# La fonction à deux points et à trois points des quadrangulations et cartes

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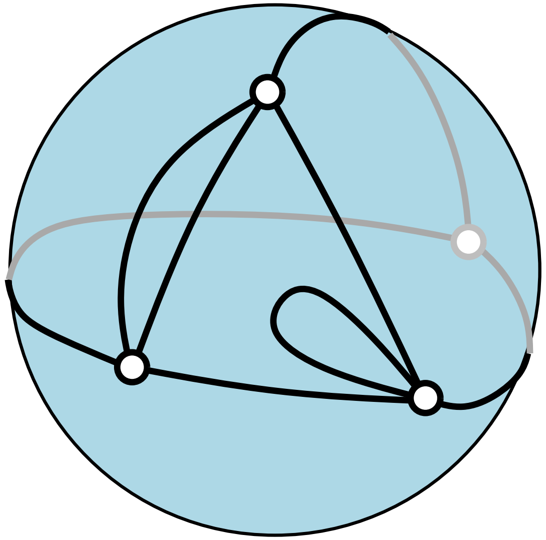
Travaux avec Jérémie Bouttier et Emmanuel Guitter

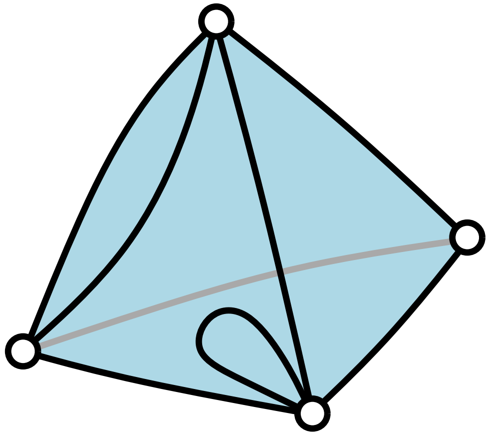
Séminaire Calin, LIPN, Mai 2014

# Maps

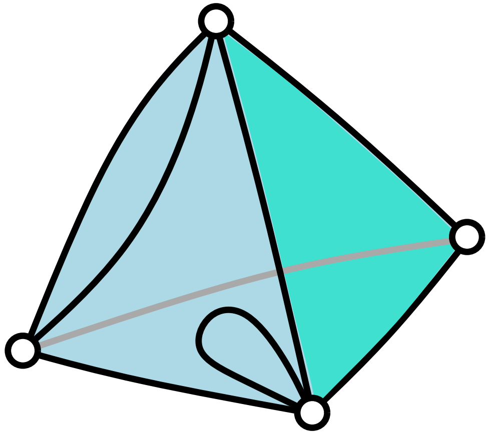
**Def.** Planar map = connected graph embedded on the sphere



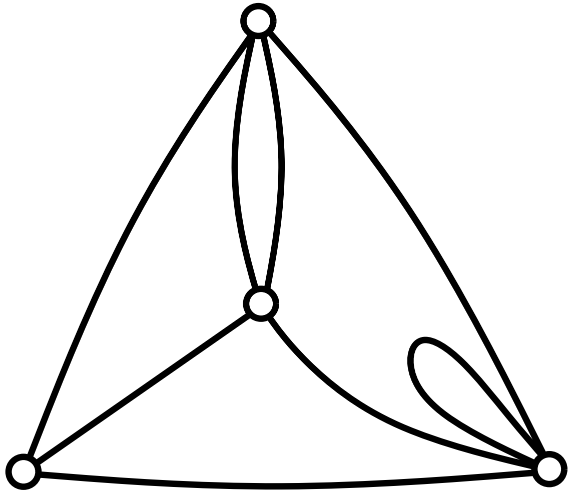
$=$



Easier to draw in the plane (by choosing a face to be the outer face)



$\Rightarrow$



# Maps as random discrete surfaces

Natural questions:

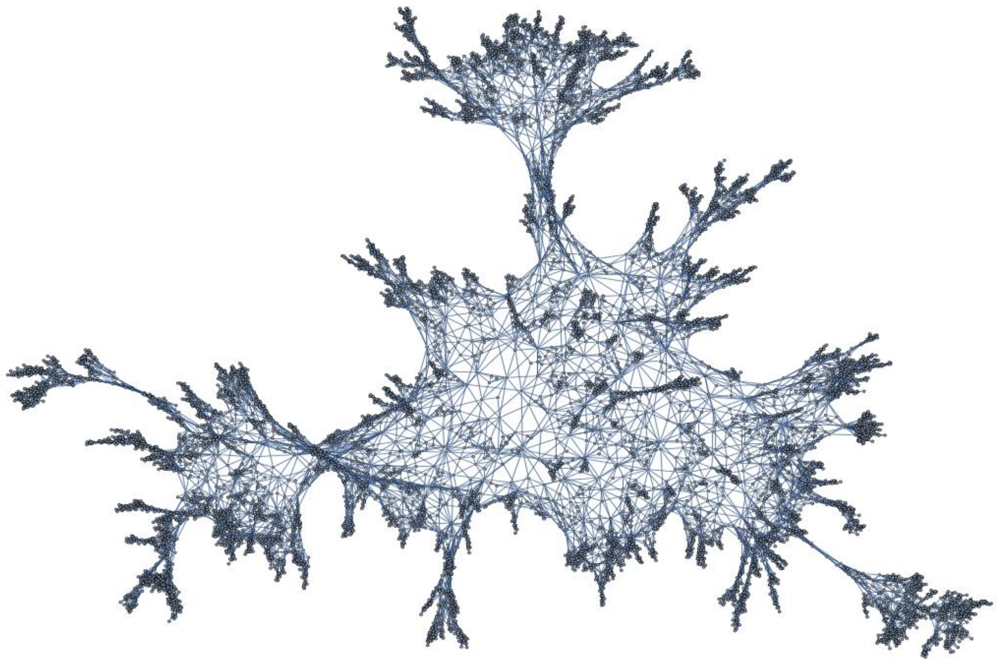
- Typical distance between (random) vertices in random maps
  the order of magnitude is $n^{1/4}$ ($\neq n^{1/2}$ in random trees)

random quadrang. $\begin{cases} \text{- [Chassaing-Schaeffer'04] probabilistic} \\ \text{- [Bouttier Di Francesco Guitter'03] exact GF expressions} \end{cases}$

- How does a random map (rescaled by $n^{1/4}$) "look like" ?

convergence to the "Brownian map"

[Le Gall'13, Miermont'13]



# Counting (rooted) maps

with a marked corner

- Very simple counting formulas ([Tutte'60s]), for instance
  Let $q_n = \#\{\text{rooted quadrangulations with } n \text{ faces}\}$
  $m_n = \#\{\text{rooted maps with } n \text{ edges}\}$

  Then $m_n = q_n = \frac{2}{n+2} 3^n \frac{(2n)!}{n!(n+1)!}$

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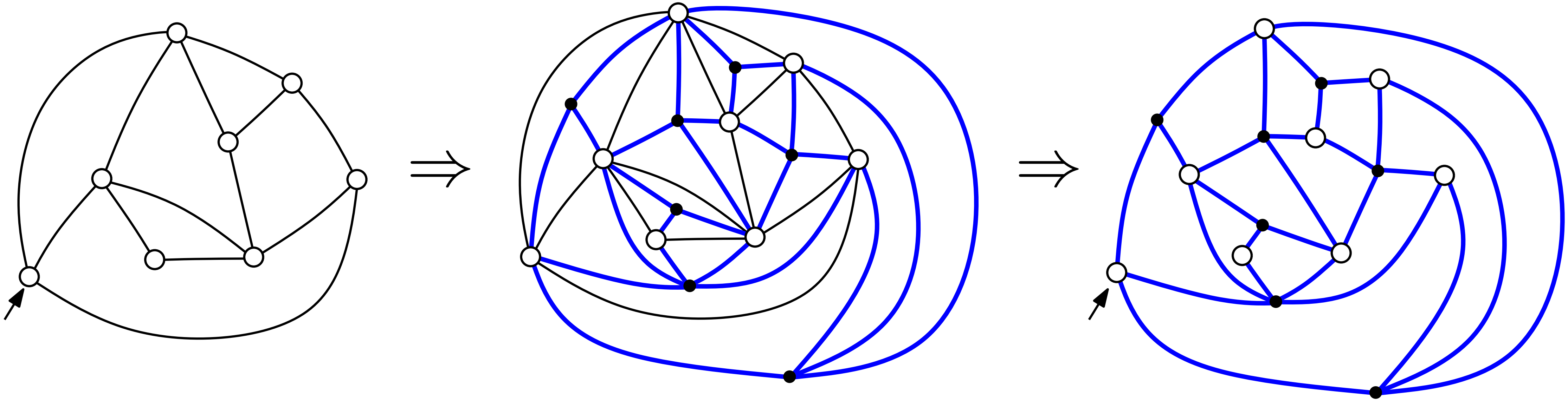

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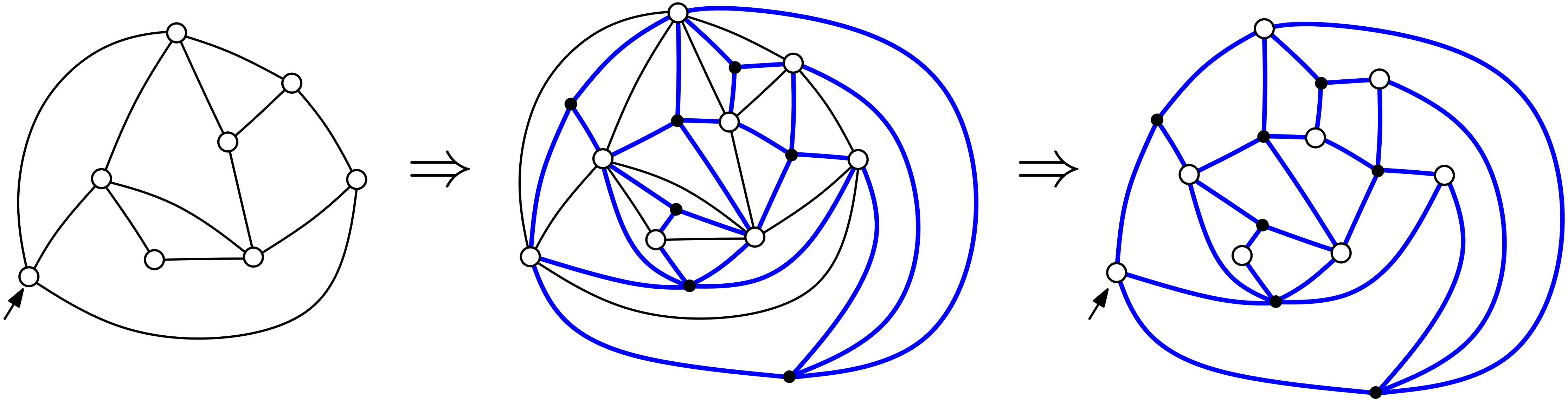

But this bijection does not preserve distance-parameters (only bounds)

# The $k$-point function

- Let $\mathcal{M} = \cup_n \mathcal{M}[n]$ be a family of maps (quadrangulations, general, ...) where $n$ is a size-parameter ($\#$ faces for quad., $\#$ edges for gen. maps)
- Let $\mathcal{M}^{(k)}$ = family of maps from $\mathcal{M}$ with $k$ marked vertices $v_1, \ldots, v_k$

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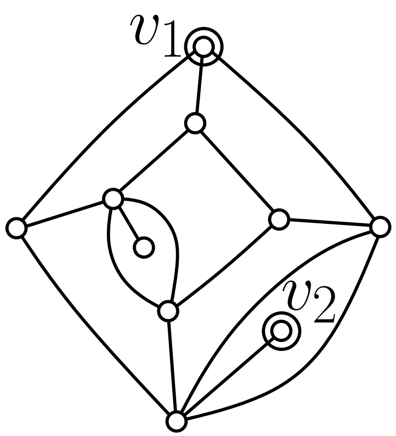
**Refinement by distances :**

For $D = (d_{i,j})_{1 \leq i < j \leq k}$ any $\binom{k}{2}$-tuple of positive integers

let $\mathcal{M}_D^{(k)} :=$ subfamily of $\mathcal{M}^{(k)}$ where $\mathrm{dist}(v_i, v_j) = d_{ij}$ for $1 \leq i < j \leq k$

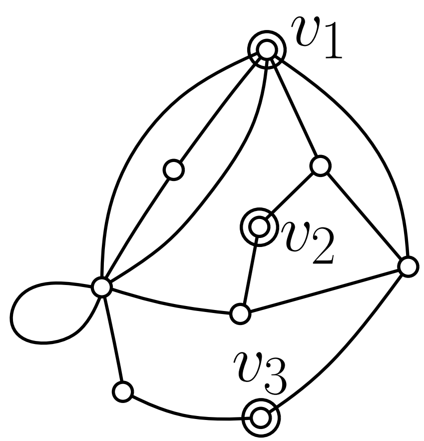
The counting series $G_D \equiv G_D(g)$ of $\mathcal{M}_D^{(k)}$ with respect to the size is called the **$k$-point function** of $\mathcal{M}$




$k = 2$

$d_{12} = 3$

quadrangulation




$k = 3$

$d_{12} = 2$

$d_{13} = 2$

$d_{23} = 3$

general map

# Exact expressions for the $k$-point function

- For the two-point functions:

  - quadrangulations **[Bouttier Di Francesco Guitter'03]**

  - maps with prescribed (bounded) face-degrees **[Bouttier Guitter'08]**

  - general maps **[Ambjørn Budd'13]**

  - general hypermaps, general constellations **[Bouttier F Guitter'13]**

- For the three-point functions

  - quadrangulations **[Bouttier Guitter'08]**

  - general maps & bipartite maps **[F Guitter'14]**

# Exact expressions for the $k$-point function

## Outline of the talk

- For the two-point functions:

① - quadrangulations **[Bouttier Di Francesco Guitter'03]**
uses Schaeffer's bijection

- maps with prescribed (bounded) face-degrees **[Bouttier Guitter'08]**

③ - general maps **[Ambjørn Budd'13]**
based on clever observation on Miermont's bijection

- general hypermaps, general constellations **[Bouttier F Guitter'13]**

- For the three-point functions

② - quadrangulations **[Bouttier Guitter'08]**
uses Miermont's bijection

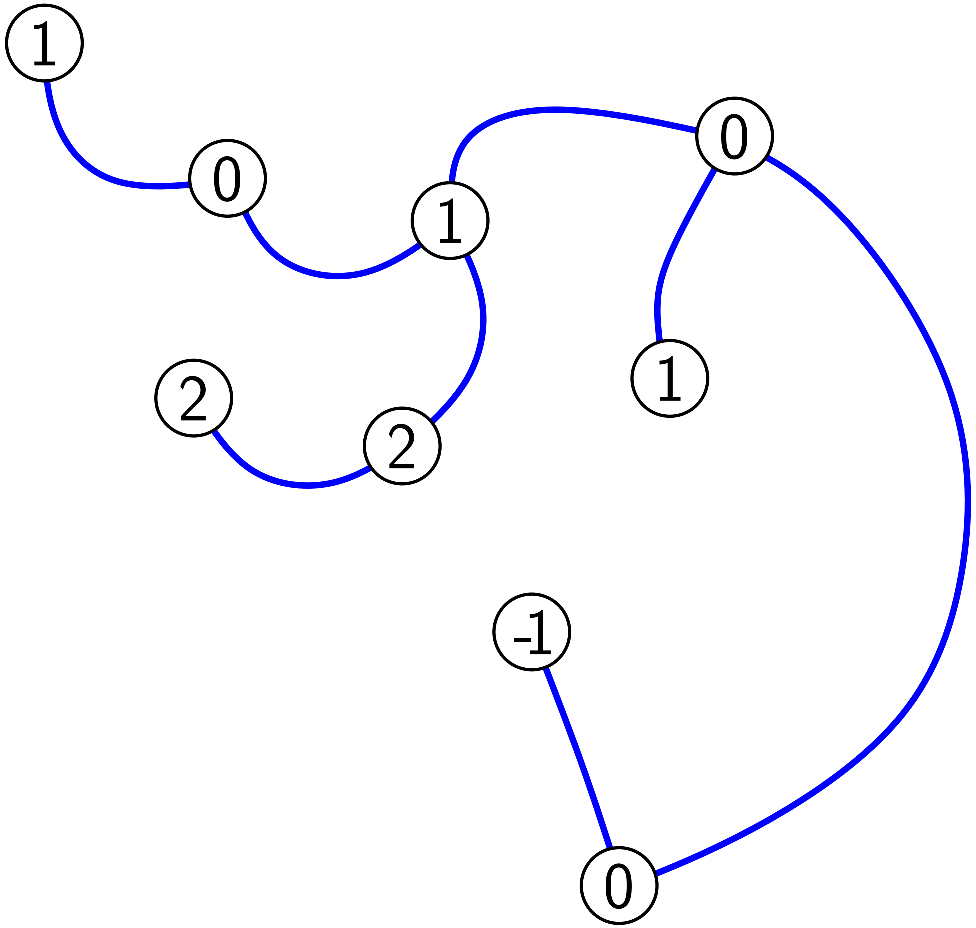
④ - general maps & bipartite maps **[F Guitter'14]**
uses AB bijection

## Computing the two-point function of quadrangulations using the Schaeffer bijection

# Well-labelled trees

Well-labelled tree = plane tree where

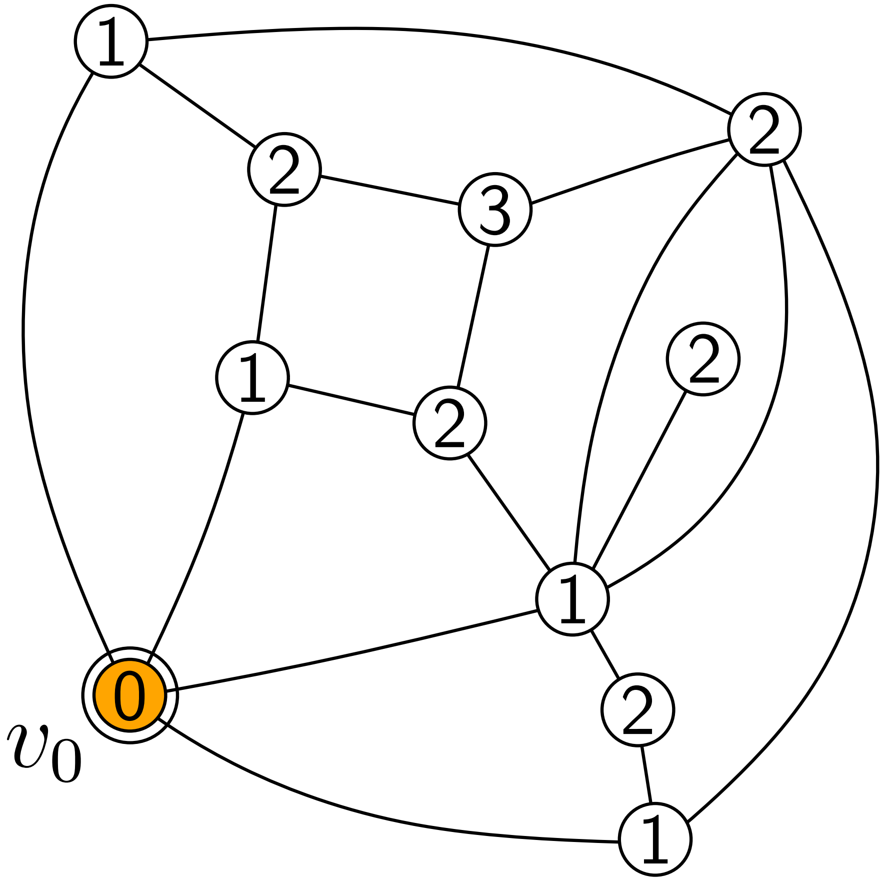
- each vertex $v$ has a label $\ell(v) \in \mathbb{Z}$
- each edge $e = \{u, v\}$ satisfies $|\ell(u) - \ell(v)| \leq 1$



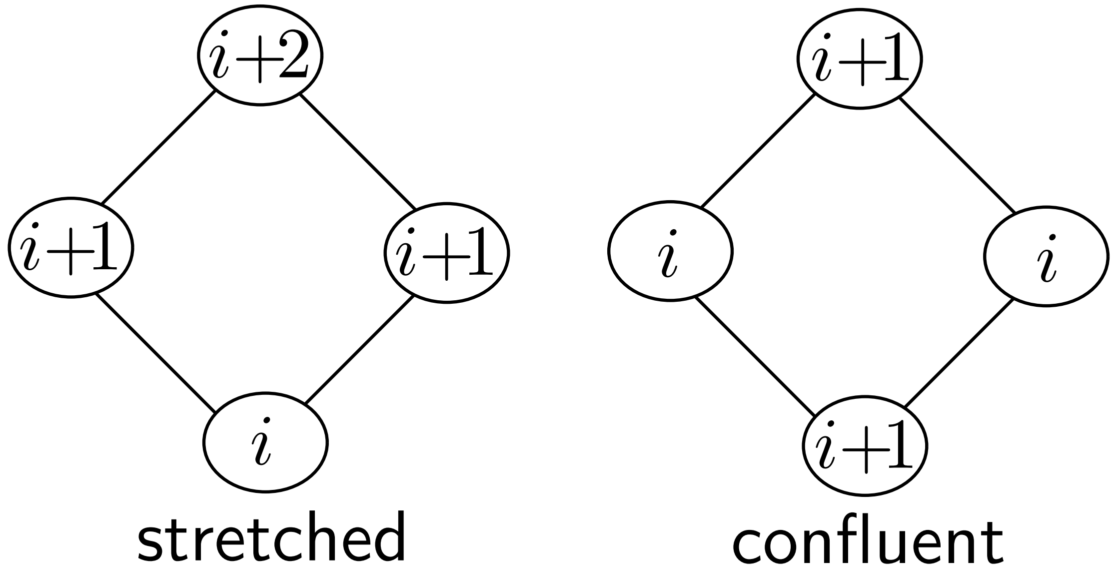
# Pointed quadrangulations, geodesic labelling

