Peeking at quantum gravity with self-overlapping curves

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From the point of view of quantum field theory, we would like to write and solve:

$$Z = \int_{\mathcal{M}} \mathcal{D}g_{\mu\nu} \mathcal{D}\Phi \exp(-S[g_{\mu\nu}, \Phi])$$

$g_{\mu\nu}$: metric structure on $\mathcal{M}$; Φ: matter content.

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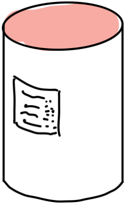
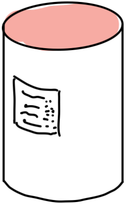
What are $\mathcal{D}$, $S[\cdot]$, Z?

Postulate (['t Hooft 1993, Susskind 1995]):

Quantum Gravity ($D+1$ dimensions)

$=$ **Quantum Theory (D dimensions)**





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Hints for the emergence of gravity:

- Black hole entropy [Bekenstein 1972]:

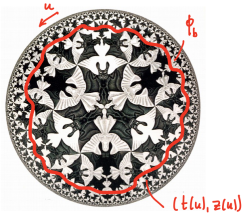
$$S_{BH} = \frac{c^3}{4G\hbar} A_{\text{horizon}}$$

- Laws of black hole thermodynamics [Bardeen, Bekenstein, Carter, Hawking 1973]

A simple model of holography

Bulk: Near-horizon limit of (near-extremal) black holes
$\rightarrow$ **Jackiw-Teitelboim theory** ($D = 2$):

$$Z = \int \mathcal{D}g_{\mu\nu}\mathcal{D}\phi \exp\left(\frac{1}{16\pi G_N}\int \mathrm{d}^2x\sqrt{g}\phi(R+2) + \frac{\phi_b}{8\pi G_N}\oint \mathrm{d}sk\right) .$$



Source: math.slu.edu

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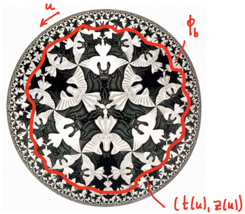

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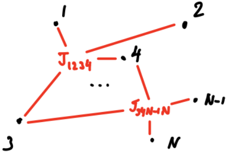

Boundary: **Sachdev-Ye-Kitaev model** ($D = 1$)

$$H = \sum_{1\leq i<j<k<l\leq N} J_{ijkl}\psi_i\psi_j\psi_k\psi_l, \quad \left\langle J_{ijkl}^2\right\rangle \propto \frac{J^2}{N^3} .$$

Same effective action on the boundary (a comprehensive review is [Mertens 2022]):

$$\frac{\phi_b}{8\pi G_N} \oint_{S^1} \mathrm{d}u Sch[t,u] \ ,$$

with $t : S^1 \to S^1,\ t' > 0$ (reparametrization) and $Sch[t,u] = \left(\frac{t''}{t'}\right)' - \frac{1}{2}\left(\frac{t''}{t'}\right)^2$.

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Also equivalent to a random matrix theory [Saad, Shenker, Stanford 2019]:

$$Z = \int \mathrm{d}H \exp\left(- \operatorname{Tr} V(H)\right) \ ,$$

$$\rho_0(E) = \frac{\gamma}{2\pi^2} \sinh\left(2\pi\sqrt{2\gamma E}\right) \ .$$

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But... no Hilbert space interpretation!

We are interested in metrics on the disk.

Conformal gauge:

$$\mathrm{d}s^2 = e^{2\Sigma}|\mathrm{d}z|^2 \,, \qquad z = x + iy \,.$$

Metrics of constant curvature:

$$4\partial_z\partial_{\bar{z}}\Sigma = -\kappa e^{2\Sigma} \,, \quad \kappa = \pm 1, 0 \,.$$

