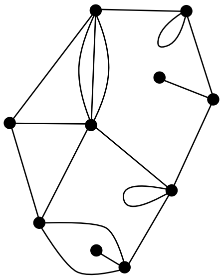
Brownian disks

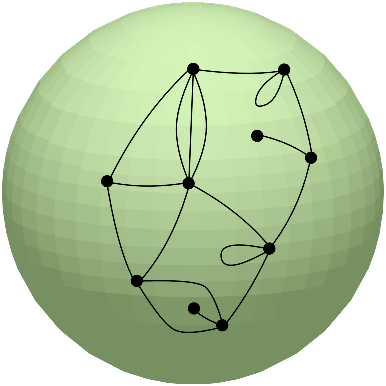
Jérémie BETTINELLI

based on joint work with Grégory Miermont

Feb. 20, 2018

Plane maps

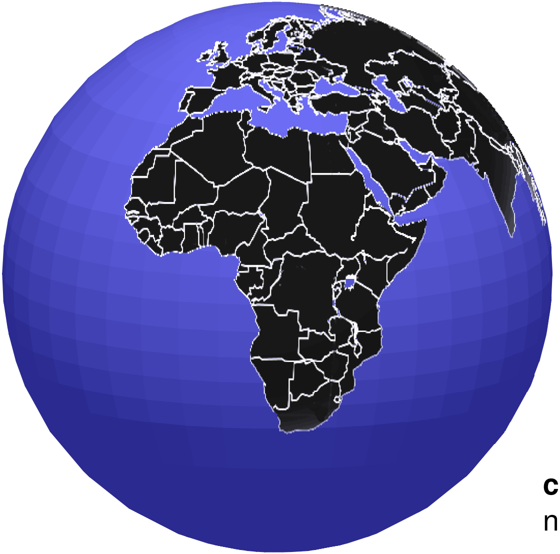




plane map: finite connected graph embedded in the sphere

faces: connected components of the complement

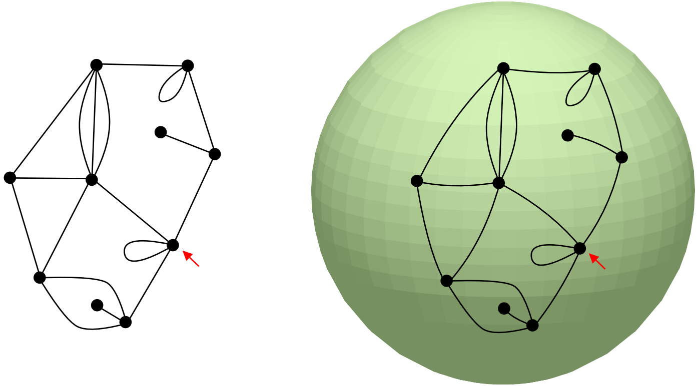
Example of plane map



faces:
countries and
bodies of water

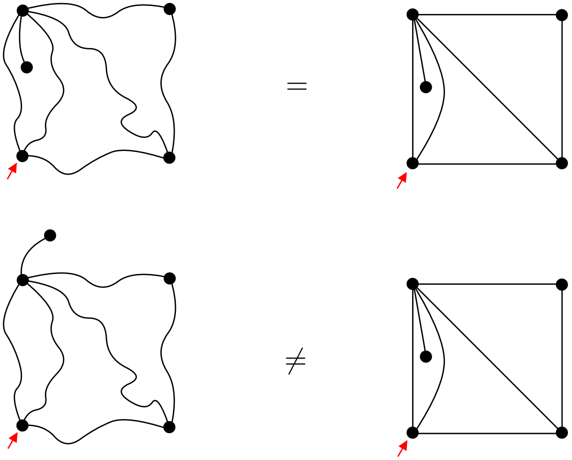
connected graph
no “enclaves”