Pointed quadrangulation = quadrangulation with a marked vertex $v_0$

Geodesic labelling with respect to $v_0$: $\ell(v) = \mathrm{dist}(v_0, v)$




**Rk:** two types of faces

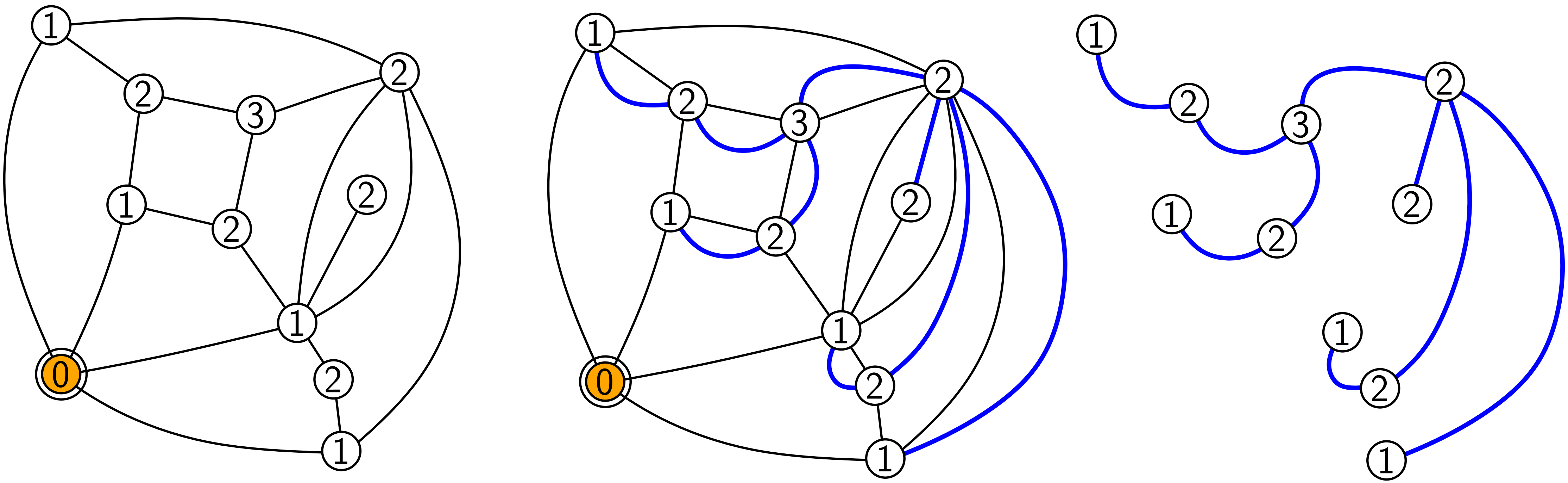


# The Schaeffer bijection

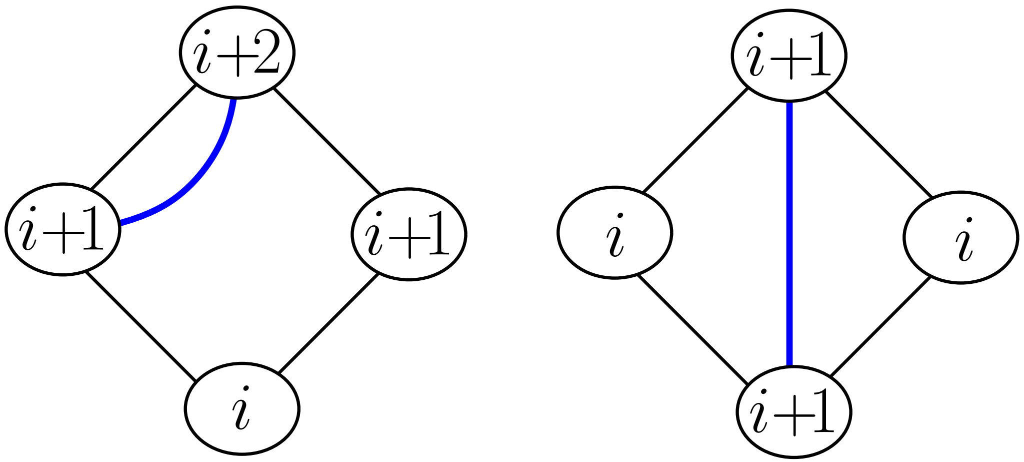
**[Schaeffer'99]**, also **[Cori-Vauquelin'81]**

Pointed quadrangulation $\Rightarrow$ well-labelled tree with min-label=1

$n$ faces $\qquad$ $n$ edges



Local rule in each face:

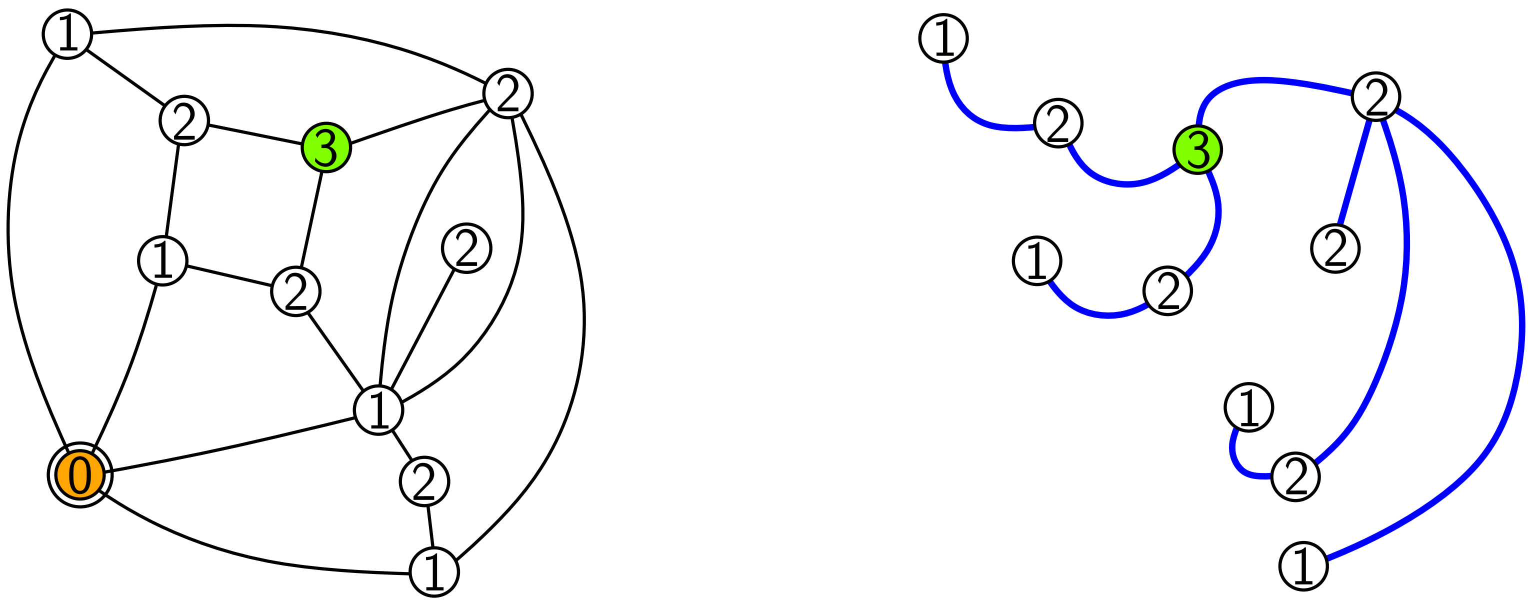


# The 2-point function of quadrangulations (1)

Denote by $G_d \equiv G_d(g)$ the two-point function of quadrangulations

bijection $\Rightarrow G_d(g) =$ GF of well-labelled trees with min-label=1
and with a marked vertex of label $d$

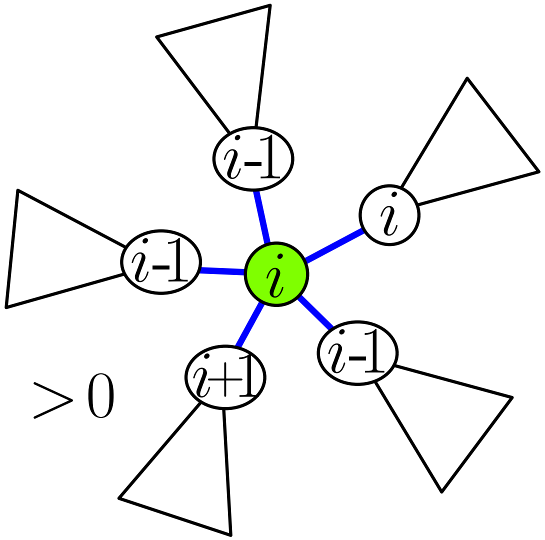
$d = 3$



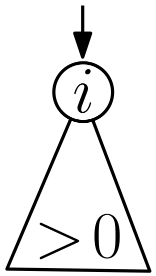

**Rk:** $G_d = F_d - F_{d-1} = \Delta_d F_d$

where $F_d \equiv F_d(g) =$ GF of well-labelled trees with positive labels
and with a marked vertex of label $d$

# The 2-point function of quadrangulations (2)




$$\Rightarrow \quad F_i = \log \frac{1}{1 - g(R_{i-1} + R_i + R_{i+1})}$$

with $R_i =$ 


GF rooted well-labelled trees with positive labels and label $i$ at the root

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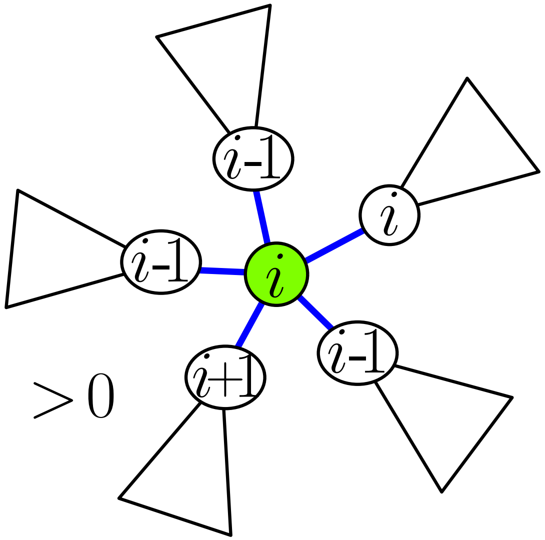


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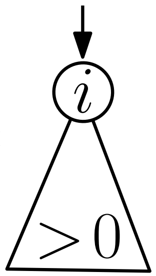
with $R_i =$ 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Equ. for $R_i$: $\boxed{R_i = \frac{1}{1-g(R_{i-1}+R_i+R_{i+1})}}$ (so $F_i = \log(R_i)$, $G_d = \log(\frac{R_d}{R_{d-1}})$)

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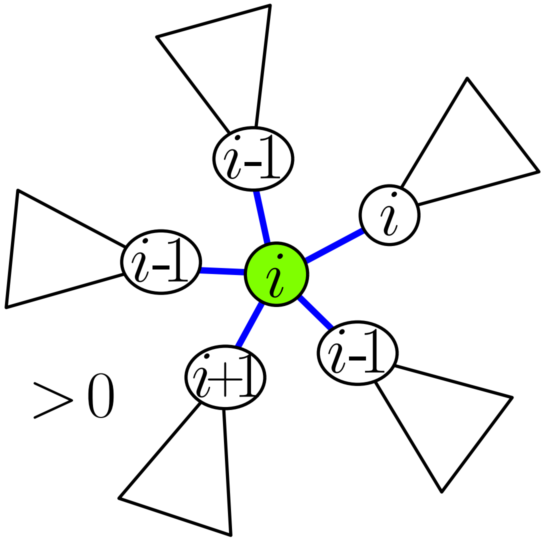


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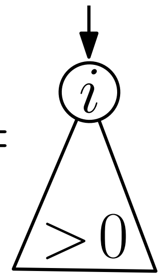
with $R_i =$ 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- Exact expression for $R_i$ **[BDG'03]**

$$\boxed{R_i = R\frac{[i]_x[i+3]_x}{[i+1]_x[i+2]_x}}$$

with the notation $[i]_x = \frac{1-x^i}{1-x}$

with $R \equiv R(g)$ and $x \equiv x(g)$ given by $\begin{cases} R = 1 + 3gR^2 \\ x = gR^2(1+x+x^2) \end{cases}$

$R(g) = \frac{1-S}{6g} \qquad x(g) = \frac{\sqrt{6}}{2} \frac{S^{1/2}\sqrt{1-(1+6g)S}-S-24g+1}{-1+S+6g}$ with $S = \sqrt{1-12g}$

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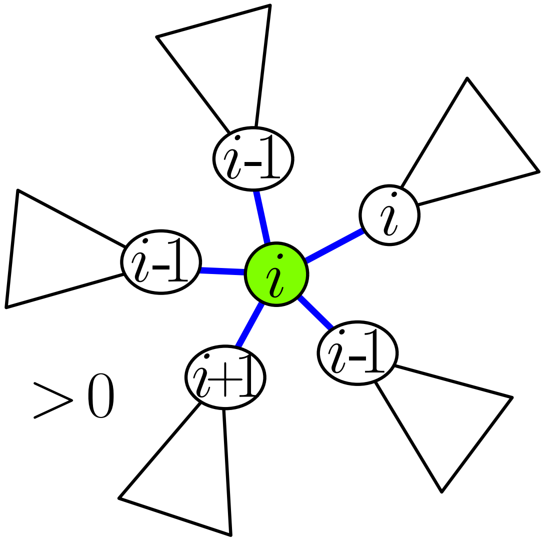


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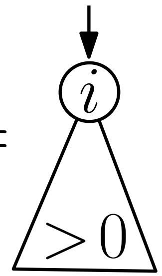
with $R_i =$ 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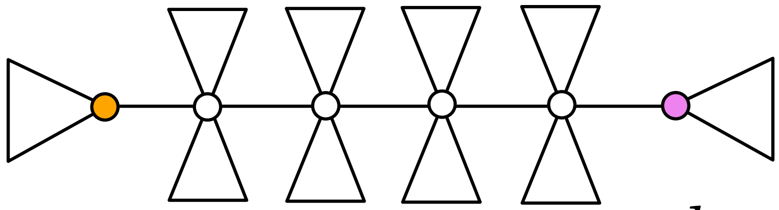
Final 2-point function expression: $\boxed{G_d = \log\left(\frac{[d]_x^2[d+3]_x}{[d-1]_x[d+2]_x^2}\right)}$

# Asymptotic considerations

- Two-point function of (plane) trees:

$$\boxed{G_d(g) = (gR^2)^d}$$

with $R = 1 + gR^2 = \frac{1-\sqrt{1-4g}}{2g}$



$d = 5$

$G_d$ is the $d$ th power of a series having a square-root singularity

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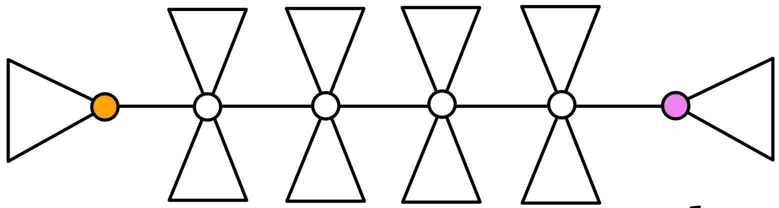


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- Two-point function of quadrangulations:

$$\boxed{G_d(g) \sim_{d\to\infty} a_1 x^d + a_2 x^{2d} + \cdots}$$

where $x = x(g)$ has a quartic singularity

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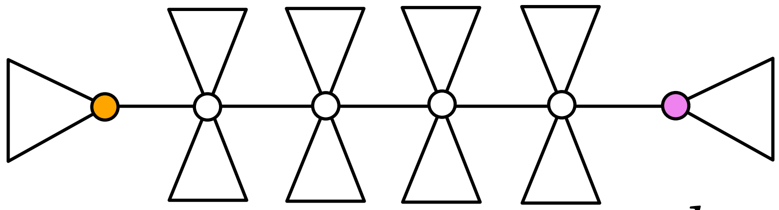

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Convergence in the two cases "follows" from (proof by Hankel contour) **[Banderier, Flajolet, Louchard, Schaeffer'03]**: for $0 < s < 1$,

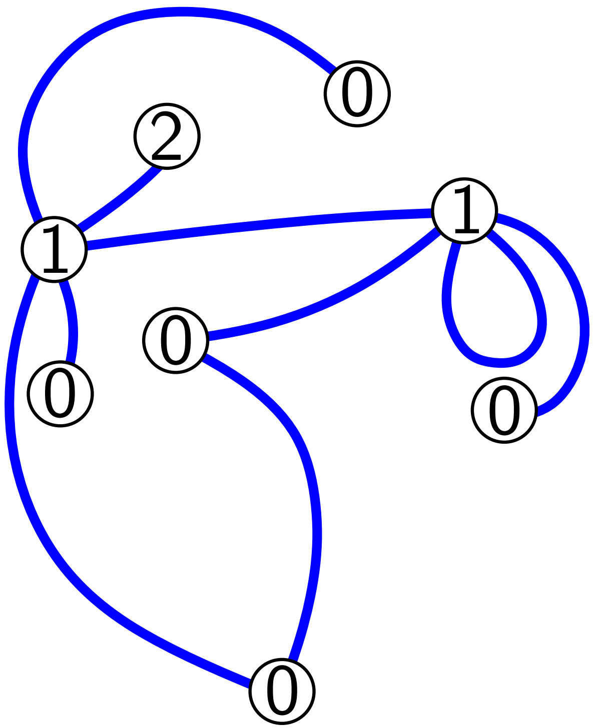
$$x(g) \underset{g\to 1}{\sim} 1 - (1-g)^s \Rightarrow [g^n]x^{\alpha n^s} \sim \frac{1}{2\pi n}\int_0^\infty e^{-t}\mathrm{Im}(\exp(-\alpha t^s e^{i\pi s}))\mathrm{d}t$$

# Computing the two-point and three-point function of quadrangulations using Miermont's bijection

# Well-labelled maps

Well-labelled map $=$ map where

- each vertex $v$ has a label $\ell(v) \in \mathbb{Z}$
- each edge $e = \{u, v\}$ satisfies $|\ell(u) - \ell(v)| \leq 1$




a well-labelled map $M$ with $3$ faces

**Rk:** Well-labelled tree $=$ well-labelled map with one face

# Very-well-labelled quadrangulations

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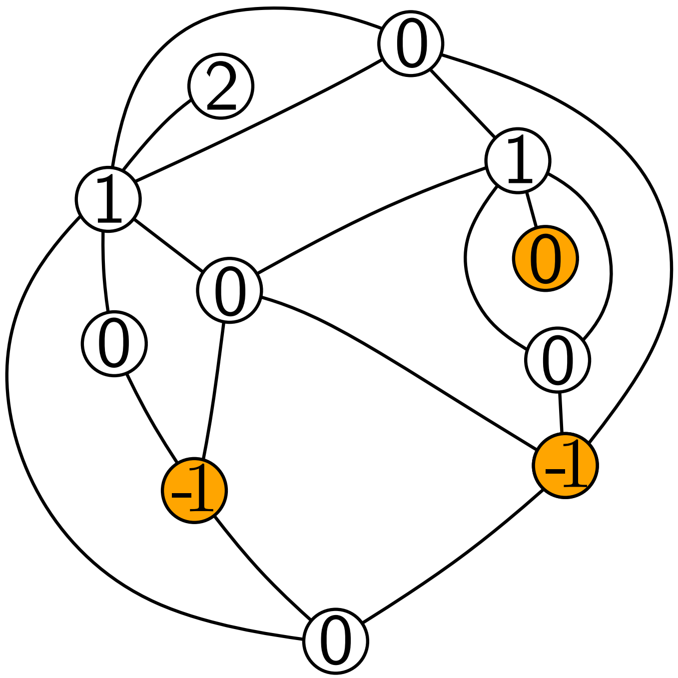
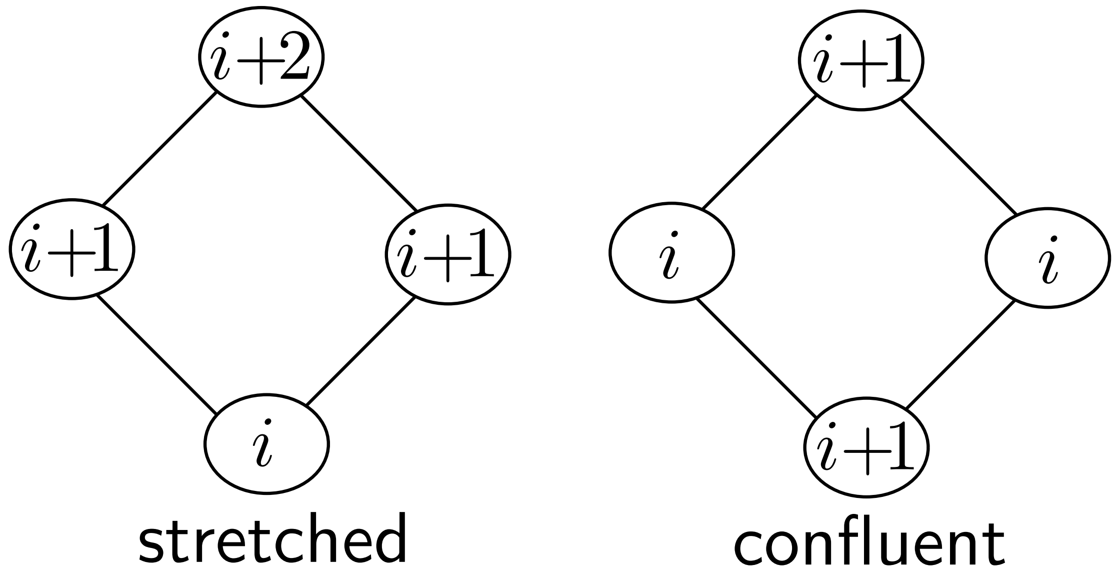



a very-well-labelled quadrangulation $Q$ with $3$ local min

**Rk:** two types of faces



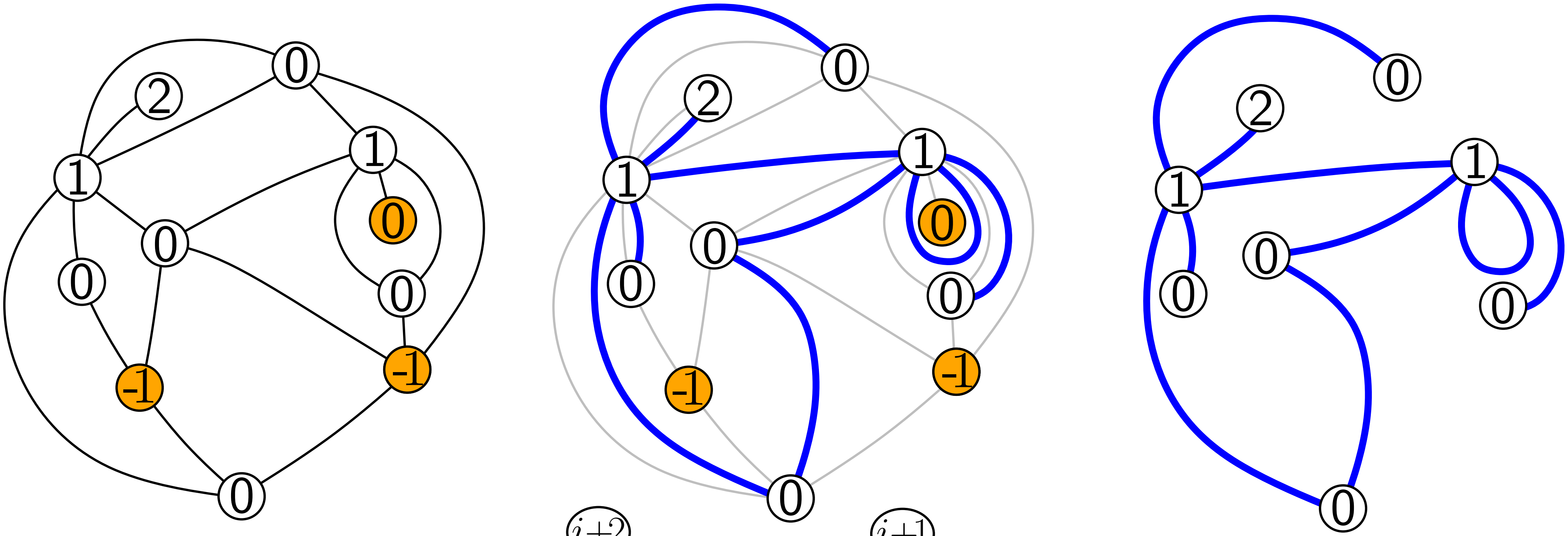

**Def:** local min= vertex with all neighbours of larger label

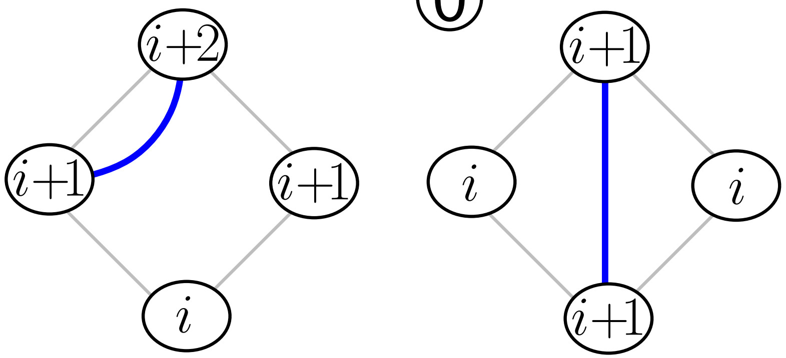
**Rk:** Geodesic labelling $\Leftrightarrow$ there is just one local min, of label $0$

