1}^{r} i \binom{2i+2}{i}$$

where $L = a + \sum_{i=1}^{r} k_i$ (total boundary length)

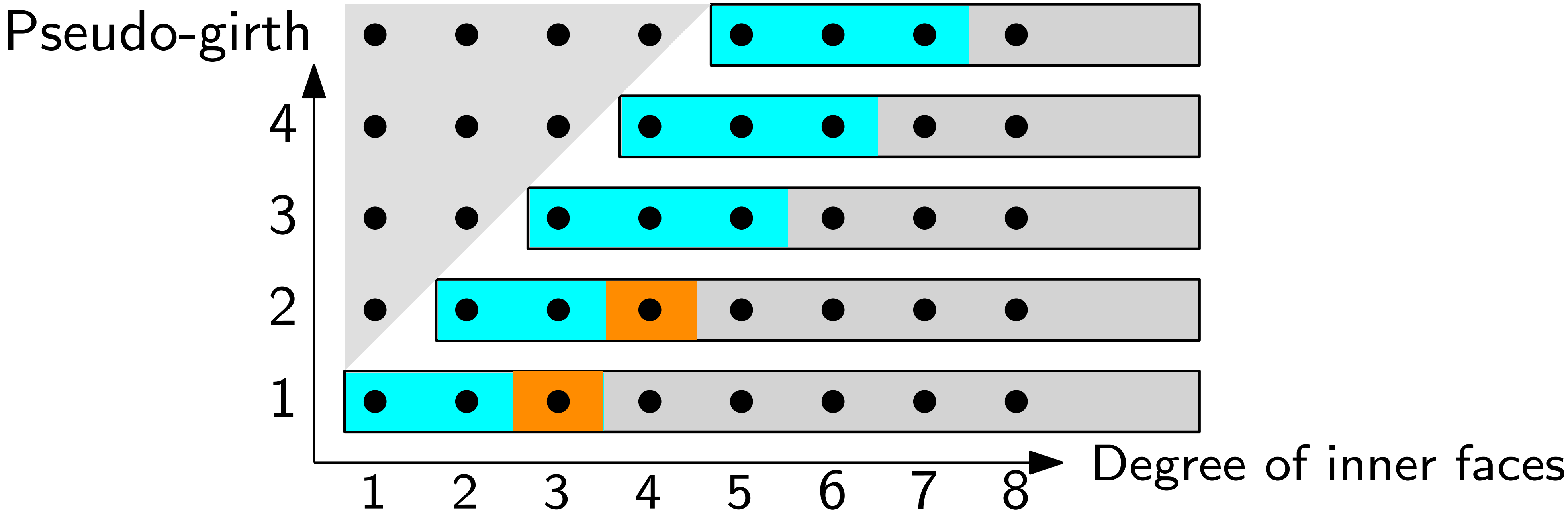
Results in any given pseudo-girth

We have a bijection in each pseudo-girth $d \geq 1$
for maps with boundaries, with inner face degrees in $\{d, d+1, d+2\}$



Results in any given pseudo-girth

[Bernardi, F'15] We have a bijection in each pseudo-girth $d \geq 1$ for maps with boundaries, with inner face degrees in $\{d, d+1, d+2\}$



Pseudo-girth-constraint is void for

$d = 1$ (recover Krikun's formula)

$d = 2$ bipartite case (new formula for quadrangulations with boundaries)