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Theorem

a) *Let $\Sigma_b : S^1 \to \mathbb{R}$ be a continuous function defined on the boundary of the disk. Then there exists a unique solution $\Sigma \in C^\infty(D)$ of the Liouville equation such that $\Sigma = \Sigma_b$ on the boundary.*

b) *The most generic solution (up to disk automorphisms, $PSL(2,\mathbb{R})$) takes the form:*

$$e^{\Sigma} = \frac{2|F'(z)|}{1 + \kappa|F(z)|^2} \,.$$

Immersions of the disk

We parametrize metrics on the disk $\mathcal{D}$:

$$\mathrm{d}s^2 = \frac{4|F'(z)|^2}{\left(1+\kappa|F(z)|^2\right)^2}|\mathrm{d}z|^2, \quad \begin{cases} F : \mathcal{D} \to H^2 \text{ holomorphic,} \\ F'(z) \neq 0 \ \forall z \in \mathcal{D}. \end{cases}$$

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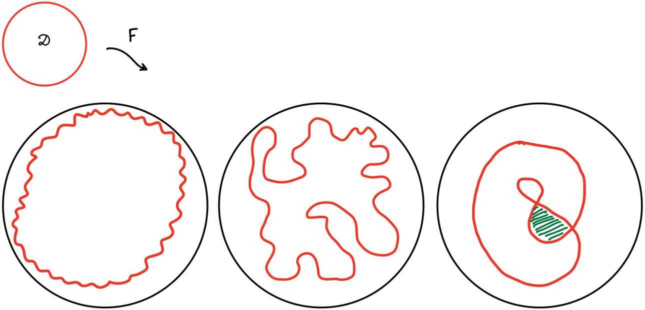

Figure: Reparametrization embedding – General embedding – Immersion

Questions

- How does considering immersions change the previous results?
- What are the properties of those immersions?
  Number of self-overlaps, fractals,...?
- Of their boundaries?
  Do they characterise the whole immersion?
  Minimal combinatorial properties that lead to an immersion?

Self-overlapping curves $=$ curves that are boundary of an immersed disk.

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Evans & Wenk [2020] (interior boundary, minimum homotopy area).

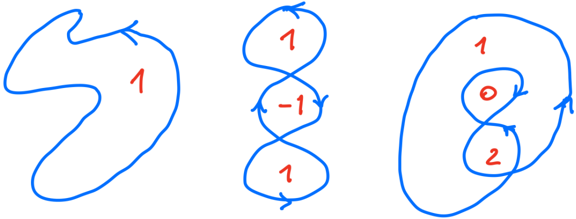
Self-overlapping curves: Numbers

- Turning number

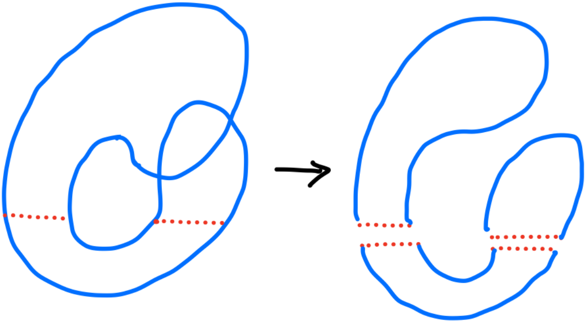
$$\mathrm{turn}(\gamma) = \frac{1}{2\pi}\oint_\gamma k \mathrm{d}s = 1\,, \qquad k = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}\,.$$

- Winding number (number of overlaps)

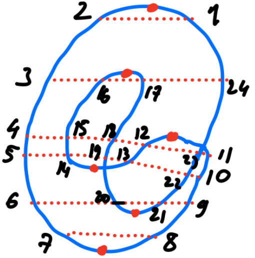
$$\mathrm{wind}_\gamma(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{\gamma'(t)}{\gamma(t) - z_0}\mathrm{d}t \geq 0\,.$$

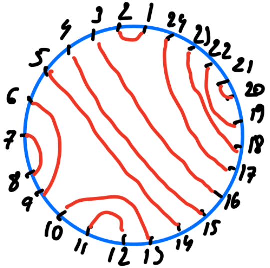


Curves that can be decomposed into simple curves through well-chosen cuts.