# The Miermont bijection [Miermont'07], [Ambjørn, Budd'13]

Very-well labelled quadrangulation $Q$ $\Rightarrow$ well-labelled map $M$

$n$ faces $\qquad\qquad$ $n$ edges

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local min $v$ $\quad \overset{}{\underset{\ell(v)=\min(f)-1}{\longleftrightarrow}} \quad$ face $f$

non-local min $\quad \underset{\text{same label}}{\longleftrightarrow} \quad$ vertex

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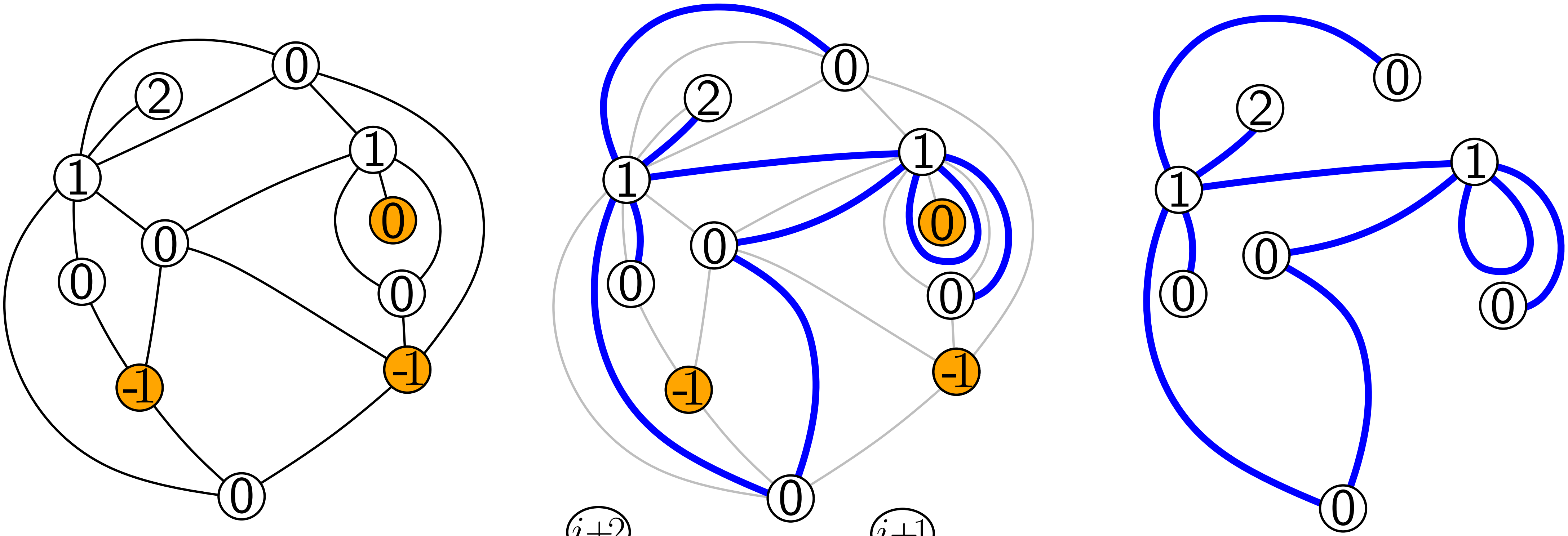

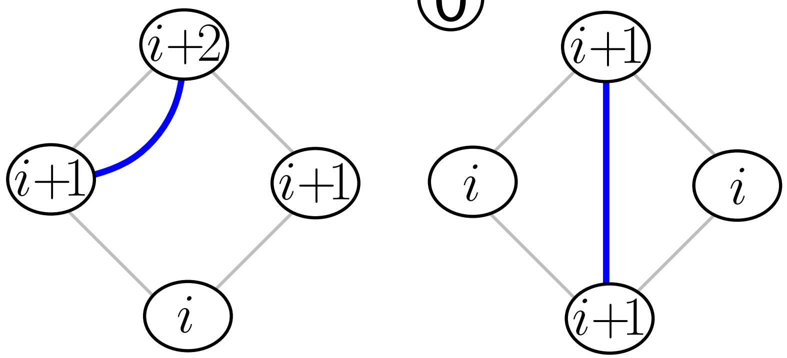


recover the Schaeffer bijection (case of one local min, of label $0$)

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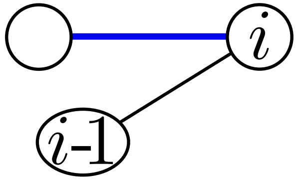
local min $v$ $\quad \overset{\longleftrightarrow}{\ell(v) = \min(f) - 1}$ $\quad$ face $f$

non-local min $\quad \underset{\text{same label}}{\longleftrightarrow}$ $\quad$ vertex

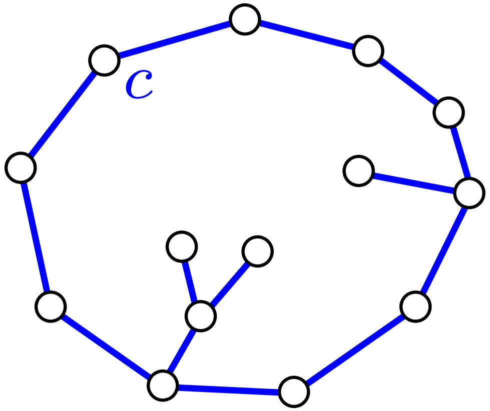
# Proof of the stated properties


implies




(follows from the local rules)

$\Rightarrow$




From each corner $c$ in a “face” of $M$ starts a label-decreasing path of $Q$ that stays in the face and ends at a local min of $Q$

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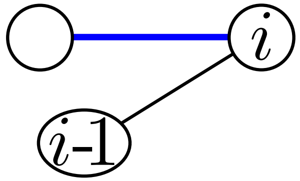
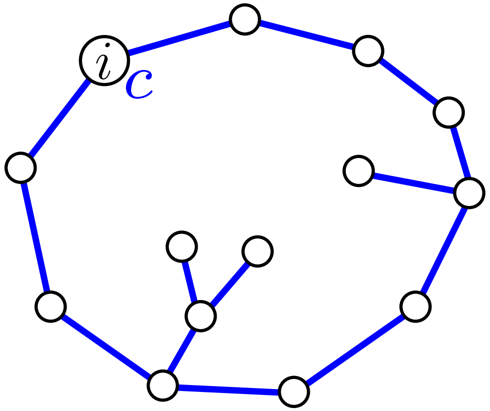
 implies  (follows from the local rules)

$\Rightarrow$ 

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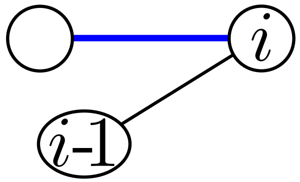
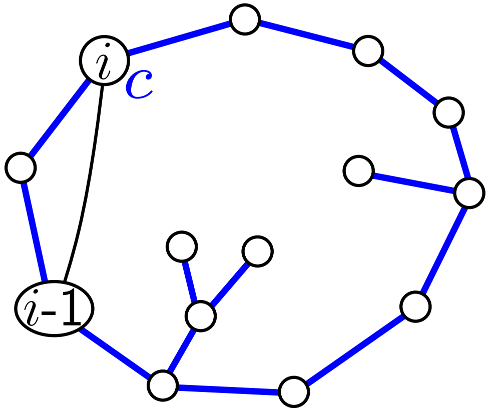


(follows from the local rules)

$\Rightarrow$

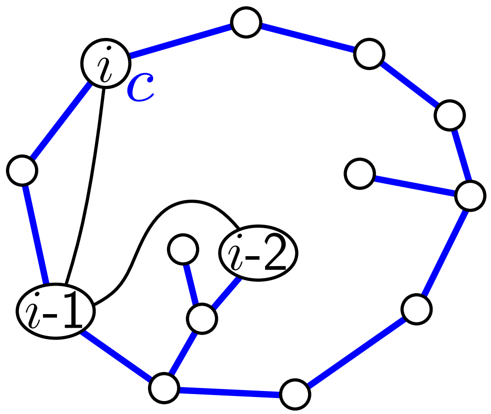



From each corner $c$ in a "face" of $M$ starts a label-decreasing path of $Q$ that stays in the face and ends at a local min of $Q$

# Proof of the stated properties


(follows from the local rules)

$\Rightarrow$

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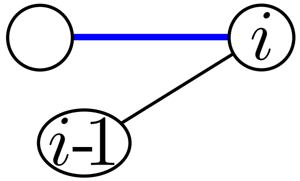


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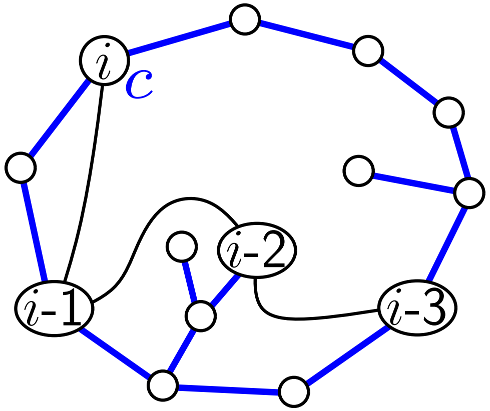


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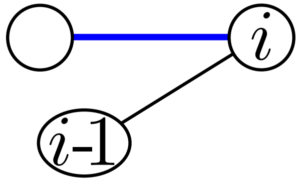


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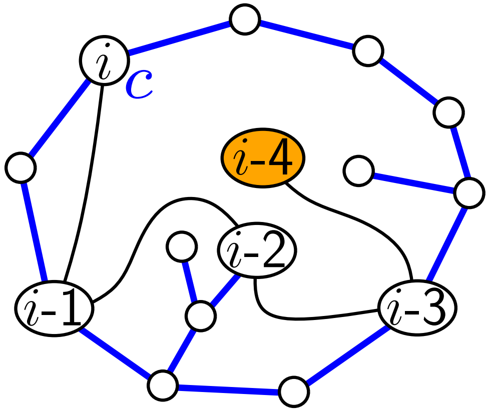


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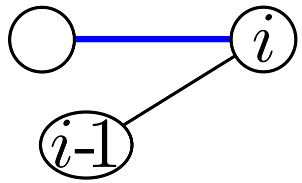


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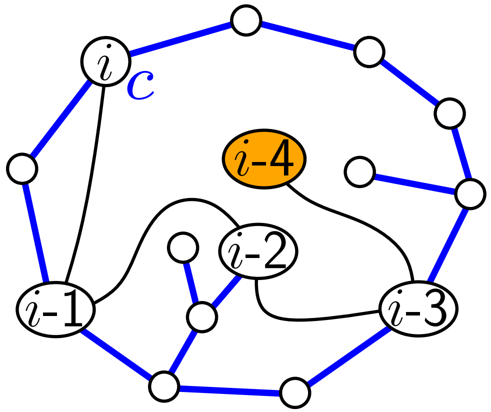


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Let $n = \#$ faces of $Q$, $p = \#$ local min of $Q$, $f = \#$ “faces” of $M$

| | $\#V$ | $\#E$ | $\#F$ |
|---|---|---|---|
| $Q$ | $n+2$ | $2n$ | $n$ |
| $M$ | $n+2-p$ | $n$ | $f = k-1+p$ |

Euler's relation, with
$k = \#$ connected comp. of $M$

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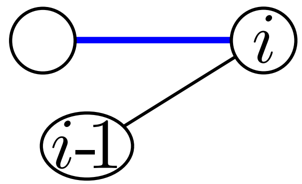


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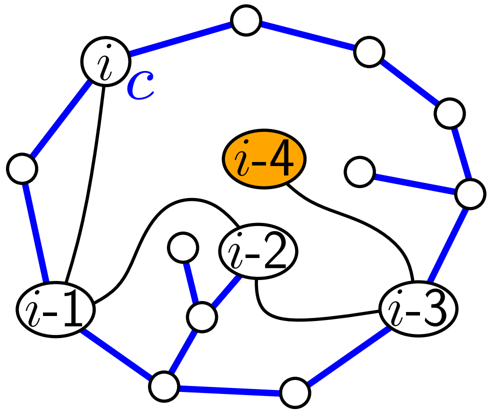


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Euler's relation, with $k = \#$ connected comp. of $M$

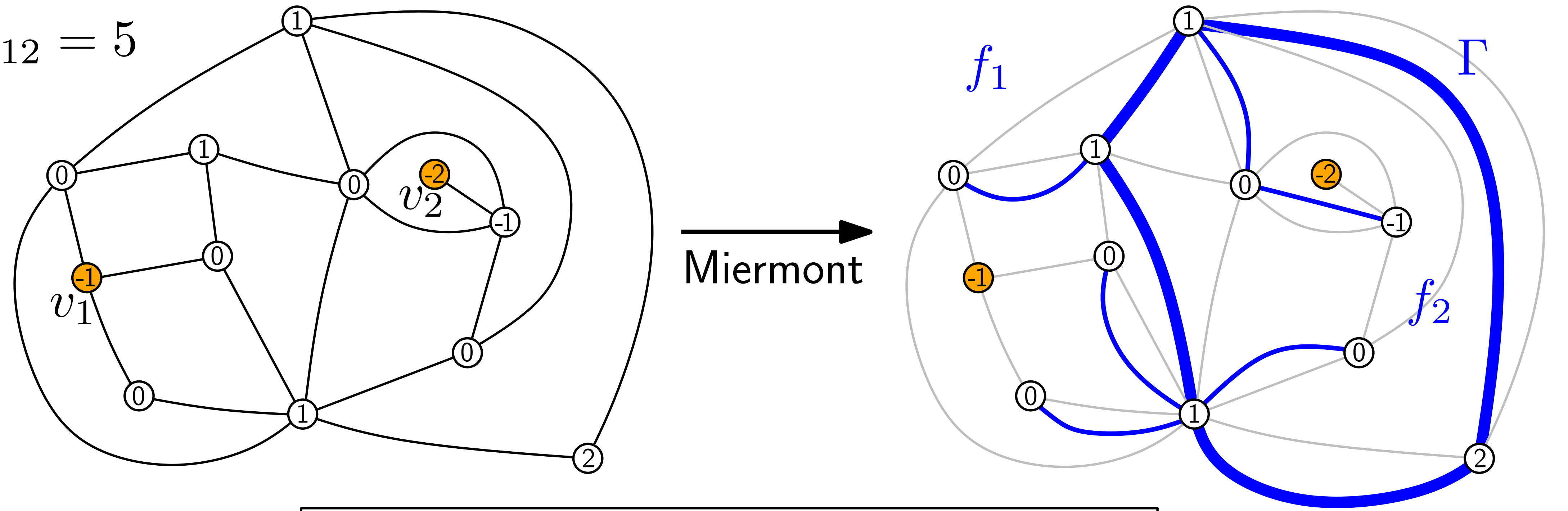
Drawing above $\Rightarrow f \leq p$

Hence $k = 1$ ($M$ connected) $f = p$, and there is exactly one local min of $Q$ in each face of $M$

# The case of two local min

$\Gamma$ the boundary, here $\min_\Gamma = 1$

$d_{12} = 5$




$$\mathrm{dist}(v_1, v_2) = 2 \cdot \min_\Gamma - \ell(v_1) - \ell(v_2)$$

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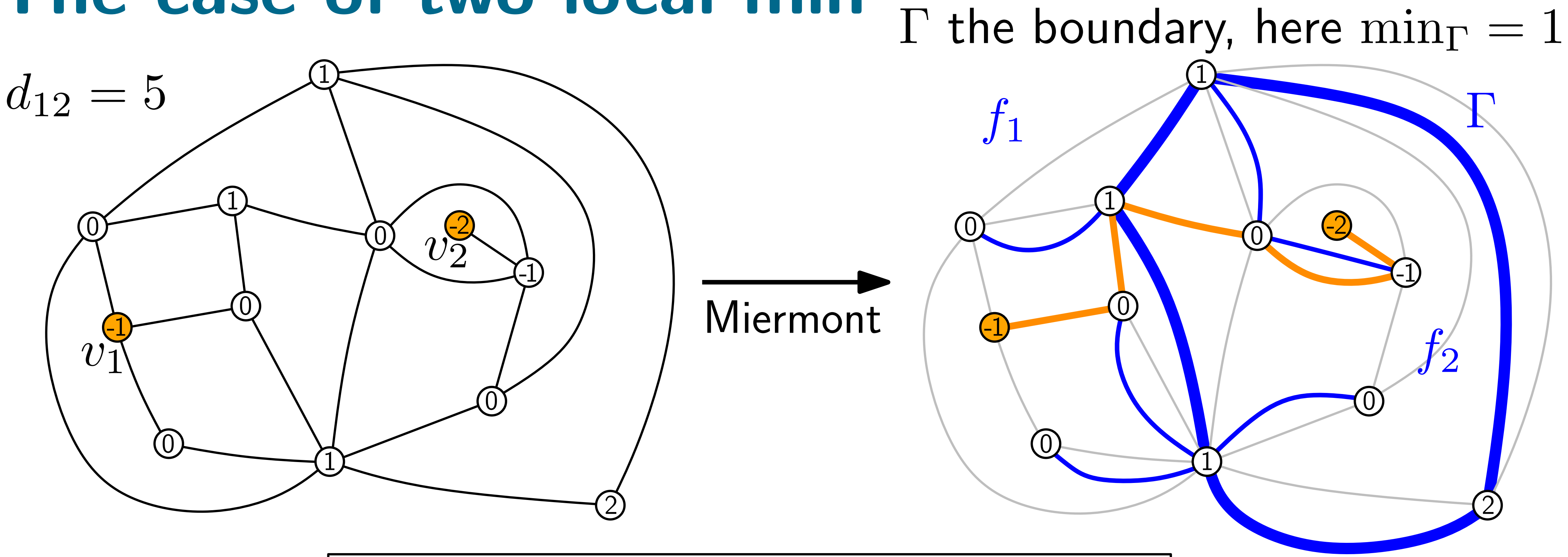


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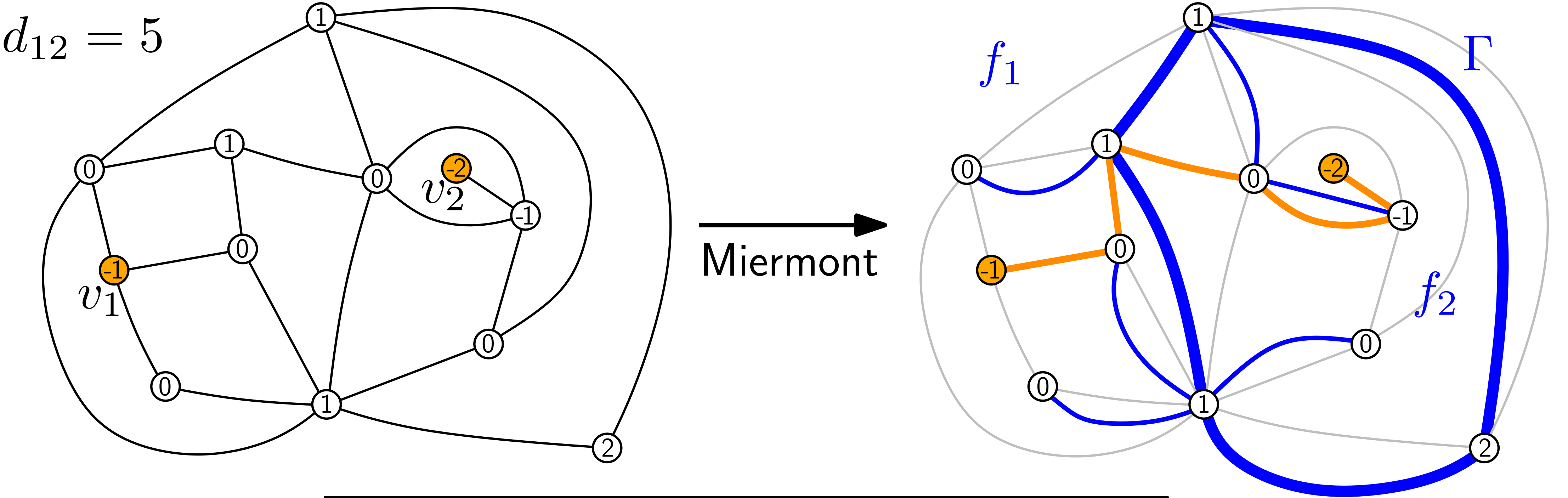
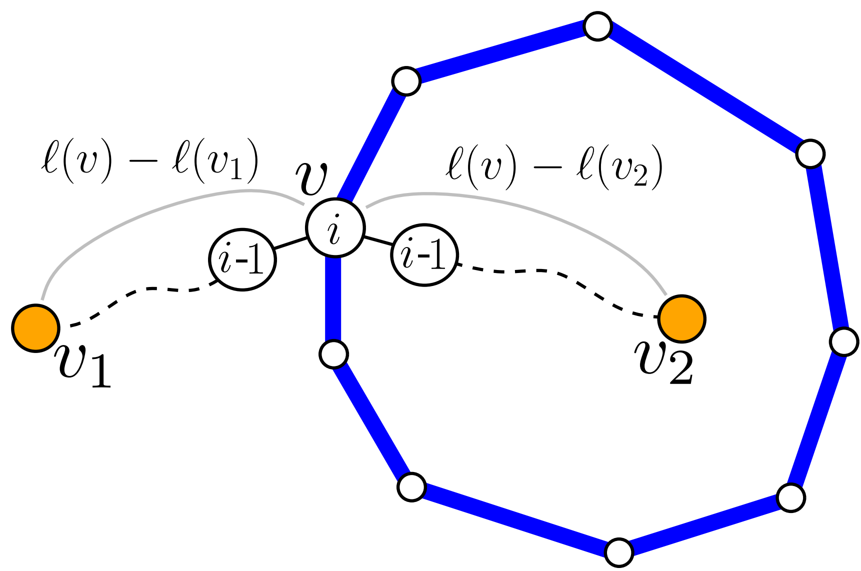



$$\text{dist}(v_1, v_2) = 2 \cdot \min_\Gamma - \ell(v_1) - \ell(v_2)$$

**Proof:** $\forall v \in \Gamma$, a shortest path $v_1 \to v \to v_2$ has length $2\ell(v) - \ell(v_1) - \ell(v_2)$ (because of the existence of a label-decreasing path on each side)



# Another way of computing the 2-point function

**[Bouttier, Guitter'08]** Let $d \geq 2$ and let $s, t \geq 1$ such that $s + t = d$

A bi-pointed quadrangulation $Q$ where $d_{12} = d$ has a unique very-well labelling $\ell(.)$ with two local min, at $v_1, v_2$, and $\ell(v_1) = -s$, $\ell(v_2) = -t$.