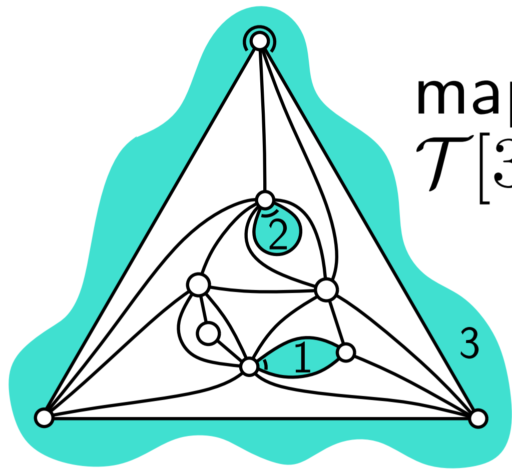
Factorized counting formulas

- Let $m \geq 0$ and $\ell_1, \ldots, \ell_r$ positive integers
- Let $\mathcal{T}[m; \ell_1, \ldots, \ell_r]$ (resp. $\mathcal{Q}[m; \ell_1, \ldots, \ell_r]$) be the set of triangulations (resp. quadrangulations) with r boundaries $B_1, \ldots, B_r$ s.t.
  - there are m internal vertices
  - every boundary B_i has length ℓ_i and a marked corner

Triangulations : Krikun's formula (2007)

$$|\mathcal{T}[m; a_1, \ldots, a_r]| = \frac{4^k (e-2)!!}{m!(2b+k)!!} \prod_{i=1}^{r} a_i \binom{2a_i}{a_i}$$

with $b = \sum_i a_i$, $k = r + m - 2$, and $e = 2b + 3k$

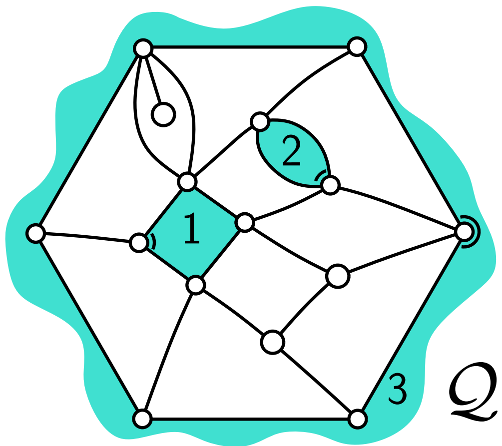


map $\in$ $\mathcal{T}[3; 2, 1, 3]$

Quadrangulations : [Bernardi, F'15]

$$|\mathcal{Q}[m; 2a_1, \ldots, 2a_r]| = \frac{3^k (e-1)!}{m!(3b+k)!} \prod_{i=1}^{r} 2a_i \binom{3a_i}{a_i}$$

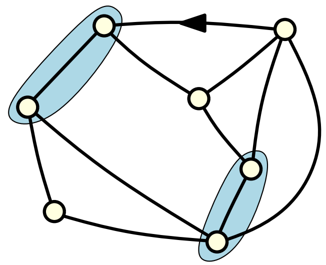
with $b = \sum_i a_i$, $k = r + m - 2$, and $e = 2b + 3k$



map $\in$ $\mathcal{Q}[3; 4, 2, 6]$

Solution of the dimer model on quadrangulations

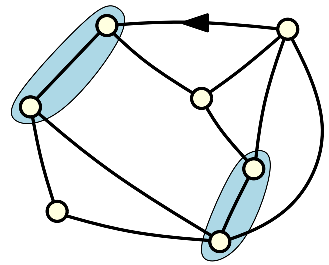
Map with dimers = pair (M, X) where M is a map and X is a subset of edges giving a partial-matching



map with 2 dimers

Solution of the dimer model on quadrangulations

Map with dimers = pair (M, X) where M is a map and X is a subset of edges giving a partial-matching

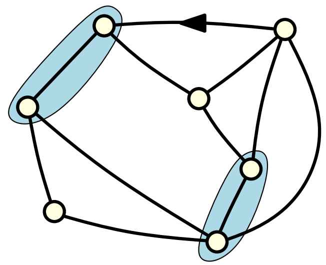


- Dimer model on (rooted) quadrangulations

Generating function : $F(t,w) = \sum_{\text{configurations}} t^{\#\text{faces}} w^{\#\text{dimers}}$