[Bonsma, Breuer 2012] Mapping the curve, together with good Blank cuts*, to a chordal graph, the problem of counting inequivalent disks corresponds to counting **Maximum Independent Sets** in the circle graph (for n vertices of the circle graph, $\mathcal{O}(n^2)$).





*such that it can define the interior of a disk

Examples of boundary curves that don't have a unique holomorphic extension:

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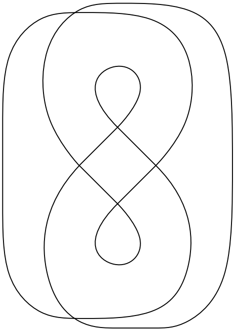
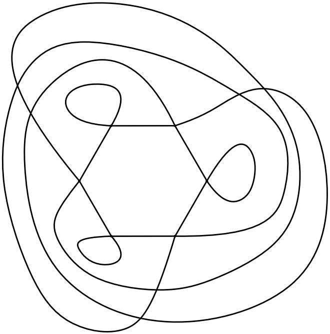


(a) Milnor (2 disks)



(b) Bennequin (5 disks)

Self-overlapping curves: Inequivalent disks

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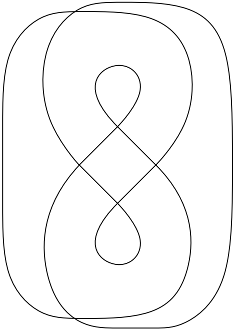

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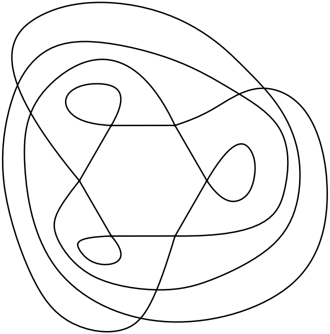
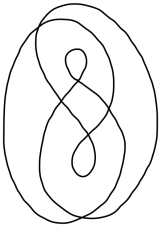
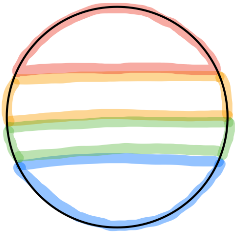


(b) Bennequin (5 disks)

NB: They can also be glued together.

Milnor's doodle





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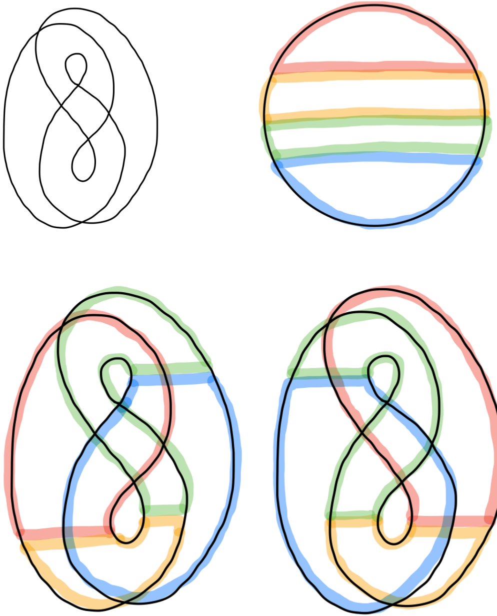
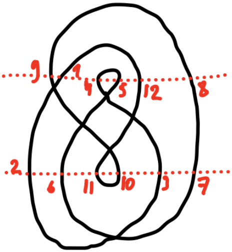




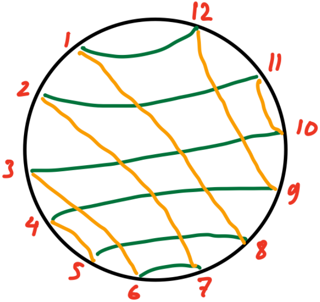


Figure: Minimal number of cuts and the associated "good" pairings.