$\ell(.)$ is given by $\boxed{\ell(v) = \min(\mathrm{dist}(v_1, v) - s,\ \mathrm{dist}(v_2, v) - t)}$

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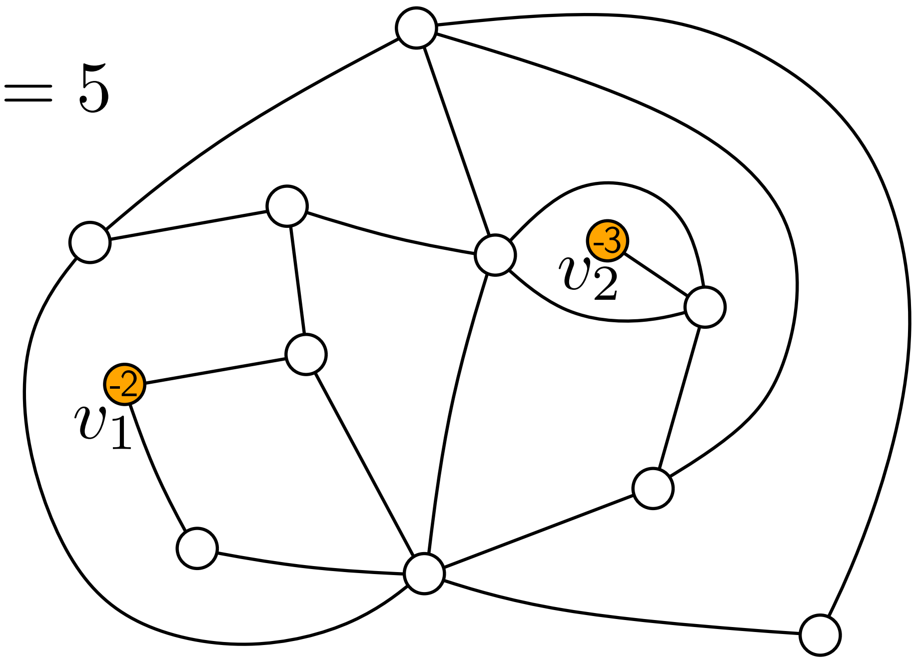

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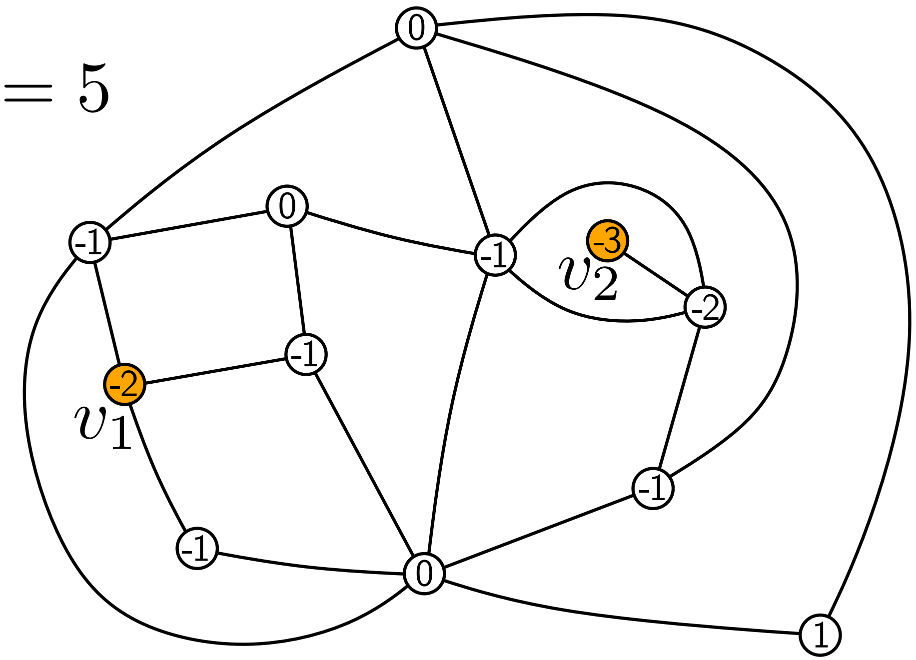

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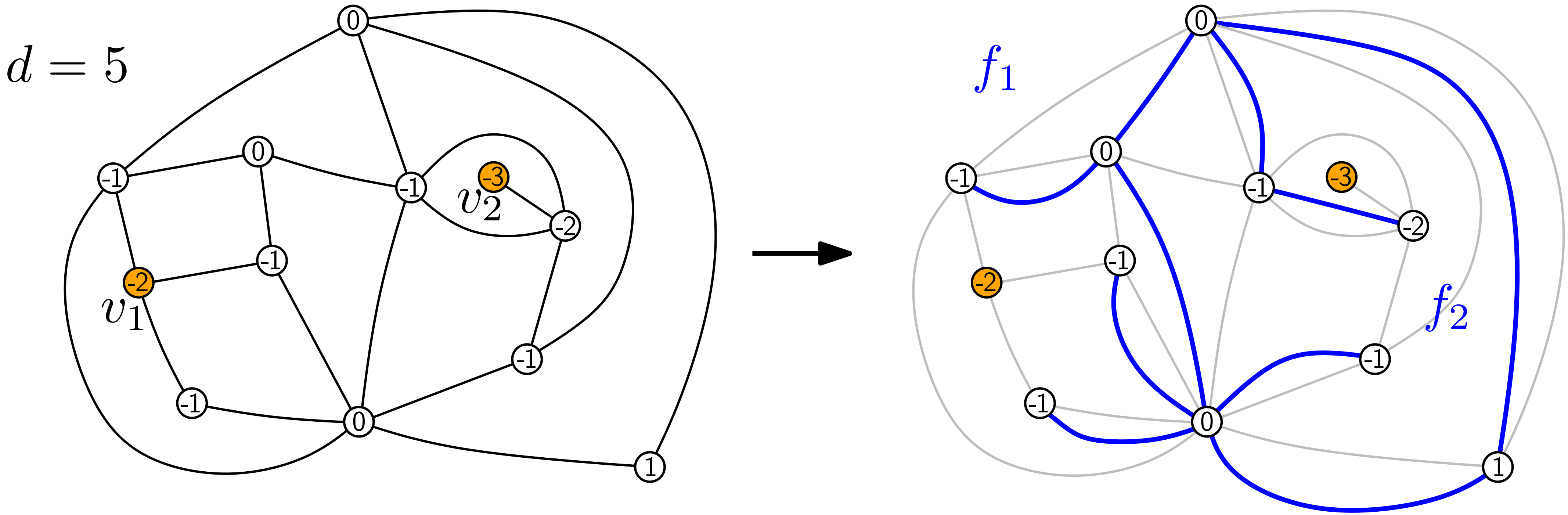

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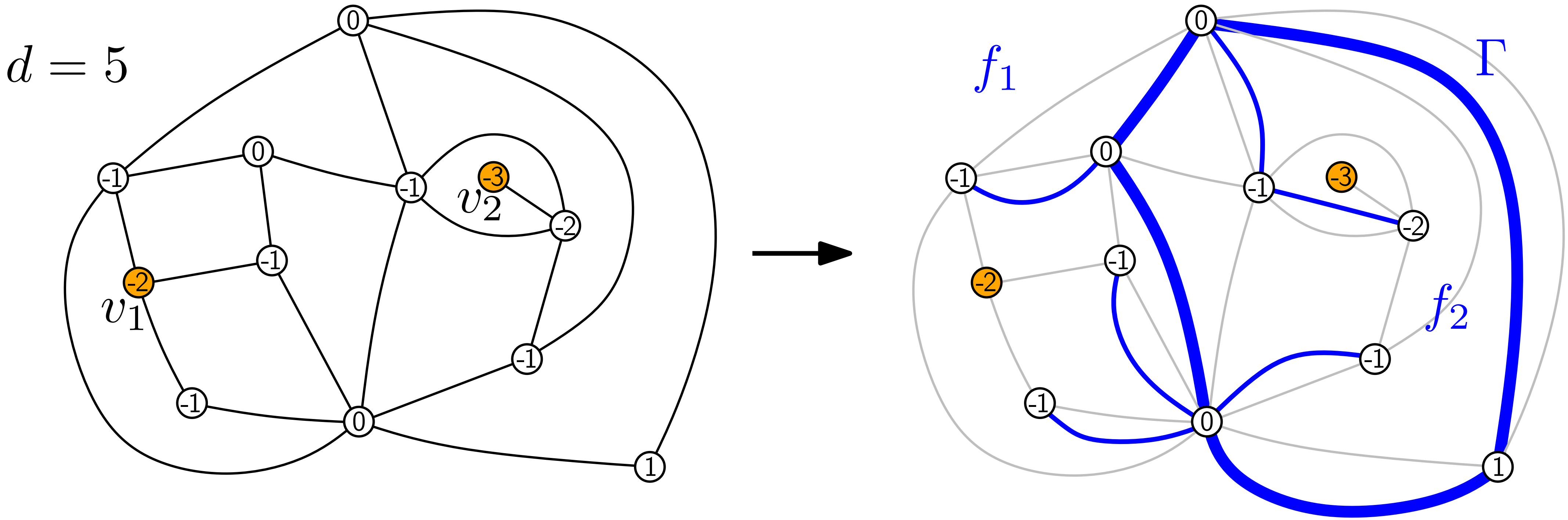


The associated well-labelled map with two faces $f_1, f_2$ satisfies:

- $\min(f_1) = -s + 1$, $\min(f_2) = -t + 1$
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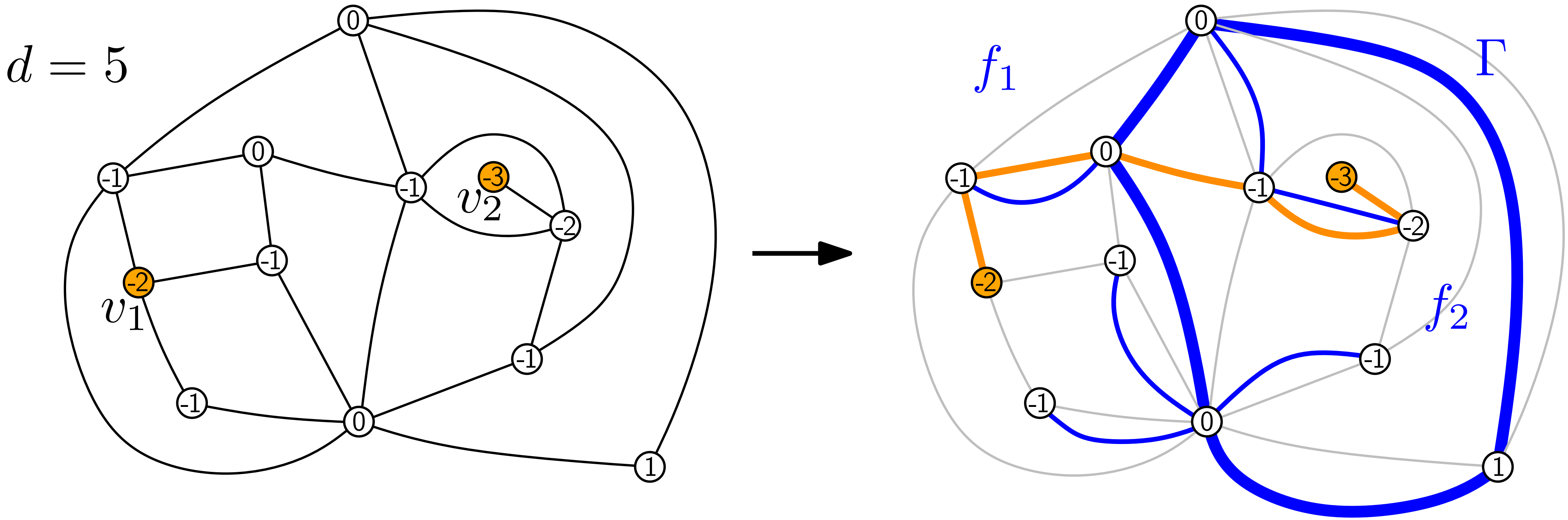


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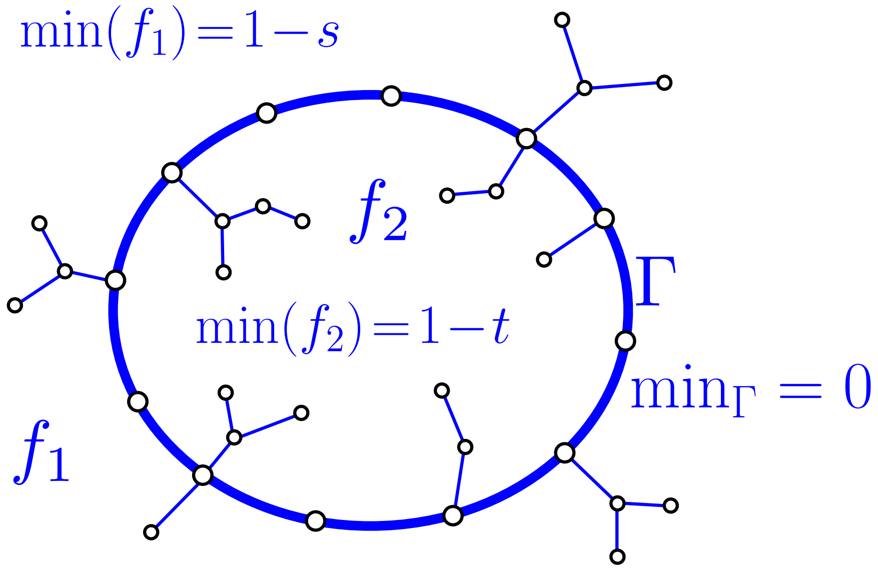
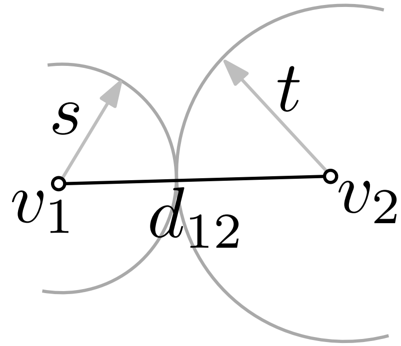
# Another way of computing the 2-point function

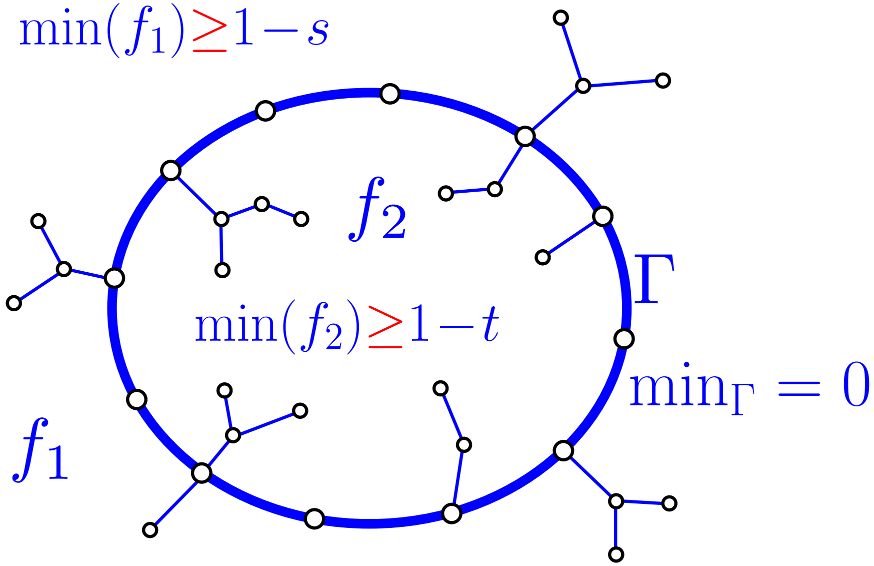
We conclude that, for $d = s + t$ $(s, t \geq 1)$ $G_d(g)$ is the series of




1st method corresponds to $t = 0$

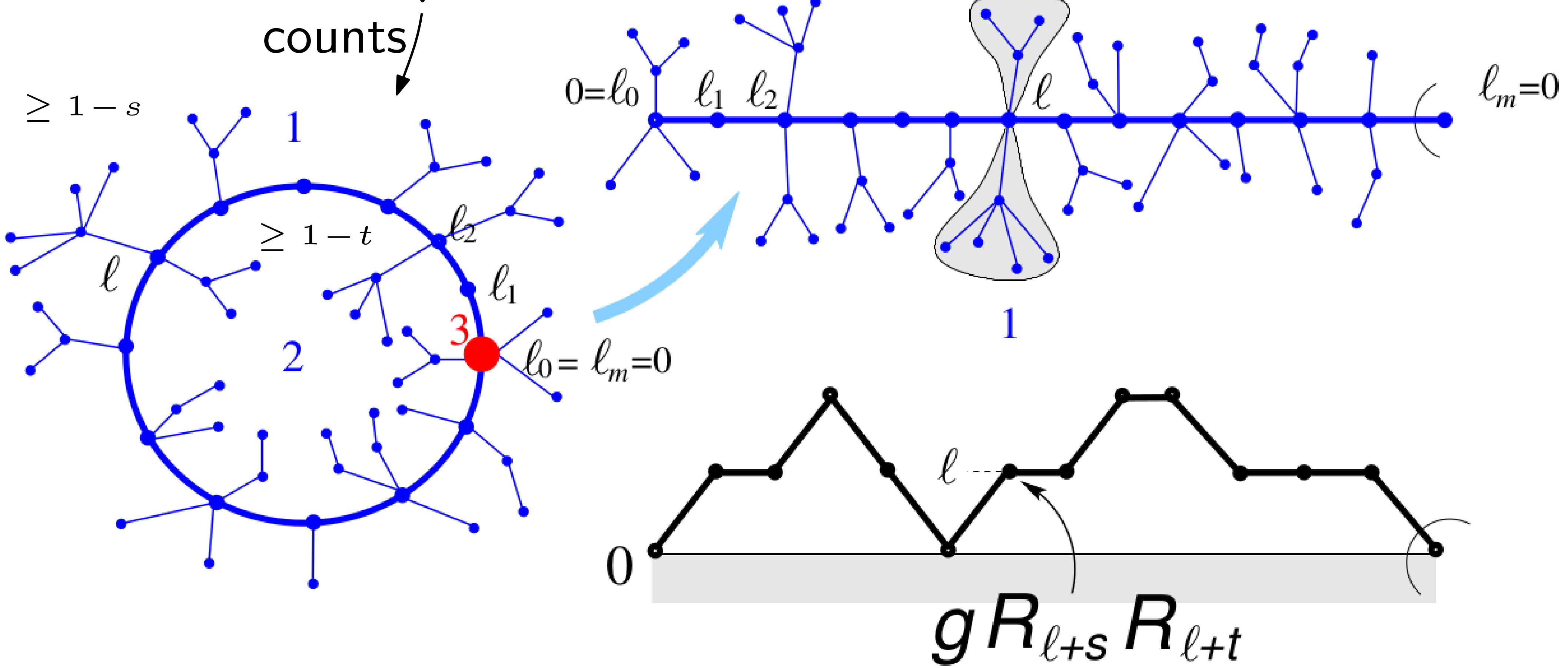
Or ($\Delta :=$ discrete differentiation) $G_d = \Delta_s \Delta_t F_{s,t}$, where $F_{s,t}$ counts



# Another way of computing the 2-point function

Then by the link between cyclic and sequential excursions:

$$F_{s,t} = \log(\underbrace{X_{s,t}})$$



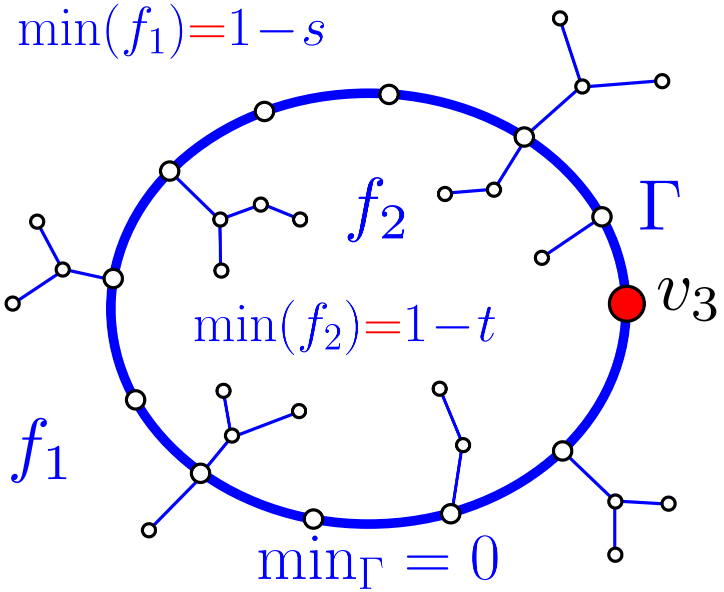

Equation for $X_{s,t}$: $X_{s,t} = 1 + g R_s R_t X_{s,t} (1 + g R_{s+1} R_{t+1} X_{s+1,t+1})$

solution (guessing/checking): $X_{s,t} = \frac{[3]_x [s+1]_x [t+1]_x [s+t+3]_x}{[1]_x [s+3]_x [t+3]_x [s+t+1]_x}$

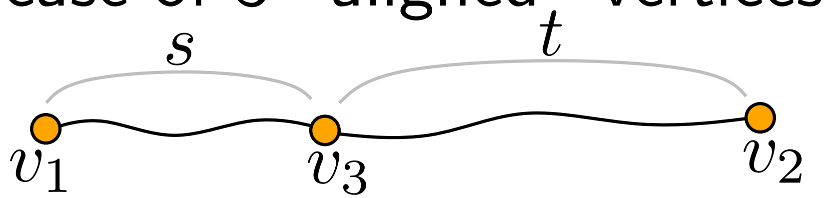
$\Rightarrow$ recover $\boxed{G_d = \log\left(\frac{[s+t]_x^2 [s+t+3]_x}{[s+t-1]_x [s+t+2]_x^2}\right)}$

# A first covered case for the 3-point function

**[Bouttier, Guitter'08]** This solves the case of 3 "aligned" vertices




$\Longleftrightarrow$




tri-pointed quadrangulations with

$d_{12} = s + t$, $d_{13} = s$, $d_{23} = t$

i.e., $v_3$ is on a geodesic path from $v_1$ to $v_2$ at respective distances $s, t$ from $v_1, v_2$

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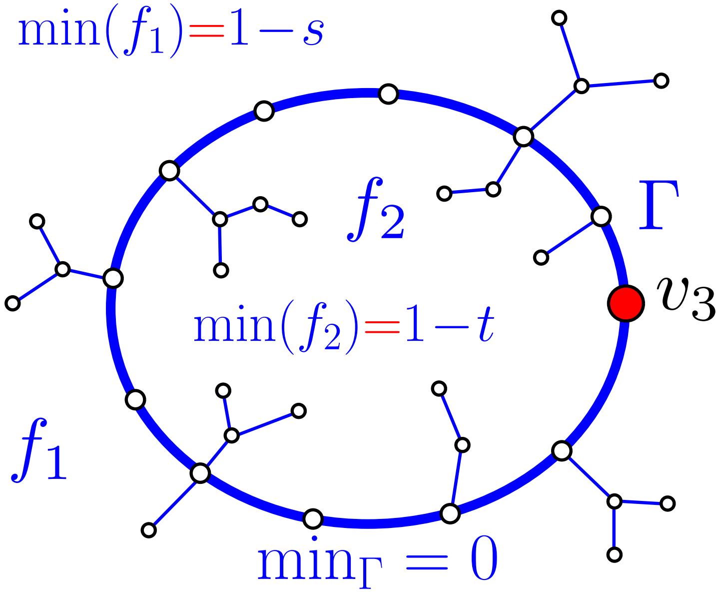


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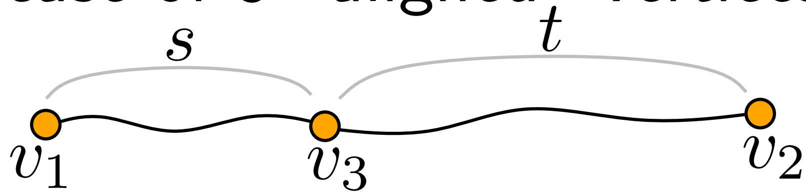


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i.e., $v_3$ is on a geodesic path from $v_1$ to $v_2$ at respective distances $s, t$ from $v_1, v_2$

Hence $G_{s+t,s,t}(g) = \Delta_s \Delta_t X_{s,t}$ where $X_{s,t} = \frac{[3]_x [s+1]_x [t+1]_x [s+t+3]_x}{[1]_x [s+3]_x [t+3]_x [s+t+1]_x}$

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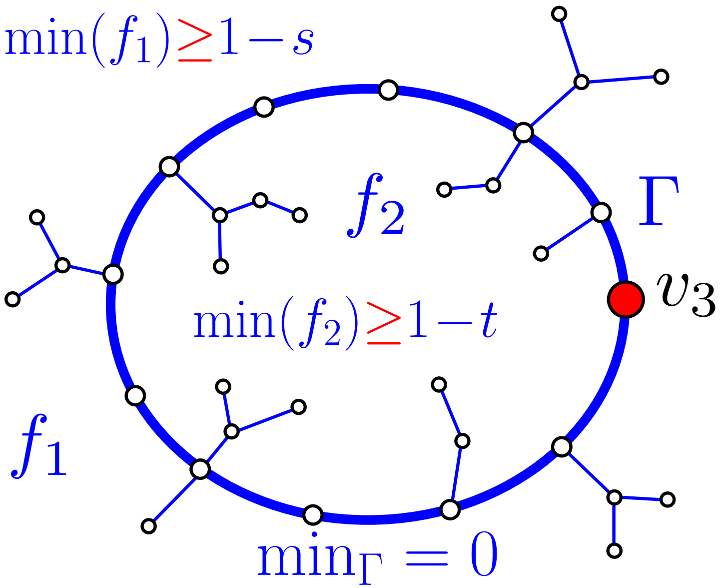