Solution of the dimer model on quadrangulations

Map with dimers = pair (M, X) where M is a map and X is a subset of edges giving a partial-matching



map with 2 dimers

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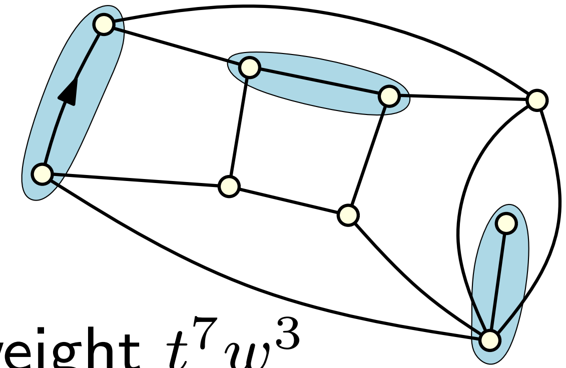
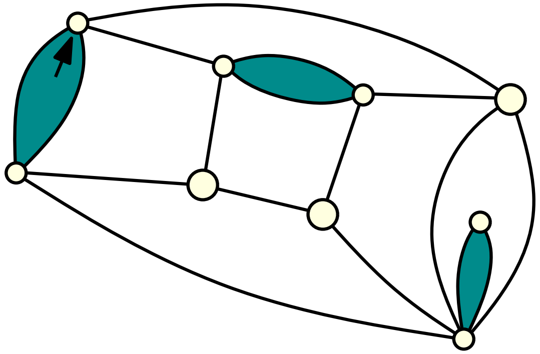
dimer ⟷ boundary of length 2



⇔



weight $t^7 w^3$

bijection

⇓

$$F(t,w) = R - 1 - t\,R^3 - 6wt^2R^6$$

où $R = 1 + 3tR^2 + 9wt^2R^5$

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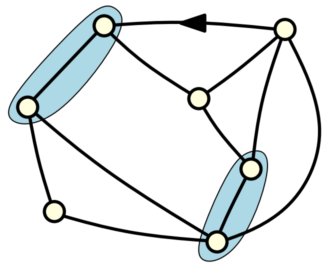
Map with dimers = pair (M, X) where M is a map and X is a subset of edges giving a partial-matching



- Dimer model on (rooted) quadrangulations

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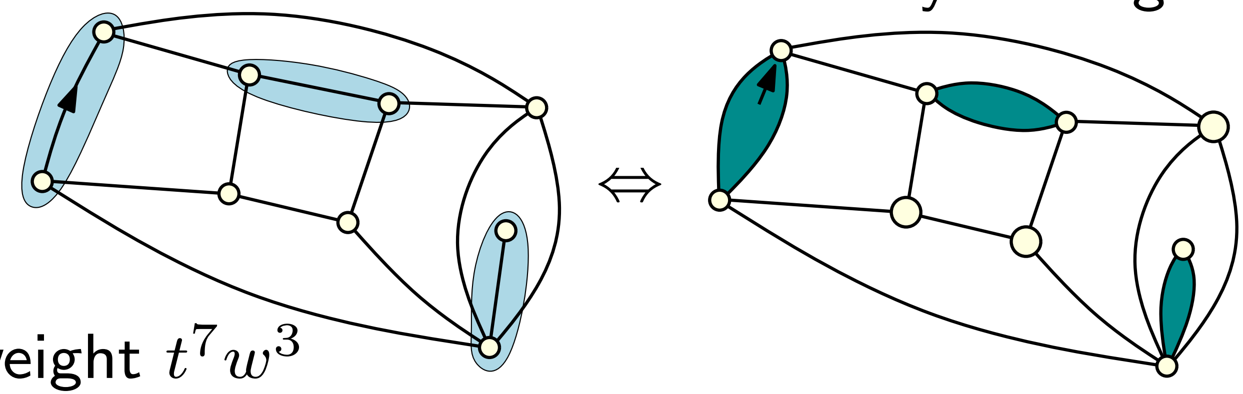


weight $t^7 w^3$

bijection $\Downarrow$

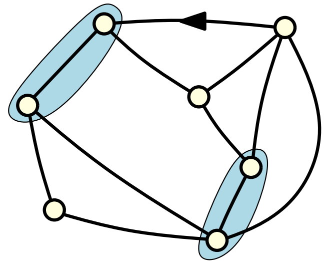
$$F(t,w) = R - 1 - t\,R^3 - 6wt^2R^6$$
où $R = 1 + 3tR^2 + 9wt^2R^5$