Self-overlapping curves: technical results

Theorem (Graver, Cargo 2011)

An oriented normal curve γ, with $0 \leq wind_\gamma(f) \leq 2$, admits a unique extension if and only if:

(i) *the number of faces with $wind_\gamma(f) = 2$ equals the number of faces with $wind_\gamma(f) = 0$,*

(ii) *all faces with $wind_\gamma(f) = 2$ have boundaries of even length.*

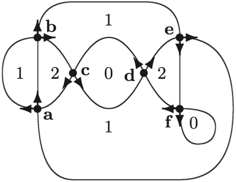
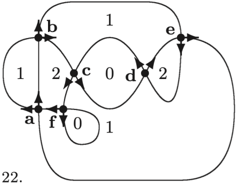


Fig. 22.



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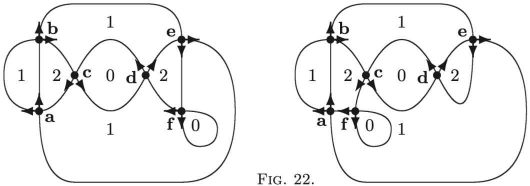

FIG. 22.

Theorem (Shor, Van Wyk 1992)

The number of incompatible decompositions is equal to the number of combinatorially inequivalent **constrained Delaunay triangulations***.*

Monte Carlo: 2D quantum “flat” gravity

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2) $\ell = \int \mathrm{d}\theta e^{\Sigma_D}$
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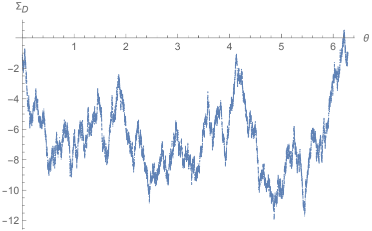
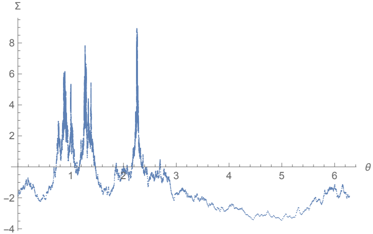
3) Redefine $\Sigma = -\Sigma_D + 2\log(\ell/2\pi)$





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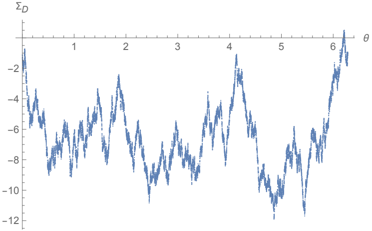

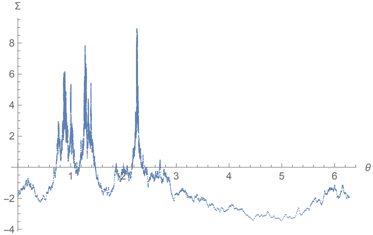

Σ has an action invariant under $PSL(2, \mathbb{R})$.

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4) Analytic continuation:

$$H(z)|_{z=e^{i\theta}} = \Sigma(\theta) + i\Gamma(\theta)\ , \quad \Gamma(\theta) = \frac{1}{2\pi}\mathrm{P}\int \mathrm{d}\theta' \frac{\Sigma(\theta')}{\tan\frac{\theta'-\theta}{2}}\ ,$$

5) Integrate the exponential of its analytic continuation:

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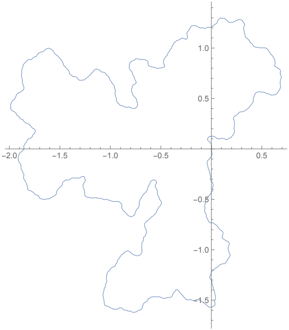
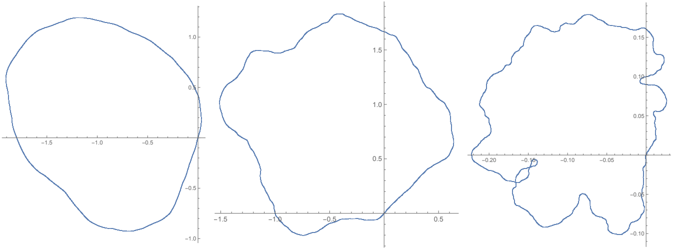
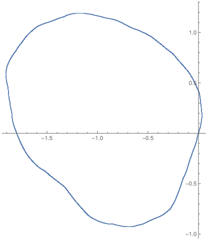