# The different cases for the 3-point function

**[Bouttier, Guitter'08]** $D = (d_{12}, d_{13}, d_{23})$ can be achieved only if

$$\begin{cases} d_{12} \leq d_{13} + d_{23} \\ d_{13} \leq d_{12} + d_{23} \\ d_{23} \leq d_{12} + d_{13} \end{cases}$$

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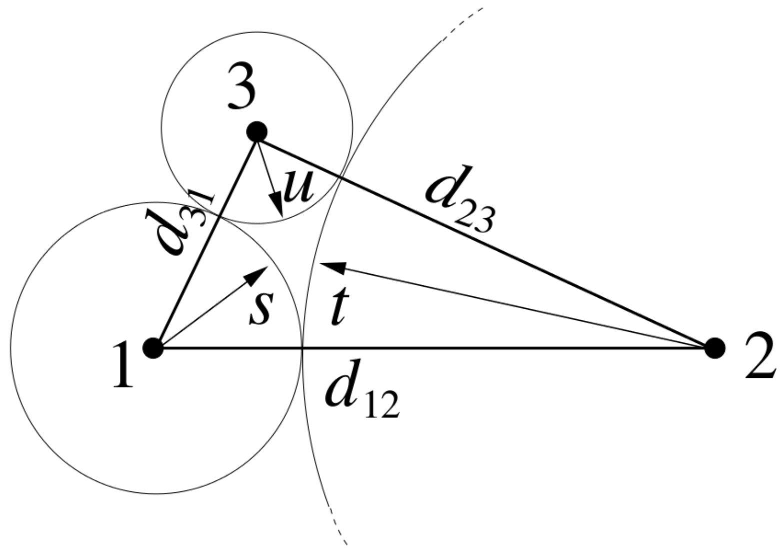
with $s, t, u \ge 0$



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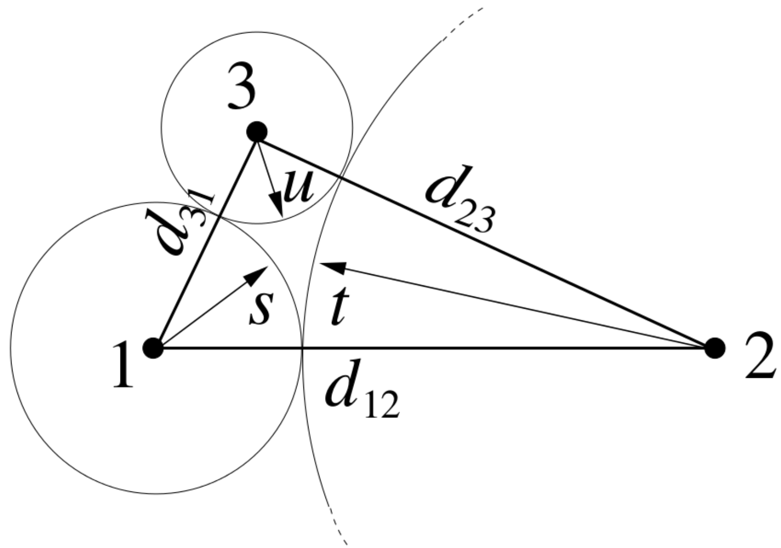


- 3 points are distinct $\Rightarrow$ at most one of $s, t, u$ is zero
- One of $s, t, u$ (say $u$) is zero $\Leftrightarrow$ aligned points (preceding slide)
- Generic case: $s, t, u > 0$ (non-aligned points)

# The generic case

**[Bouttier, Guitter'08]** write $D$ as

$$d_{12} = s + t$$
$$d_{13} = s + u$$
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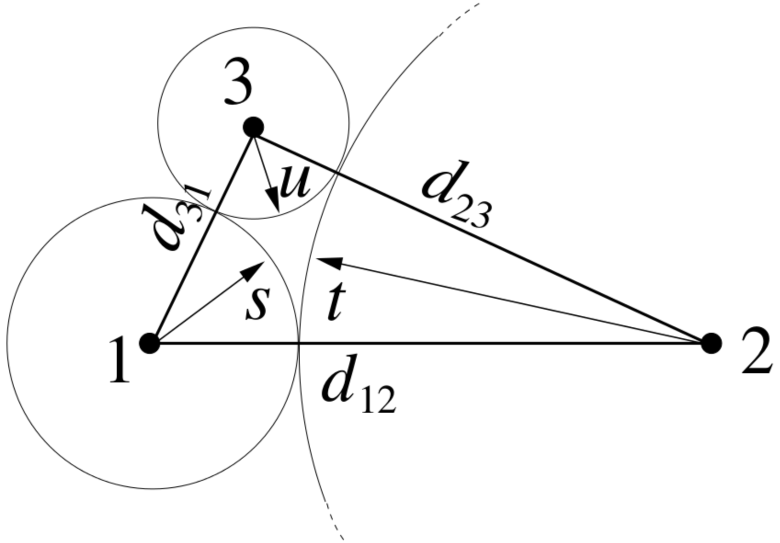
with $s, t, u > 0$




Endow $Q$ with unique very-well labelling with 3 local min at $v_1, v_2, v_3$ and where $\ell(v_1) = -s,\ \ell(v_2) = -t,\ \ell(v_3) = -u$

Apply the Miermont bijection $\Rightarrow$

obtain a 3-face well-labelled map where

$$\min(f_1) = 1 - s \qquad \min_{\Gamma_{12}} = 0$$
$$\min(f_2) = 1 - t \qquad \min_{\Gamma_{13}} = 0$$
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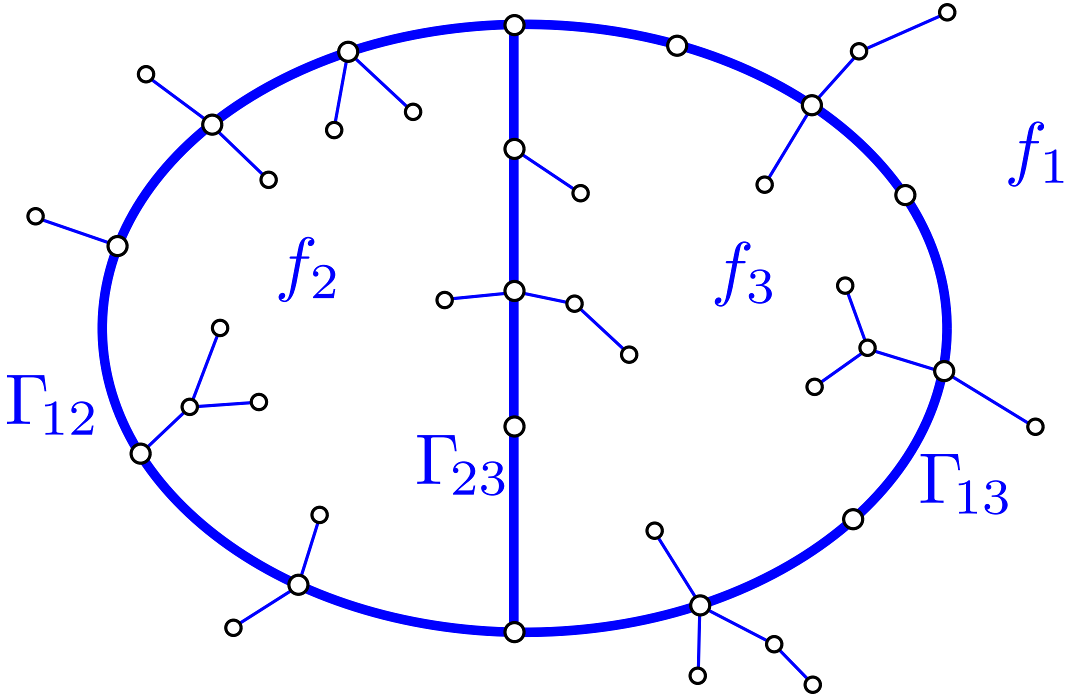

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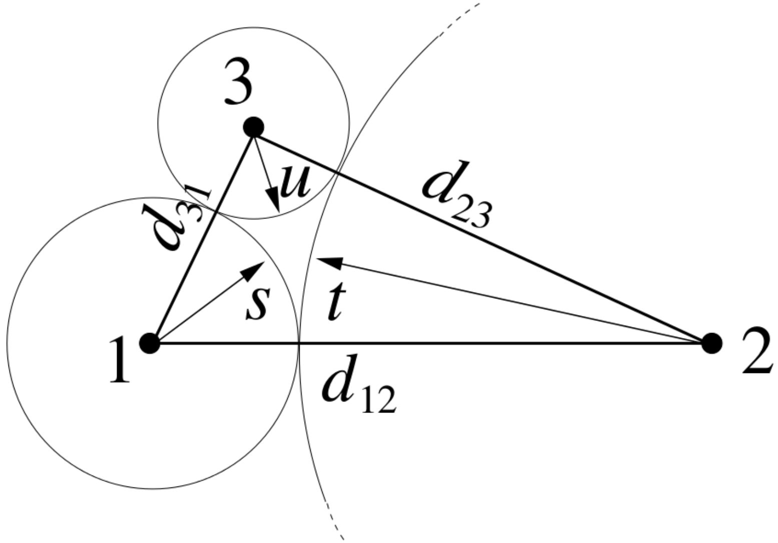


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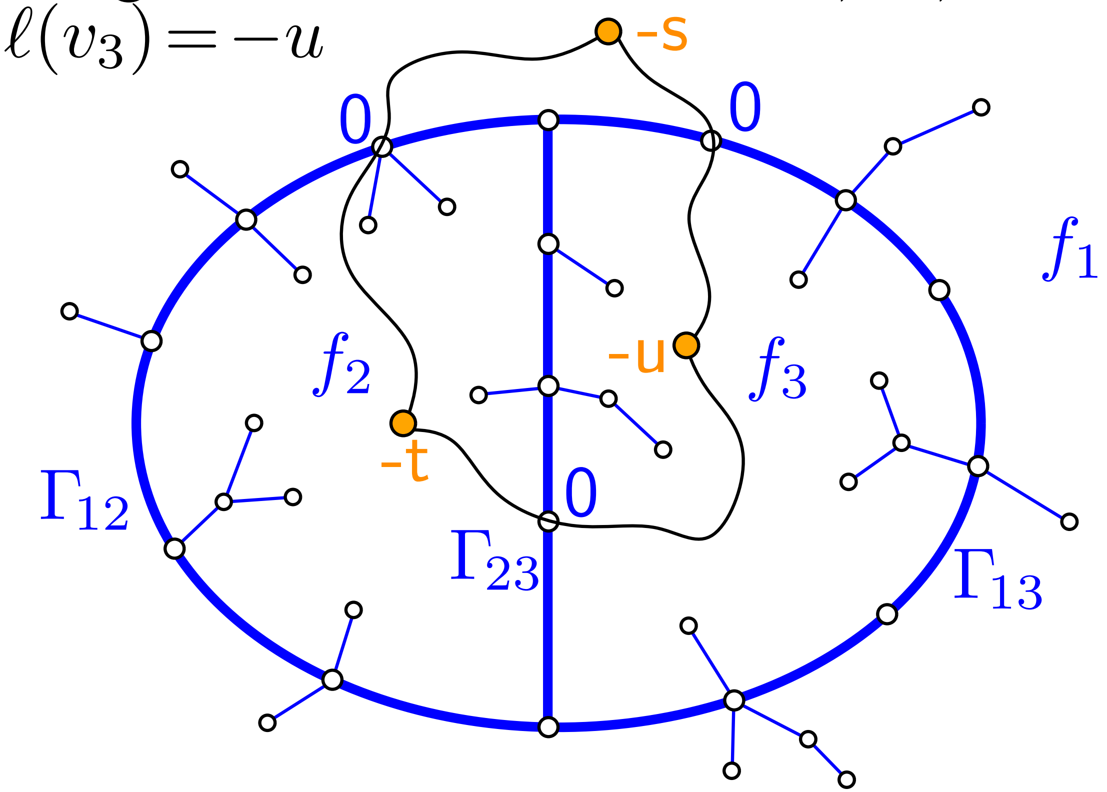


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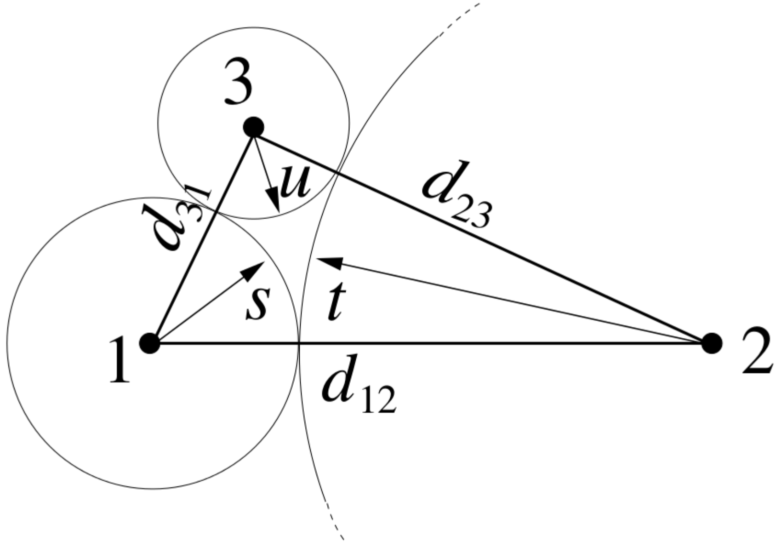


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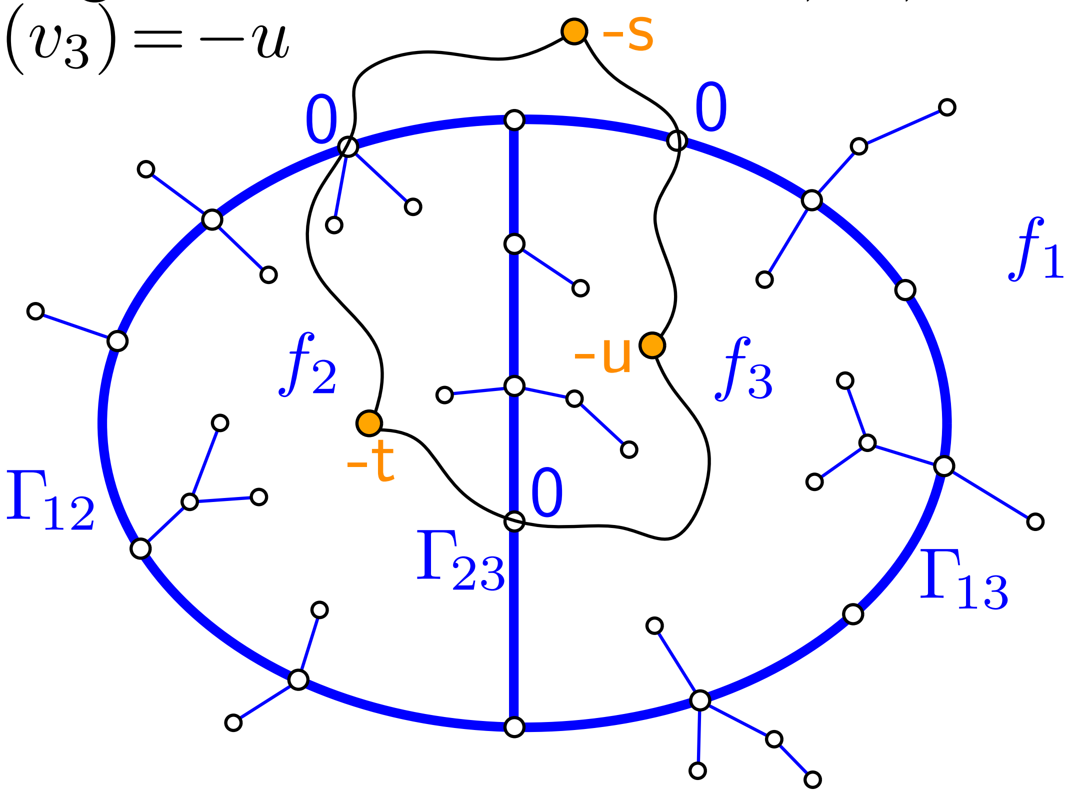


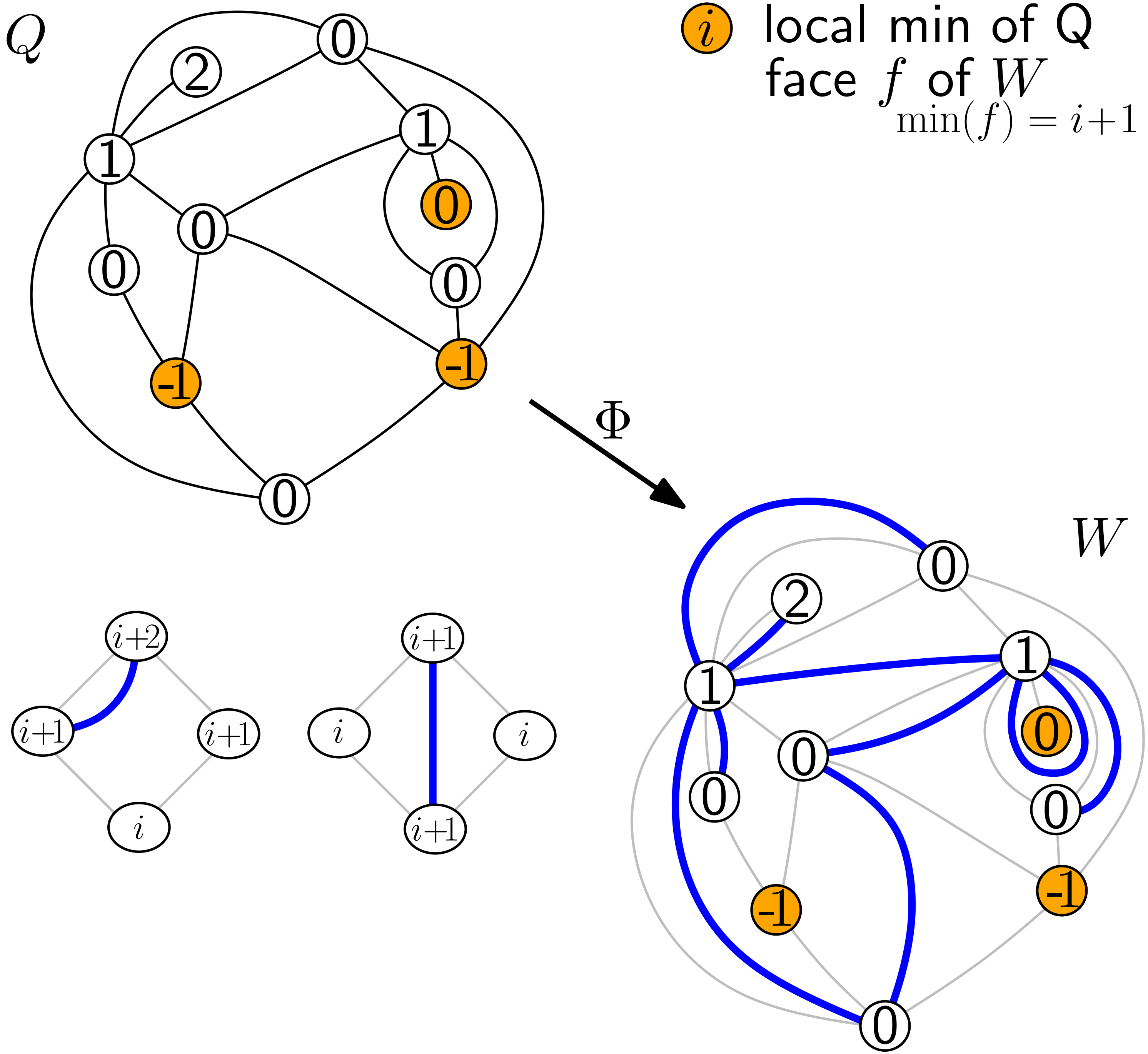

$\Rightarrow$ expression of $G_{d_{12},d_{13},d_{23}}(g)$ as $\Delta_s \Delta_t \Delta_u F_{s,t,u}$, with $F_{s,t,u}(g)$ explicit

# Computing the two-point function of general maps using the Ambjørn-Budd bijection

# The Ambjørn-Budd bijection $\Lambda$ **[Ambjørn-Budd'13]**

Recall the Miermont bijection $\Phi$ (reformulated by Ambjørn-Budd)

$i$ local min of Q
face $f$ of $W$
$\min(f) = i+1$



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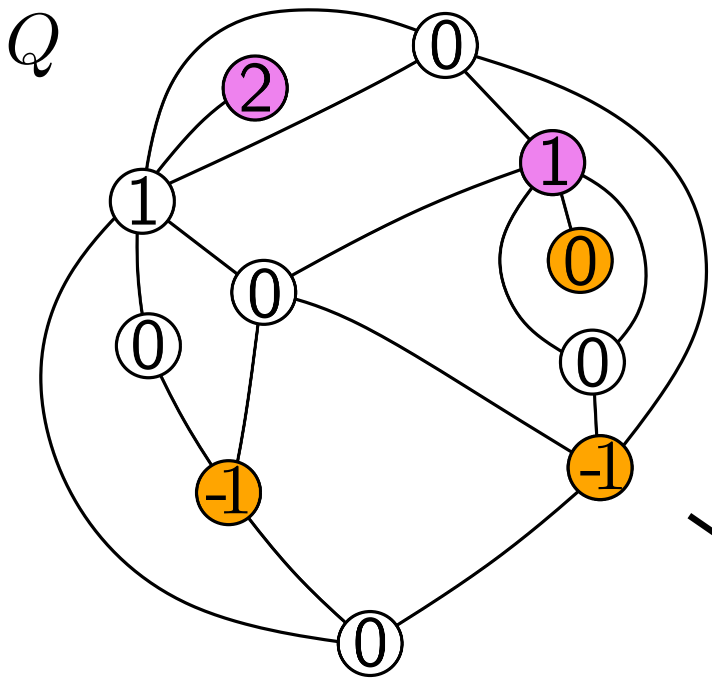
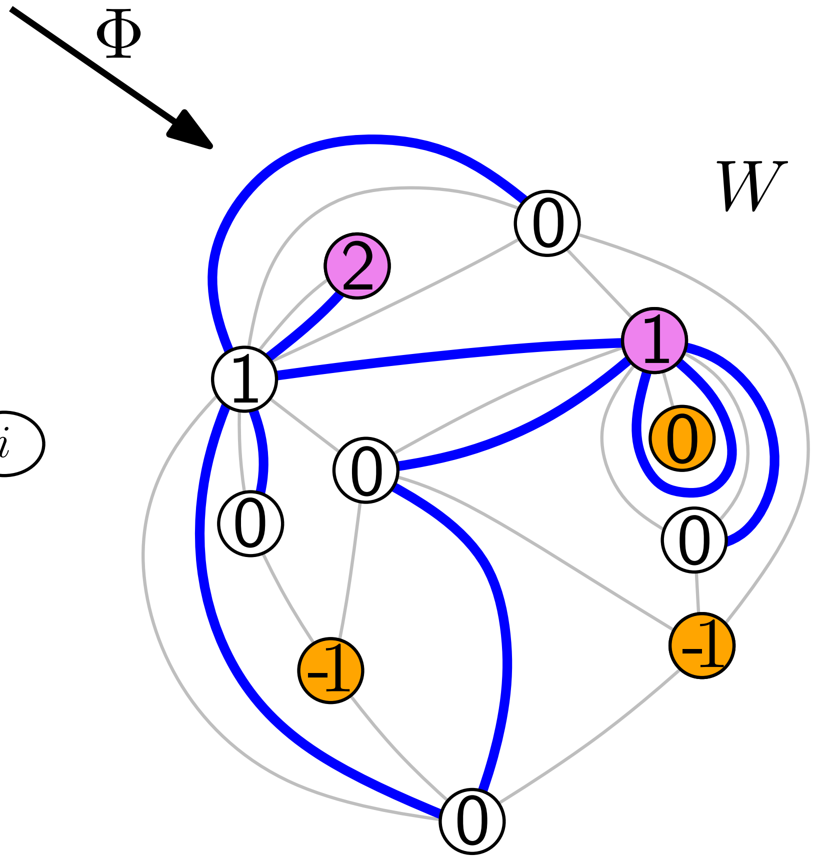
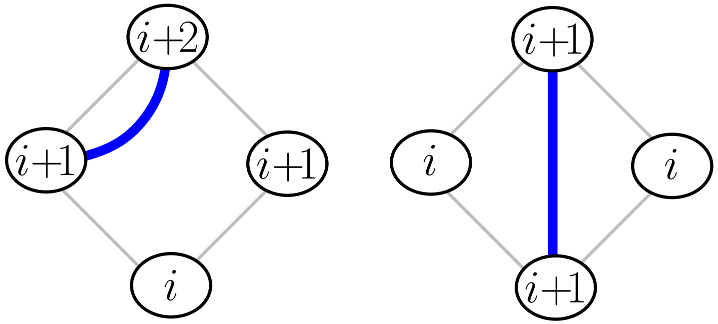