Asymptotics : for $w \in \mathbb{R}$ fixed, $[t^n]F \sim c_w\, \gamma_w^n\, n^{-5/2}$

except at critical weight $w_0 = -3/125$ where $[t^n]F \sim c_0\, \gamma_0{}^n\, n^{-7/3}$

Solution of the dimer model on quadrangulations

Map with dimers = pair (M, X) where M is a map and X is a subset of edges giving a partial-matching



map with 2 dimers

- Dimer model on (rooted) quadrangulations

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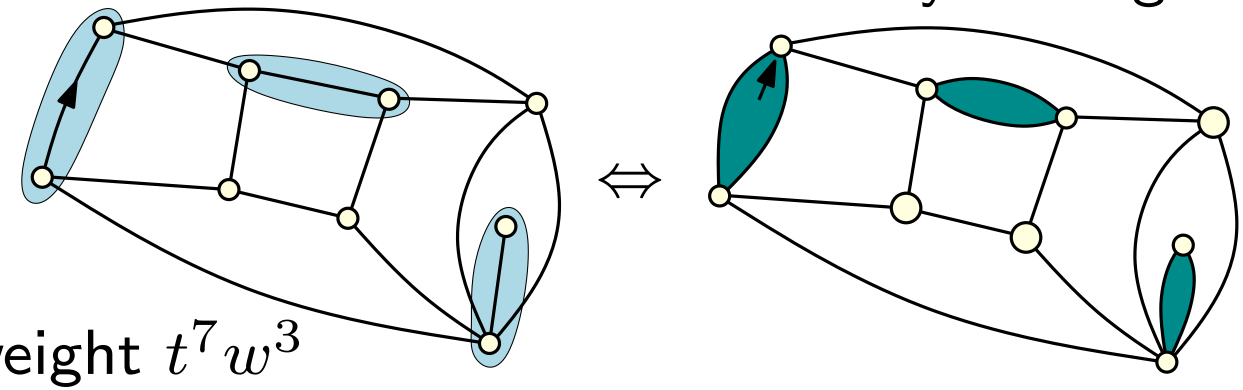

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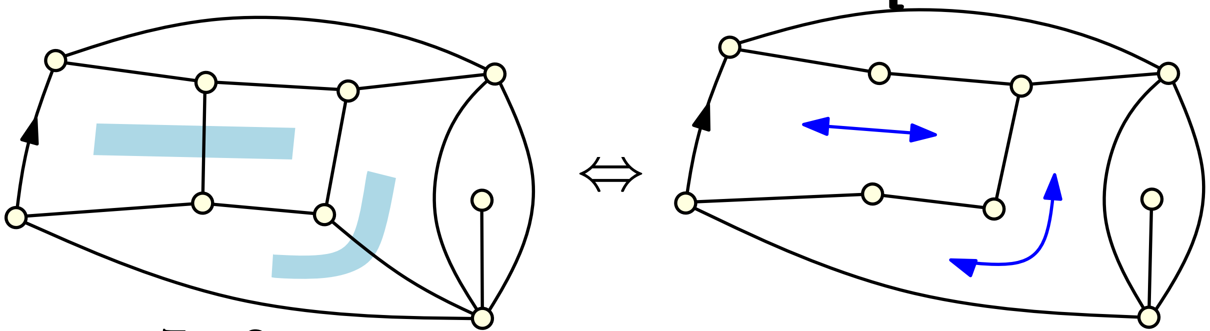
- Solution of the dual model in **[Bouttier, Di Francesco, Guitter'03]**



poids $t^7 w^2$

weight t per square face
weight $3t^2w$ per hexagonal face

$$F(t,w) = R - 1 - t\,R^3 - 15wt^2R^4$$
où $R = 1 + 3tR^2 + 30wt^2R^3$

critical weight $w_0 = -1/10$

where typical distance $\approx n^{1/6}$