(a) $\sigma = 0.5$ (b) $\sigma = 1.5$ (c) $\sigma = 3$



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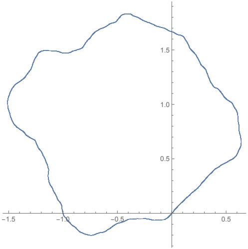
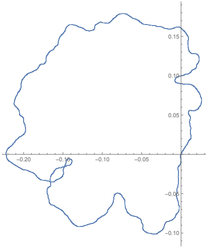


(b) $\sigma = 1.5$



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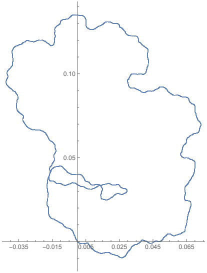
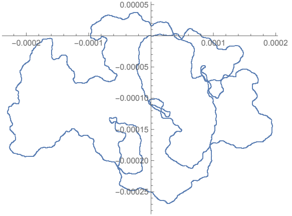
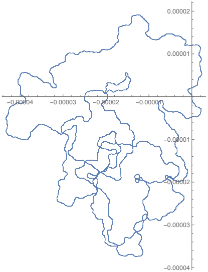


(d) $\sigma = 4$



(e) $\sigma = 7$



(f) $\sigma = 8$

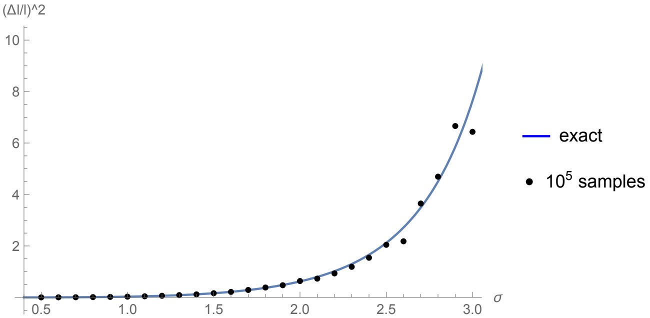
Monte Carlo: Lengths

$$\ell = \int \mathrm{d}\theta e^{\Sigma(\theta)} \ , \quad \langle \ell \rangle = 2\pi \ ,$$

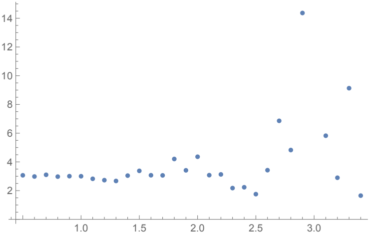
$$\left\langle \ell^2 \right\rangle = \frac{4\pi^2}{\sigma} \exp\left(-\frac{\pi\sigma^2}{12}\right) \mathrm{Erfi}\left(\frac{\sqrt{\pi}\sigma}{2}\right) \ ,$$

$$\Delta\ell = \sqrt{\langle \ell^2 \rangle - \langle \ell \rangle^2} \ ,$$

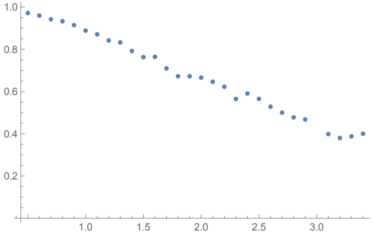
$$\mathrm{Erfi}(z) = \frac{-2i}{\sqrt{\pi}} \int_0^z \mathrm{d}t \ e^{t^2} \ .$$



$$A = \int \mathrm{d}\theta \mathrm{d}\rho \rho e^{2\Sigma(\theta,\rho)} \ .$$



(a) $\langle A \rangle$



(b) $4\pi \langle A \rangle / \langle l^2 \rangle$

Perspectives

- What is this kind of new random object?
  (fixed length, extrinsic curvature, order parameter...)
- Implement counting and identification of distinct immersions.
  (faster algorithms using minimal number of cuts?)
- Partition function.
- Hyperbolic case, other topologies...

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Thank you!