$i$ local min of Q
face $f$ of $W$
$\min(f) = i{+}1$

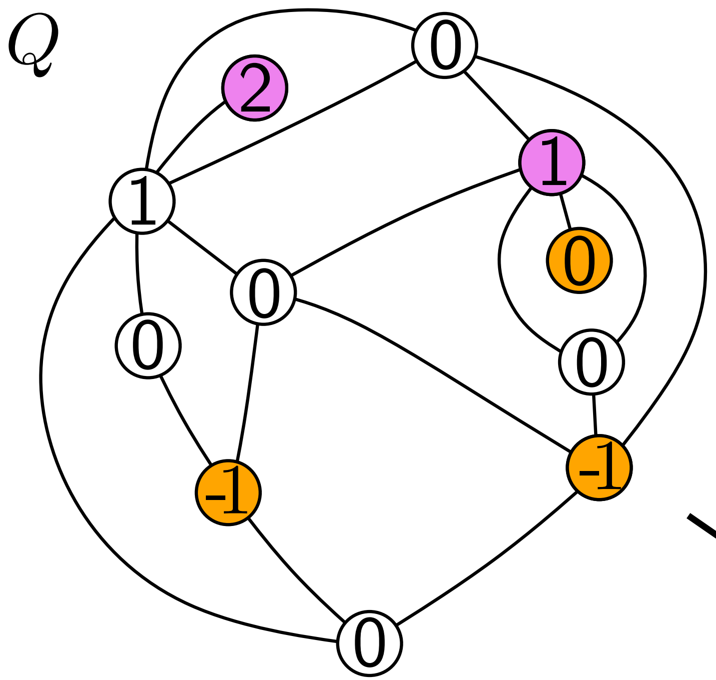
$i$ local max of Q
local max of $W$



# The Ambjørn-Budd bijection $\Lambda$ **[Ambjørn-Budd'13]**

Recall the Miermont bijection $\Phi$ (reformulated by Ambjørn-Budd)

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$i$ local max of Q
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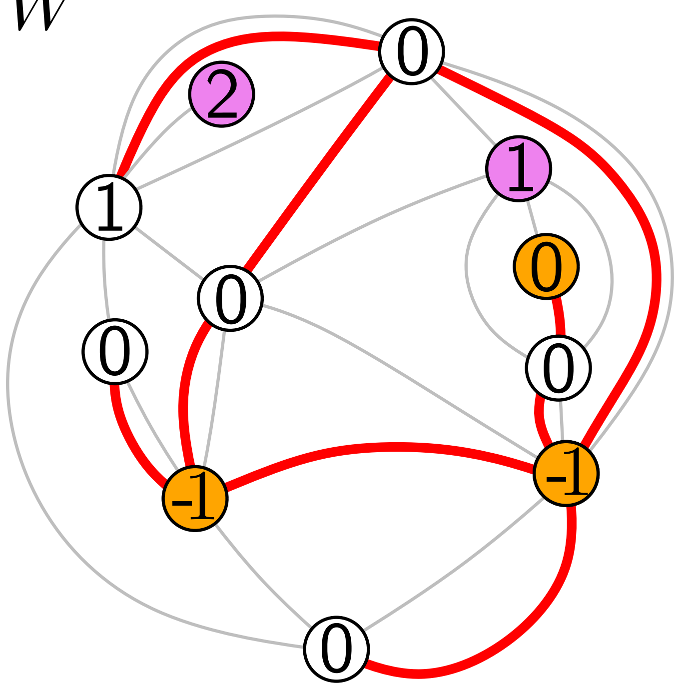
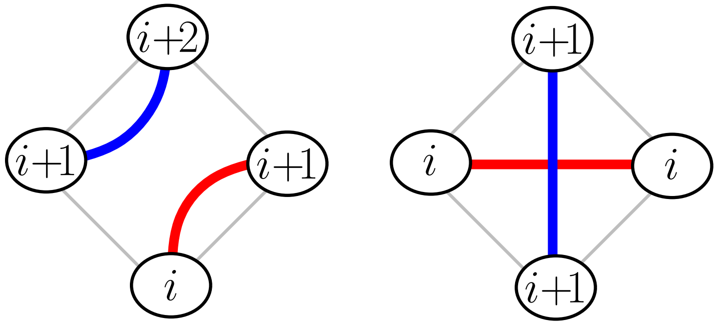
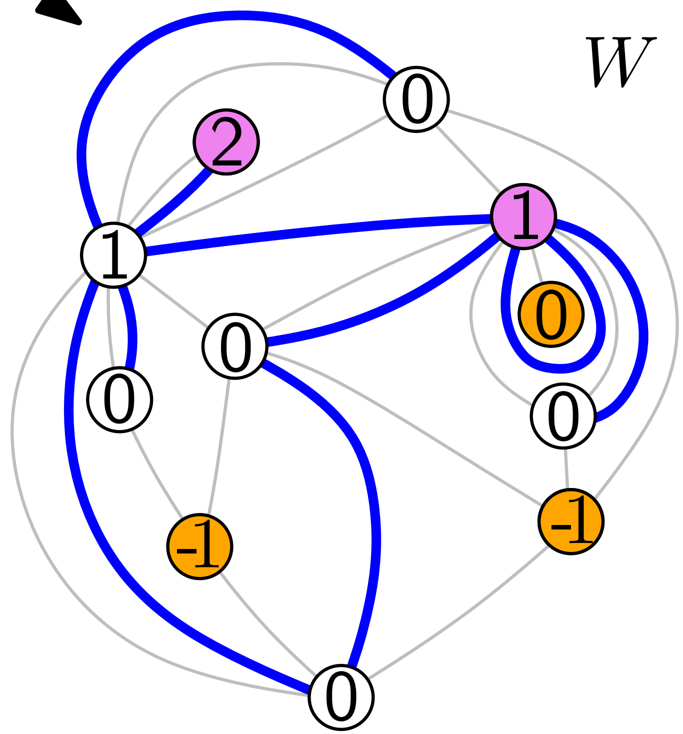
Let $op : \mathbf{Z} \to \mathbf{Z}$
$\quad i \to -i$

$\Phi^- = op \circ \Phi \circ op$

$\Phi$



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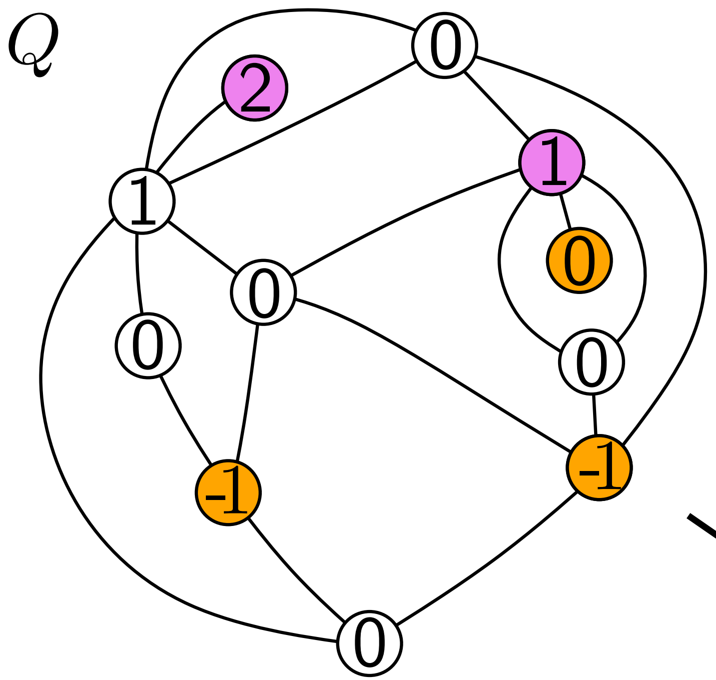
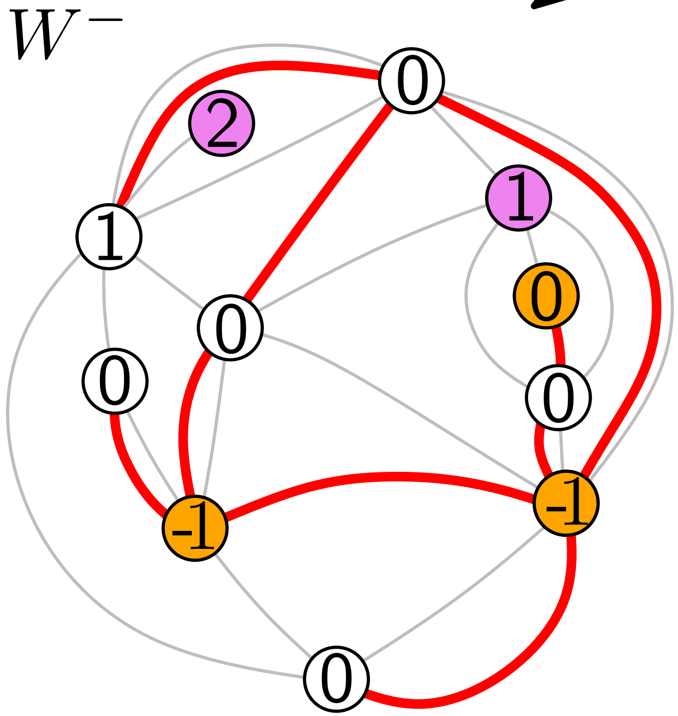
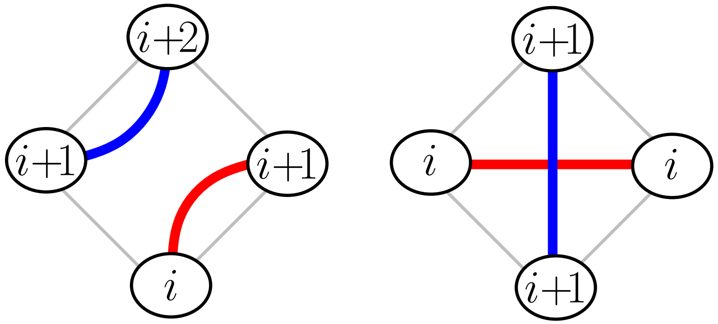
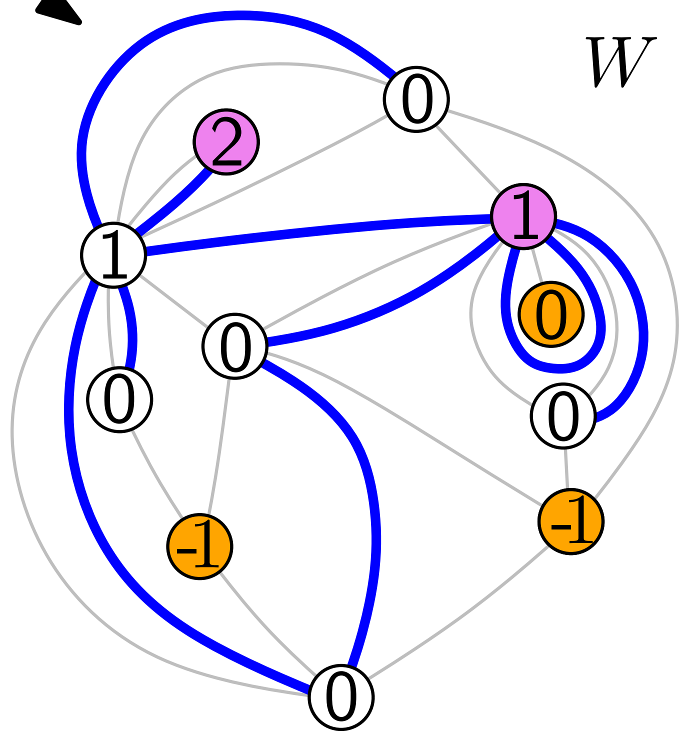



- ($i$, orange) local min of Q; face $f$ of $W$, $\min(f) = i+1$; local min of $W^-$
- ($i$, pink) local max of Q; local max of $W$; face $f$ of $W^-$, $\max(f) = i-1$



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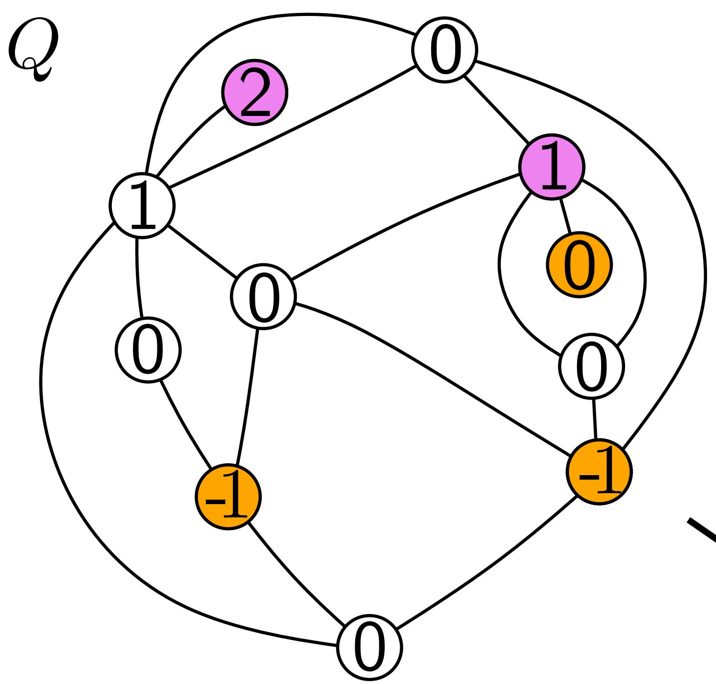


(orange $i$) local min of Q
face $f$ of $W$
$\min(f) = i+1$
local min of $W^-$

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local max of $W$
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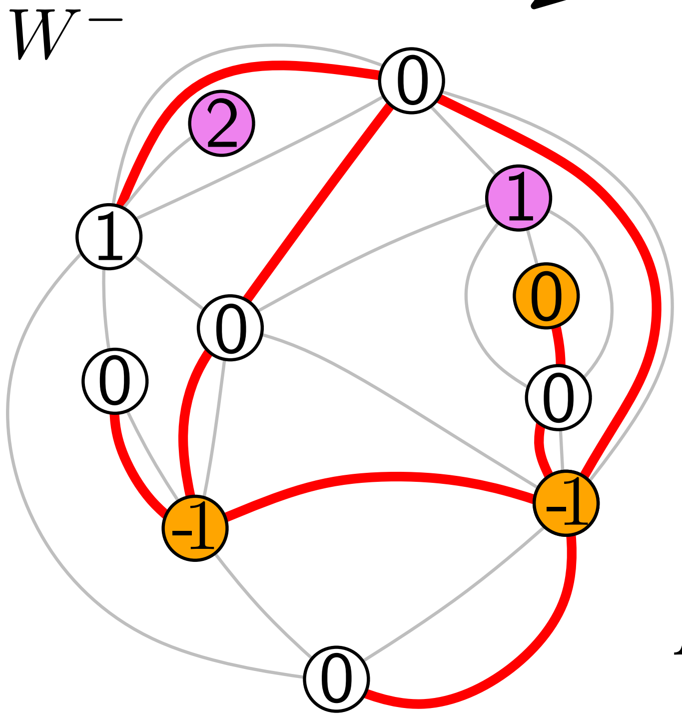
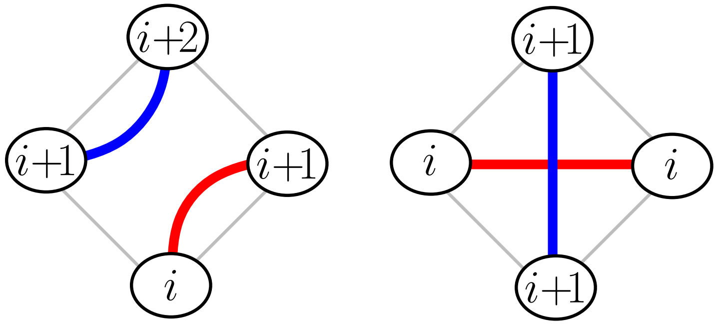
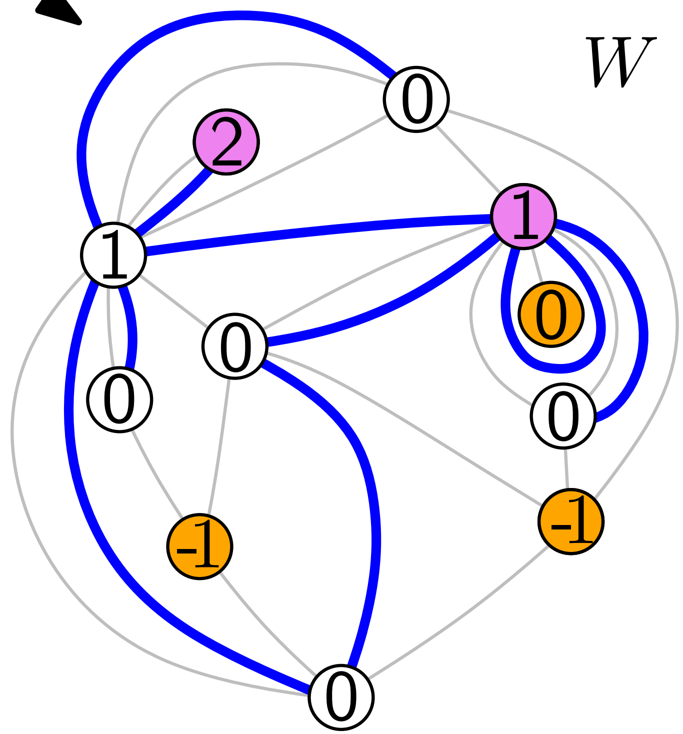


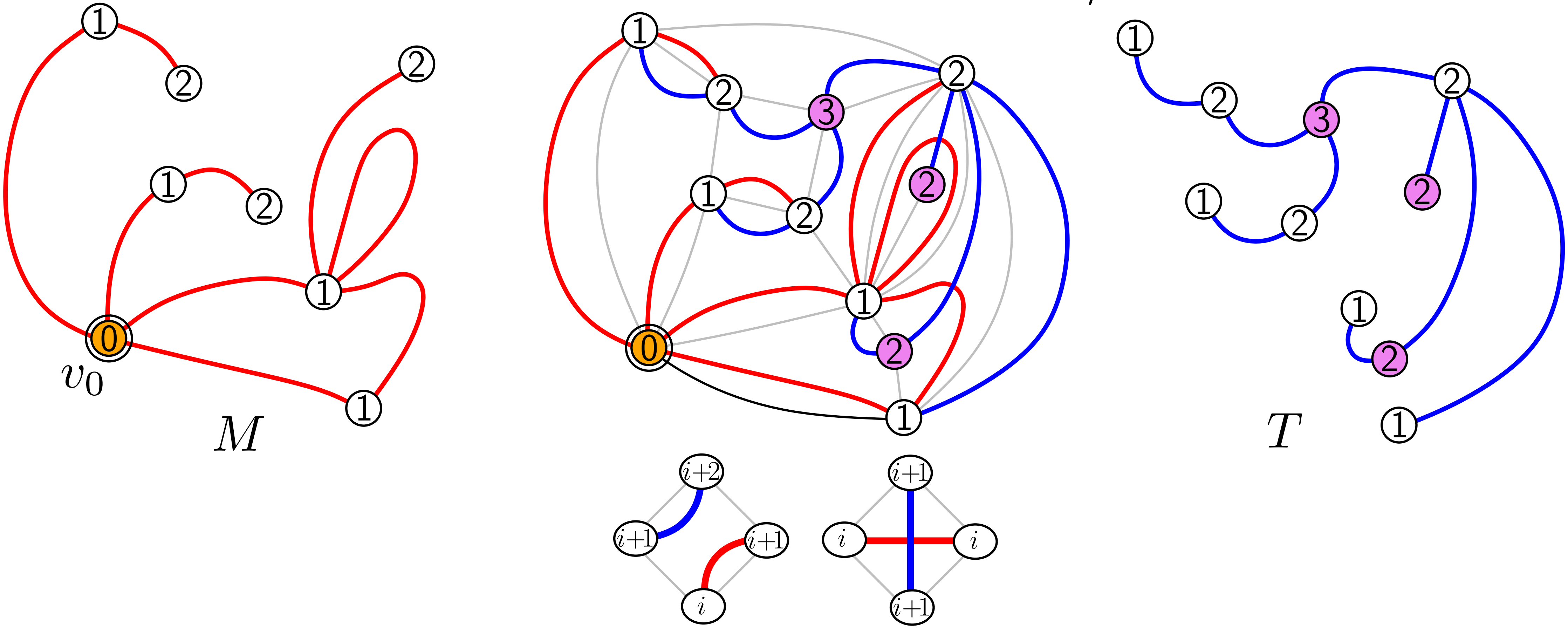




$\Lambda$

$\Lambda$ is a new "duality" relation
for well-labelled map

# The bijection $\Lambda$ applied to pointed maps

**Rk:** pointed maps+geodesic labelling $\leftrightarrow$ well-labelled maps with one local min, of label $0$




$\Rightarrow$ pointed maps $n$ edges $\leftrightarrow$ well-labelled trees min-label=1 and $n$ edges (as for quadrang., but this time vertex of $M \neq v_0 \leftrightarrow$ non-local max of $T$)

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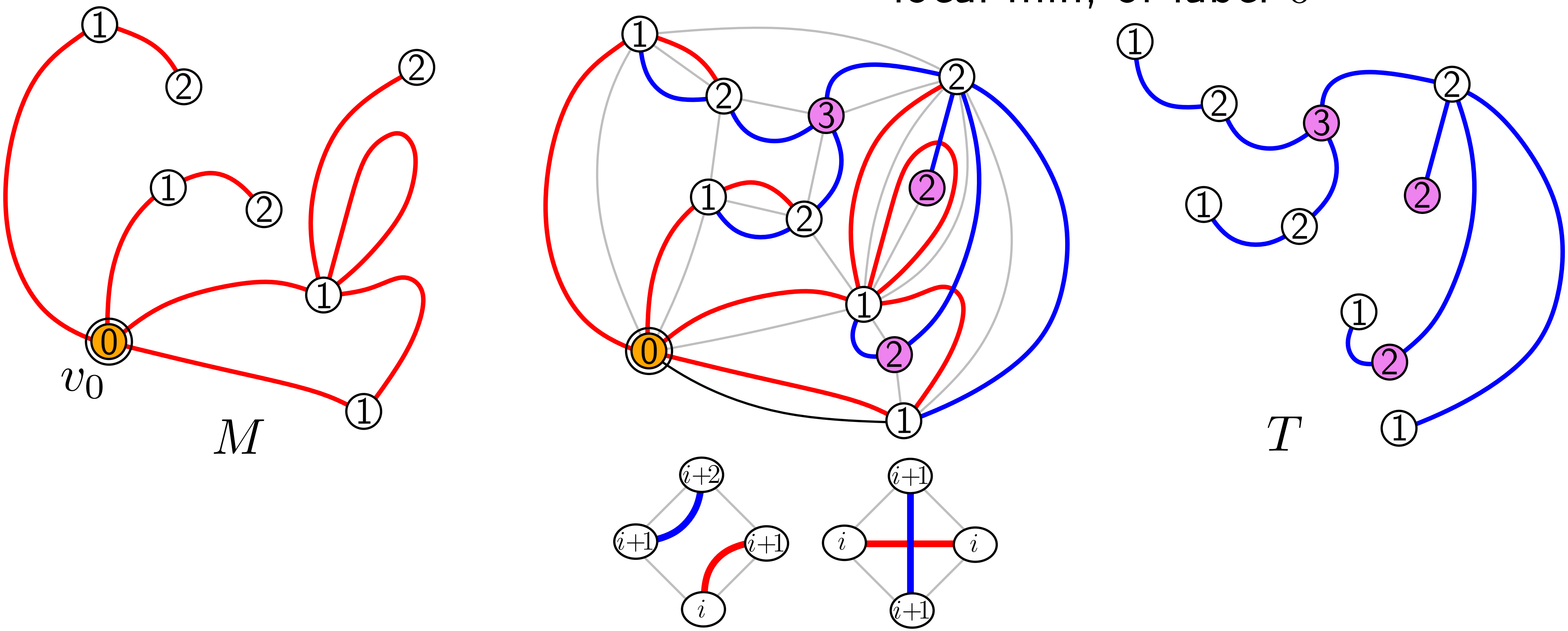


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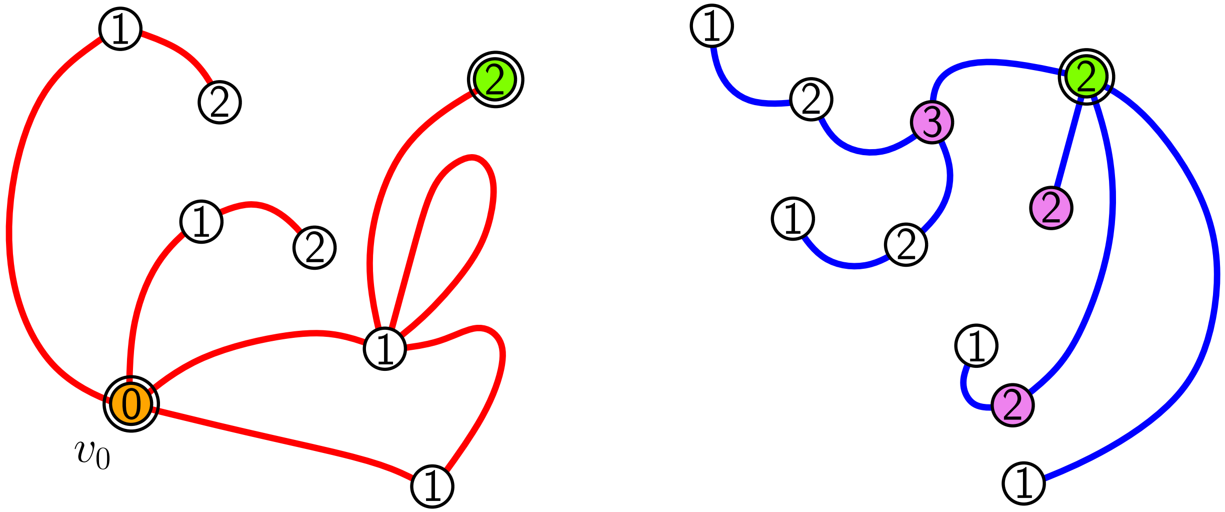
**Rk:** In that case, $\Phi^-$ gives a new bijection from pointed quadrangulations with $n$ faces to pointed maps with $n$ edges that preserves the distances to the pointed vertex (not the case with the easy local bijection)

# The two-point function of general maps

Let $G_d(g)$ the 2-point function of general maps

$d = 2$




AB bijection $\Rightarrow$ $G_d(g)$ is the series of well-labelled trees with min-label $1$ with a marked non local max of label $d$

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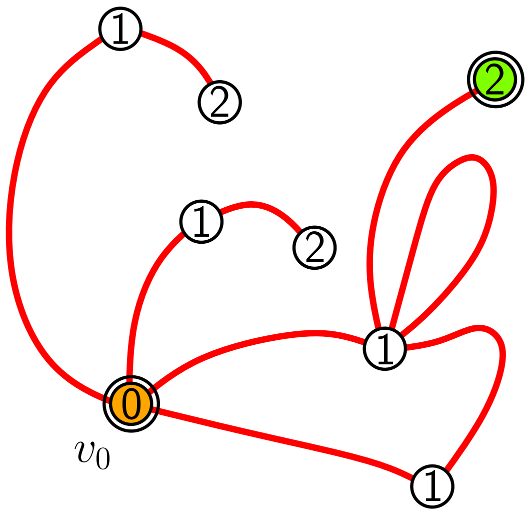


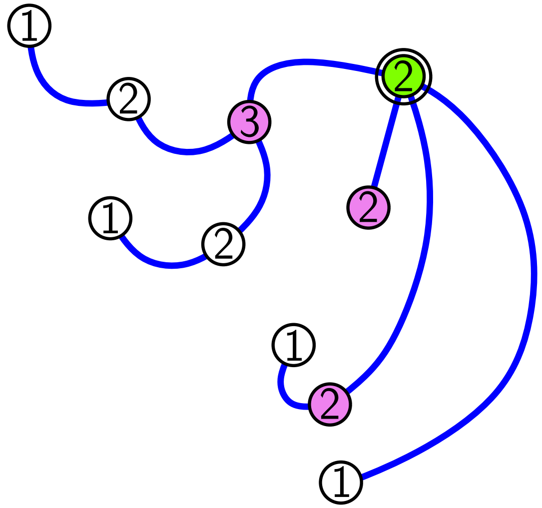


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$G_d = F_d - F_{d-1}$, with $F_d(g) :=$ the series of well-labelled trees with positive labels and a marked non local max of label $d$

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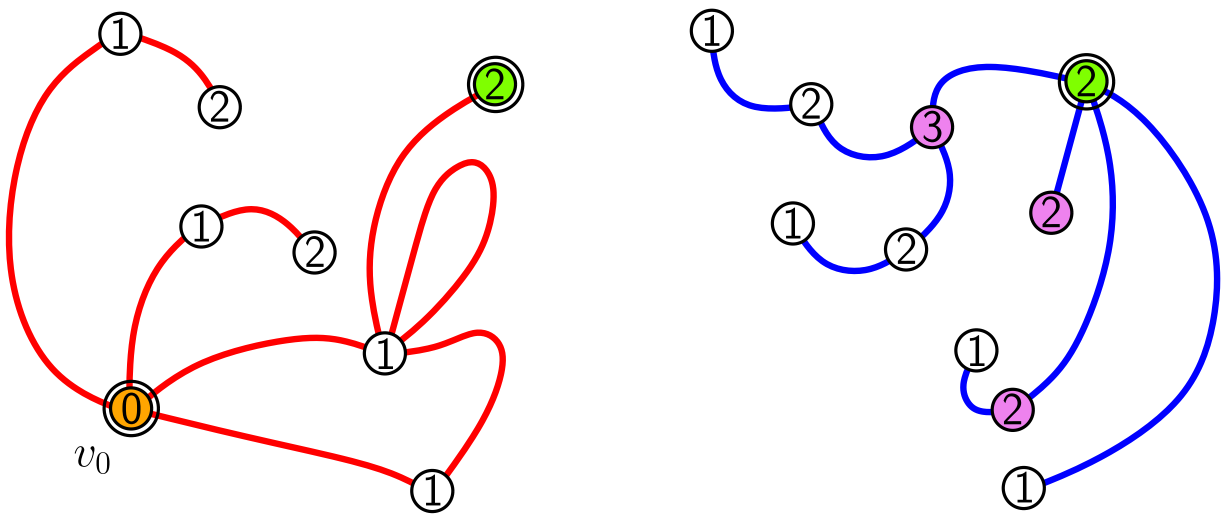


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$$F_i = \log \frac{1}{1-g(R_{i-1}+R_i+R_{i+1})} - \log \frac{1}{1-g(R_{i-1}+R_i)}$$
$$= \log(1 + gR_iR_{i+1})$$

$$\Rightarrow \boxed{G_d = \log\left(\frac{[d+1]_x^3[d+3]}{[d]_x[d+2]_x^3}\right)}$$ for general maps

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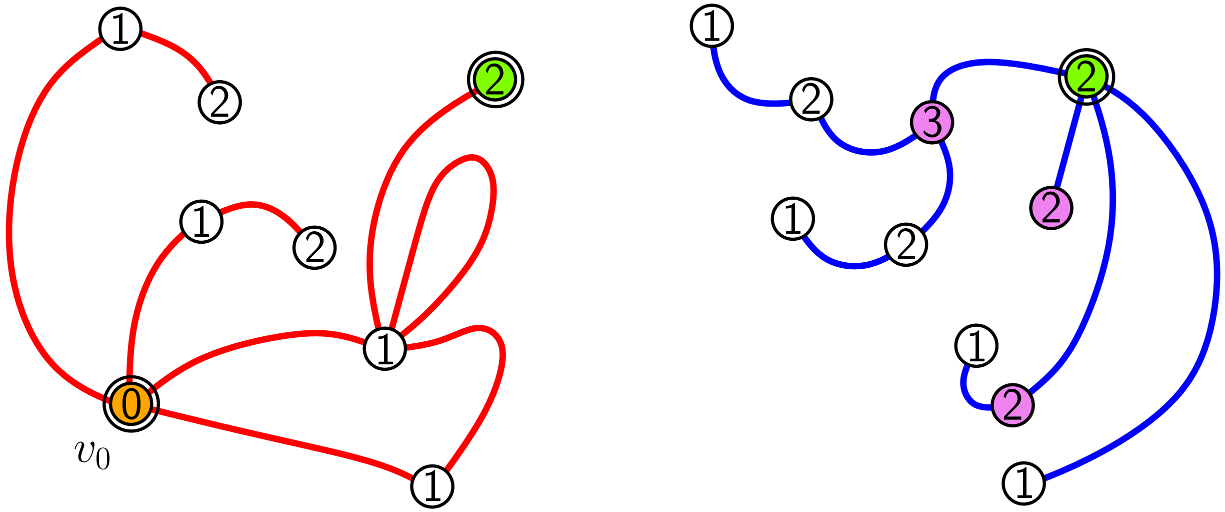


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$$\Rightarrow \boxed{G_d = \log\left(\frac{[d+1]_x^3[d+3]}{[d]_x[d+2]_x^3}\right)}$$ for general maps

recall $G_d = \log\left(\frac{[d]_x^2[d+3]_x}{[d-1]_x[d+2]_x^2}\right)$ for quadrang. (same asymptotic laws)

# The case of two local min

Let $M$ a well-labelled map with two local min $v_1, v_2$

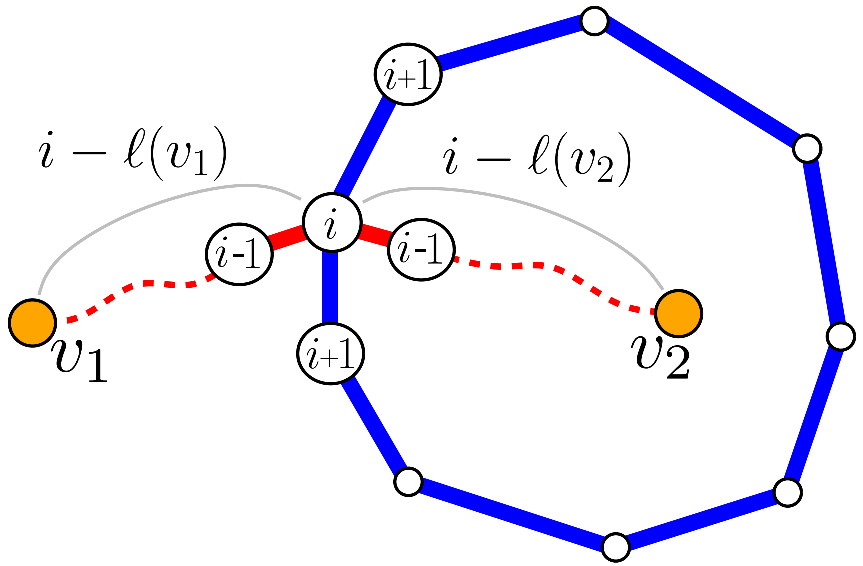
Let $M' = \Lambda(M)$, let $f_1, f_2$ the two faces of $M'$

Let $\Gamma$ the (cycle) boundary of $M'$, $i := \min_\Gamma$

**Two cases:**

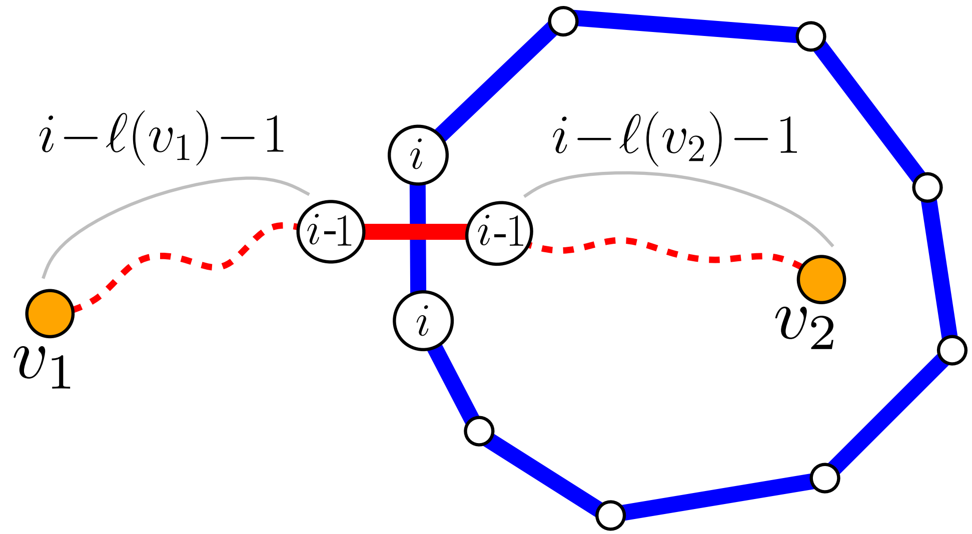
A): no edge of labels $i - i$ on $\Gamma$

$$\boxed{\mathrm{dist}_M(v_1, v_2) = 2i - \ell(v_1) - \ell(v_2)}$$




B): $\exists$ an edge of labels $i - i$ on $\Gamma$

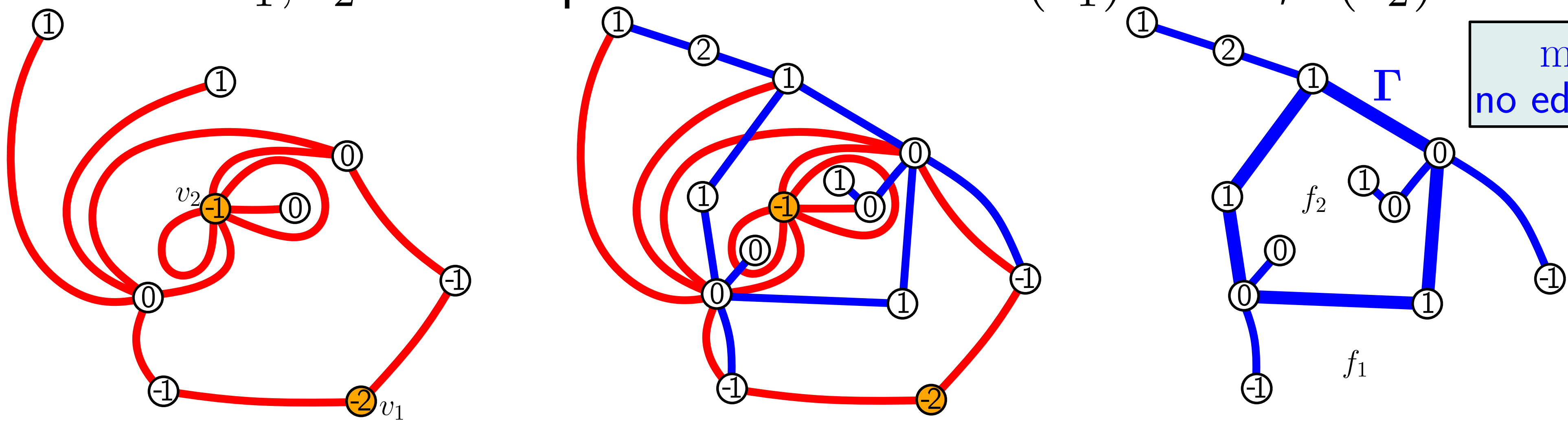
$$\boxed{\mathrm{dist}_M(v_1, v_2) = 2i - \ell(v_1) - \ell(v_2) - 1}$$



# 2 other ways to compute the 2-point function

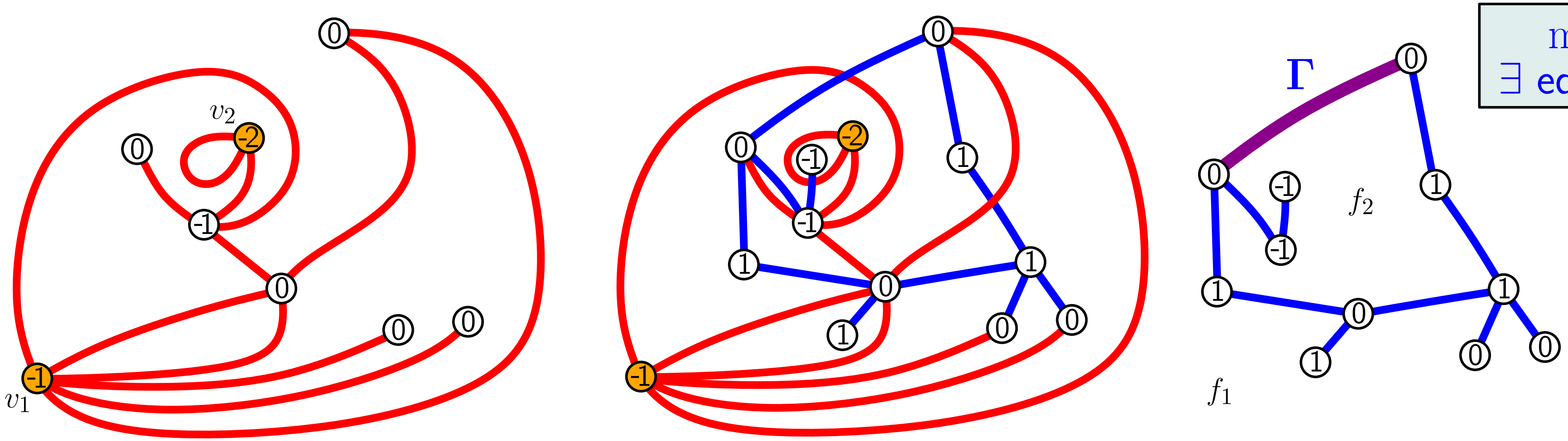
**[F, Guitter'14]** For $d \geq 1$, let $M$ a bi-pointed map with $d_{12} = d$

A) Write $d$ as $s + t$ with $s, t \geq 1$. Endow $M$ with unique well-labelling where $v_1, v_2$ are unique local min and $\ell(v_1) = -s$, $\ell(v_2) = -t$




$\min_\Gamma = 0$
no edge 0-0 on $\Gamma$

B) Write $d$ as $s + t - 1$ with $s, t \geq 1$. Endow $M$ with unique well-labelling where $v_1, v_2$ are unique local min and $\ell(v_1) = -s$, $\ell(v_2) = -t$

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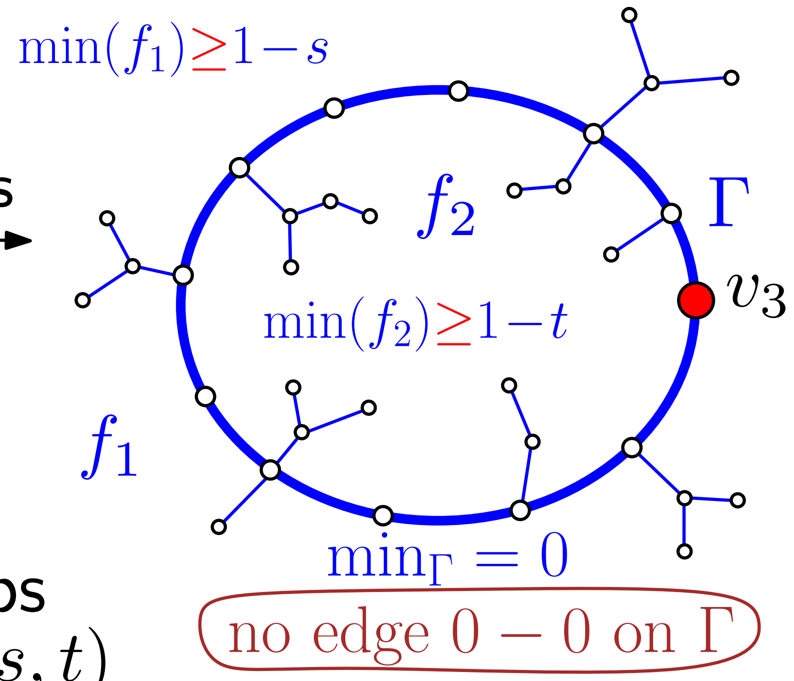
# 2 other ways to compute the 2-point function

**Case (A):** $G_{s+t}(g) = \Delta_s \Delta_t \log(\underbrace{N_{s,t}})$ counts

$$X_{s,t} = \frac{N_{s,t}}{1-gR_sR_tN_{s,t}}$$

$\Rightarrow$ exact expression for $N_{s,t}$

recover $G_{s+t} = \log\left(\frac{[s+t]_x^2[s+t+3]_x}{[s+t-1]_x[s+t+2]_x^2}\right)$




**Rk:** $\Delta_s\Delta_t N_{s,t}$ gives GF of tri-pointed maps with aligned points: $d_{12}, d_{13}, d_{23} = (s+t, s, t)$

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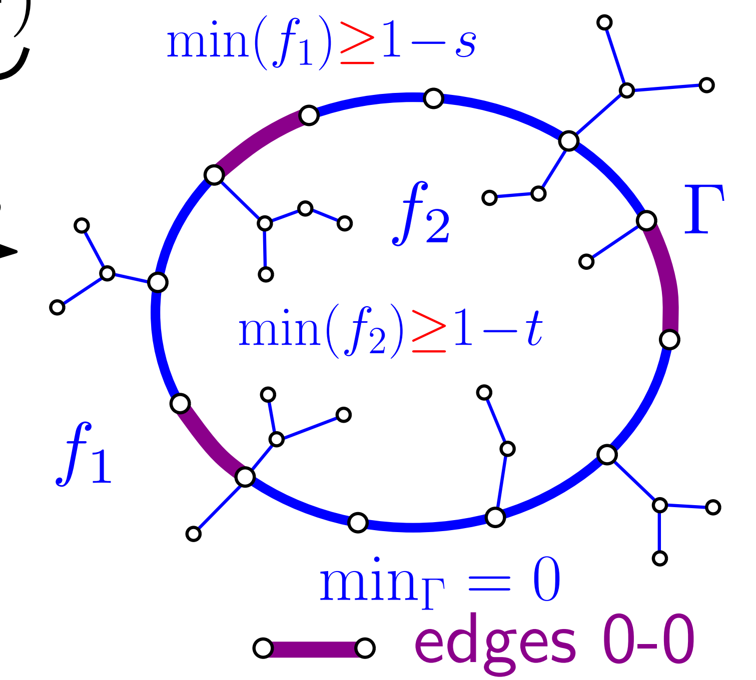
**Case (B):** $G_{s+t-1}(g) = \Delta_s \Delta_t \underbrace{\log\left(\frac{1}{1-gR_sR_tN_{s,t}}\right)}$ counts

recover $G_{s+t-1} = \log\left(\frac{[s+t-1]_x^2[s+t+2]_x}{[s+t-2]_x[s+t+1]_x^2}\right)$



# 3-point function: generic (non-aligned) case

**Case A:** $d_{12} + d_{13} + d_{23}$ even

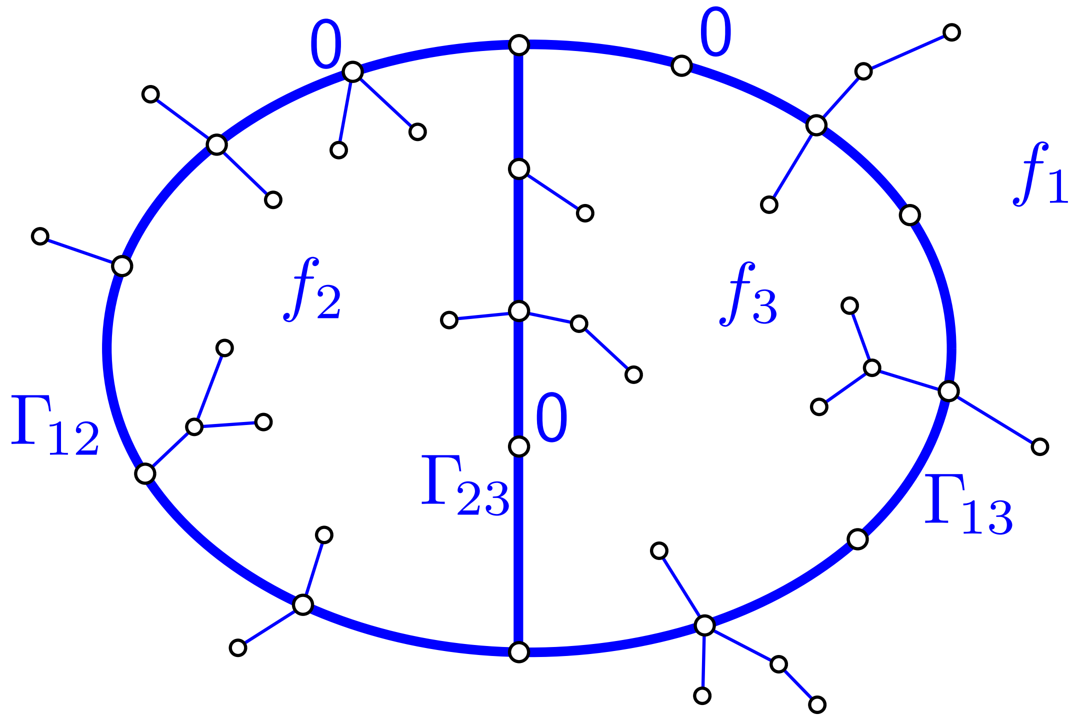
parametrize as:

$$d_{12} = s + t \qquad \text{with } s, t, u > 0$$
$$d_{13} = s + u$$
$$d_{23} = t + u$$

endow tri-pointed map with unique "$(-s, -t, -u)$-well-labelling"
and apply the AB bijection $\Lambda$

$\min(f_1) = 1 - s \qquad \min_{\Gamma_{12}} = 0$
$\min(f_2) = 1 - t \qquad \min_{\Gamma_{13}} = 0$
$\min(f_3) = 1 - u \qquad \min_{\Gamma_{23}} = 0$

and no edge 0-0 on $\Gamma$



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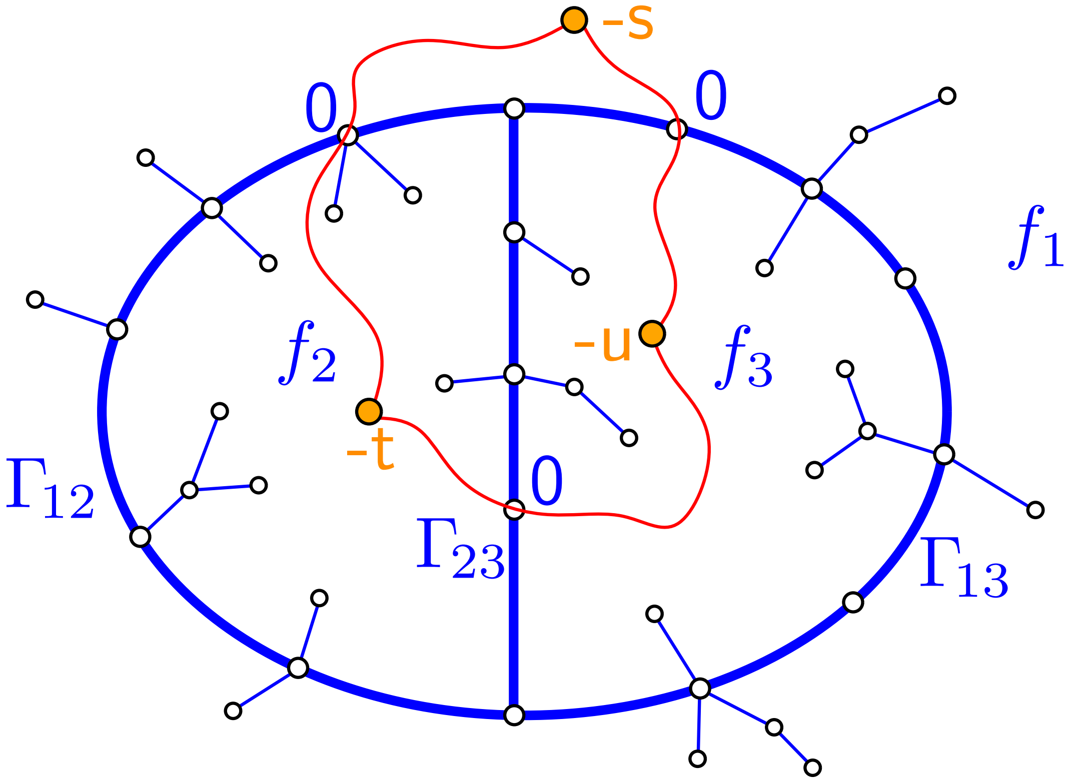

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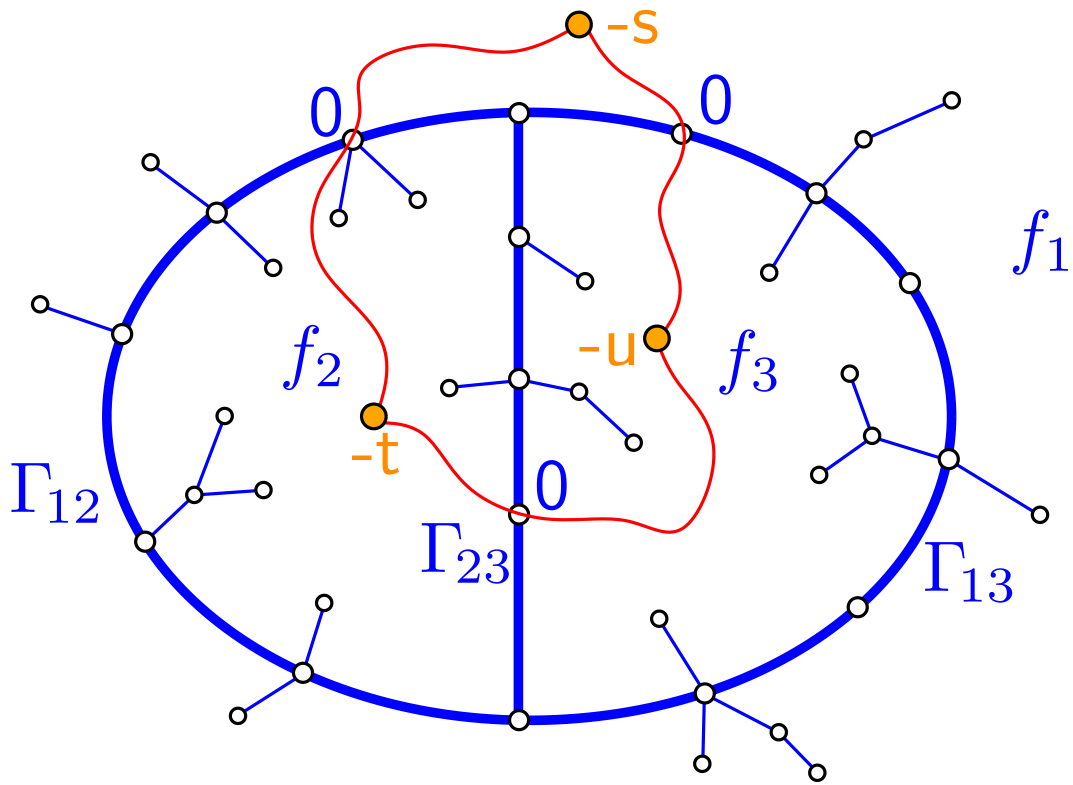


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$\Rightarrow$ expression of $G_{d_{12},d_{13},d_{23}}(g)$ as $\Delta_s\Delta_t\Delta_u F^{\text{even}}_{s,t,u}$, with $F^{\text{even}}_{s,t,u}(g)$ explicit

# 3-point function: generic (non-aligned) case

**Case B:** $d_{12} + d_{13} + d_{23}$ odd (did not exist for quadrang.)

parametrize as:

$$d_{12} = s + t - 1 \qquad \text{with } s, t, u > 0$$
$$d_{13} = s + u - 1$$
$$d_{23} = t + u - 1$$

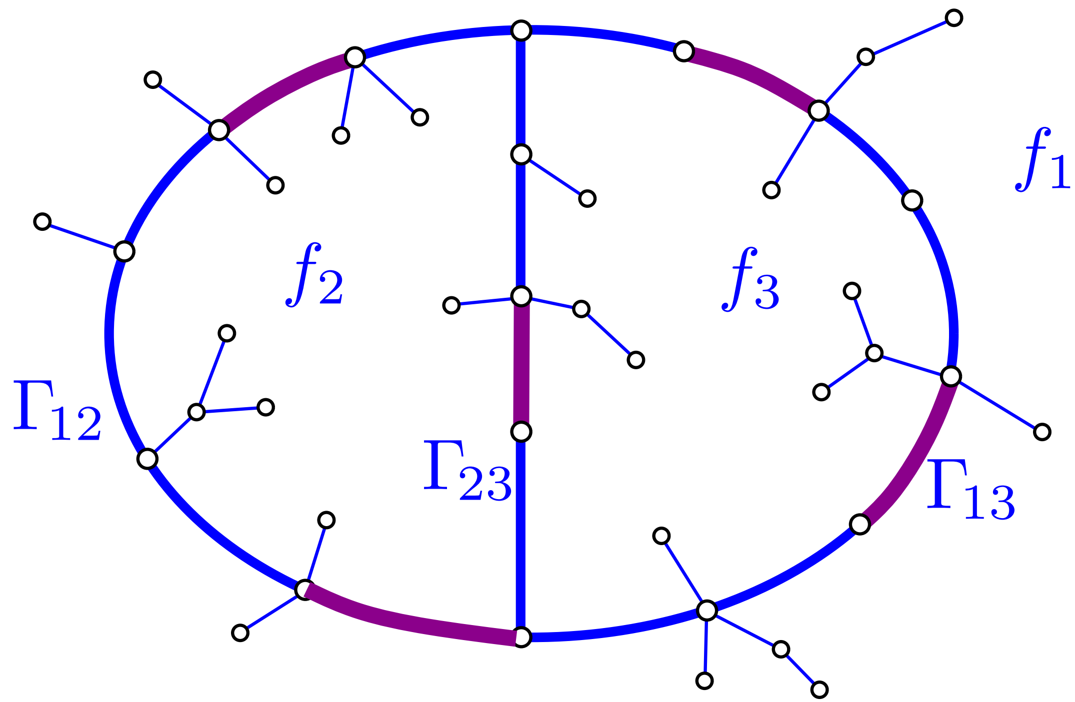
endow tri-pointed map with unique "$(-s, -t, -u)$-well-labelling" and apply the AB bijection $\Lambda$

$\min(f_1) = 1 - s \quad \min_{\Gamma_{12}} = 0$

$\min(f_2) = 1 - t \quad \min_{\Gamma_{13}} = 0$

$\min(f_3) = 1 - u \quad \min_{\Gamma_{23}} = 0$

and there is an edge 0-0 on each of $\Gamma_{12}, \Gamma_{13}, \Gamma_{23}$



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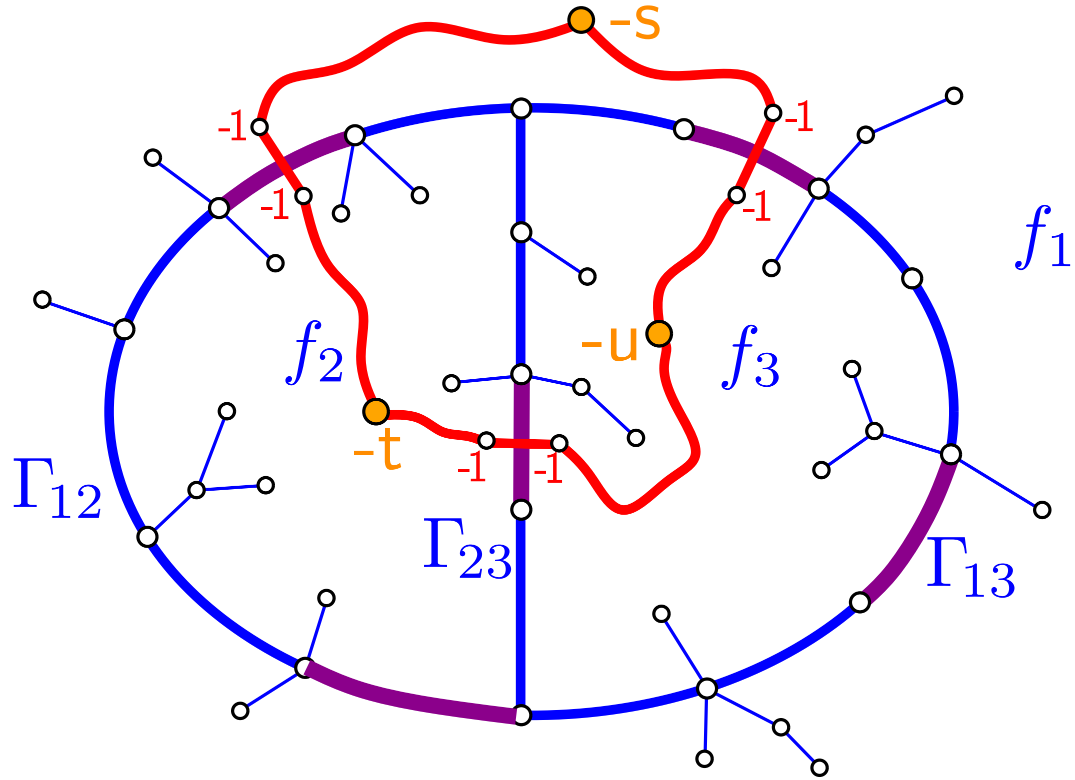

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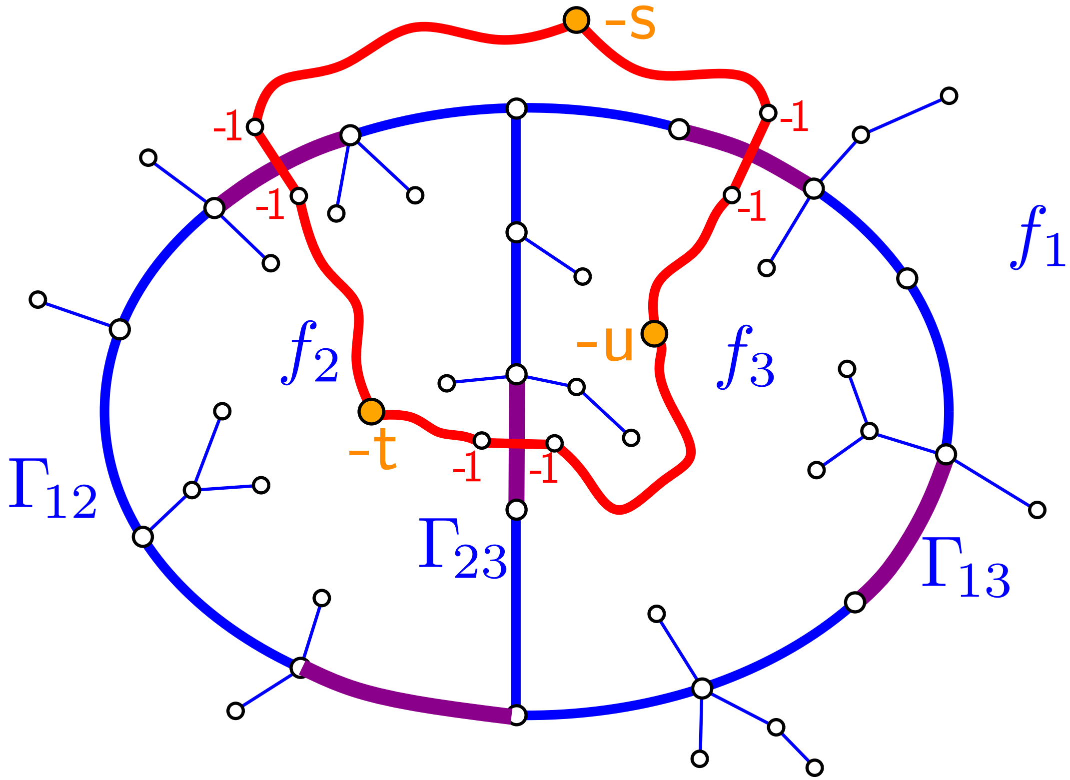
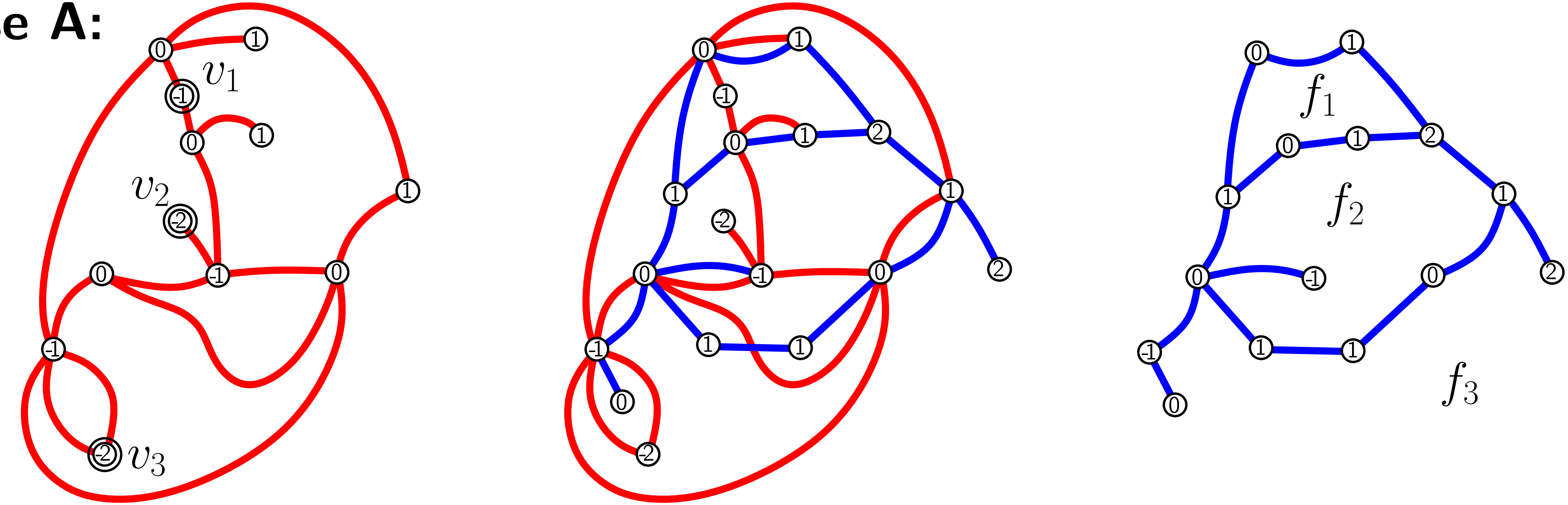



$\Rightarrow$ expression of $G_{d_{12},d_{13},d_{23}}(g)$ as $\Delta_s \Delta_t \Delta_u F^{\mathrm{odd}}_{s,t,u}$, with $F^{\mathrm{odd}}_{s,t,u}(g)$ explicit

# Examples

## Case A:

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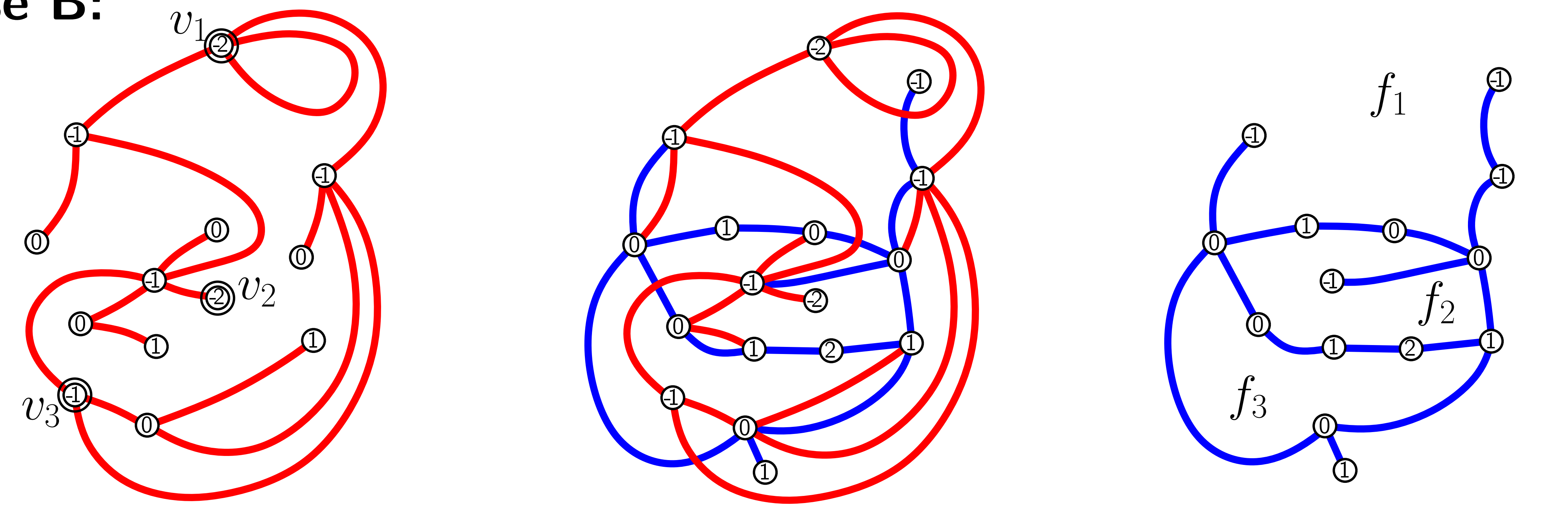
## Case B:



# Conclusion and remarks

- There are exact expressions for the 2-point and 3-point functions of quadrangulations and general maps (bijections + GF calculations)

- Asymptotically the limit laws (rescaling by $n^{1/4}$) are the same for the random quad. $Q_n$ of size $n$ as for the random map $M_n$ of size $n$

  **Rk:** also follows from **[Bettinelli, Jacob, Miermont'13]**
  $(Q_n, \mathrm{dist}/n^{1/4})$ and $(M_n, \mathrm{dist}/n^{1/4})$ are close as metric spaces, when coupling $(M_n, Q_n)$ by the AB bijection

- We can also obtain similar expressions for bipartite maps (associated well-labelled maps are restricted to have no edge $i - i$)

- The GF expressions $G_D(g)$ for maps/bipartite maps can be extended to expressions $G_D(g, z)$ where $z$ marks the number of faces