Rooted maps

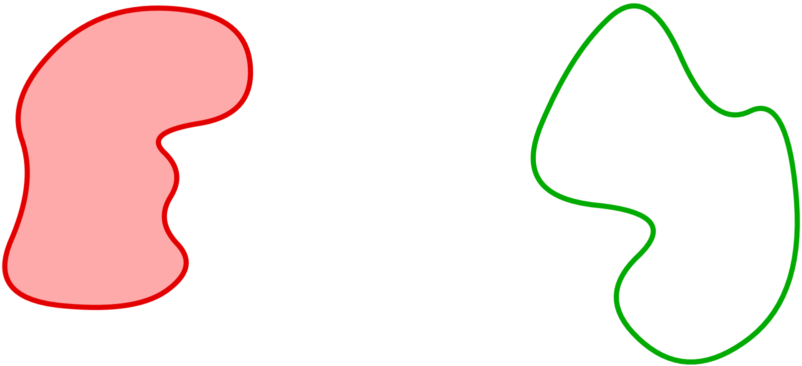


rooted map: map with one distinguished corner

Edge deformation



Gromov–Hausdorff topology: picture



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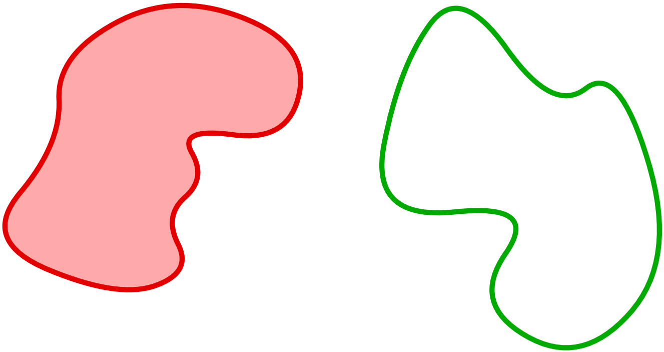

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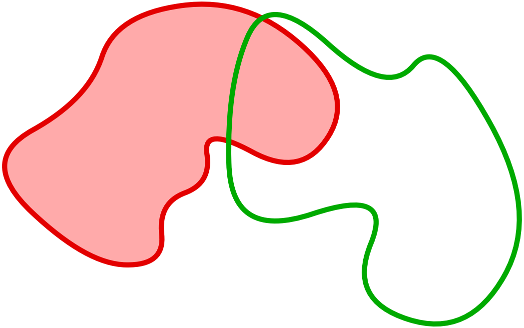

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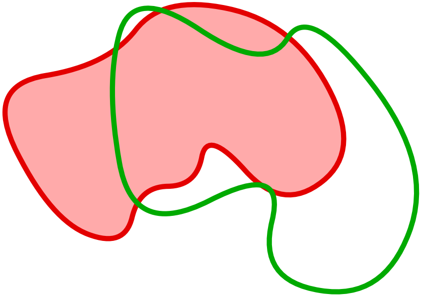

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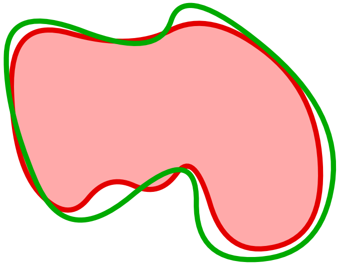

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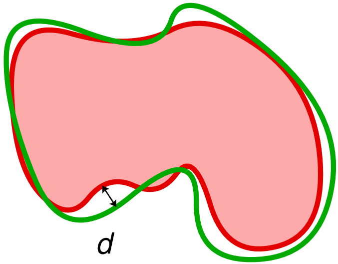

Gromov–Hausdorff topology: formal definition

- $[X, d]$: isometry class of (X, d)
- $\mathbb{M} := \{[X, d],\ (X, d) \text{ compact metric space}\}$

$$d_{\mathrm{GH}}\big([X, d], [X', d']\big) := \inf d_{\mathrm{Hausdorff}}\big(\varphi(X), \varphi'(X')\big)$$

where the infimum is taken over all metric spaces (Z, δ) and isometric embeddings $\varphi : (X, d) \to (Z, \delta)$ and $\varphi' : (X', d') \to (Z, \delta)$.

- $(\mathbb{M}, d_{\mathrm{GH}})$ is a Polish space.

Scaling limit: the Brownian map

- $a\,\mathfrak{m}$: finite metric space obtained by endowing the vertex-set of $\mathfrak{m}$ with a times the graph metric (each edge has length a).

Theorem (Le Gall '11, Miermont '11)

Let $\mathfrak{q}_n$ be a uniform plane quadrangulation with n faces. The sequence $\left((8n/9)^{-1/4}\,\mathfrak{q}_n\right)_{n\geq 1}$ converges weakly in the sense of the Gromov–Hausdorff topology toward a random compact metric space called the Brownian map.

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Definition (Convergence for the Gromov–Hausdorff topology)

A sequence $(\mathcal{X}_n)$ of compact metric spaces **converges in the sense of the Gromov–Hausdorff topology** toward a metric space $\mathcal{X}$ if there exist isometric embeddings $\varphi_n : \mathcal{X}_n \to \mathcal{Z}$ and $\varphi : \mathcal{X} \to \mathcal{Z}$ into a common metric space $\mathcal{Z}$ such that $\varphi_n(\mathcal{X}_n)$ converges toward $\varphi(\mathcal{X})$ in the sense of the Hausdorff topology.

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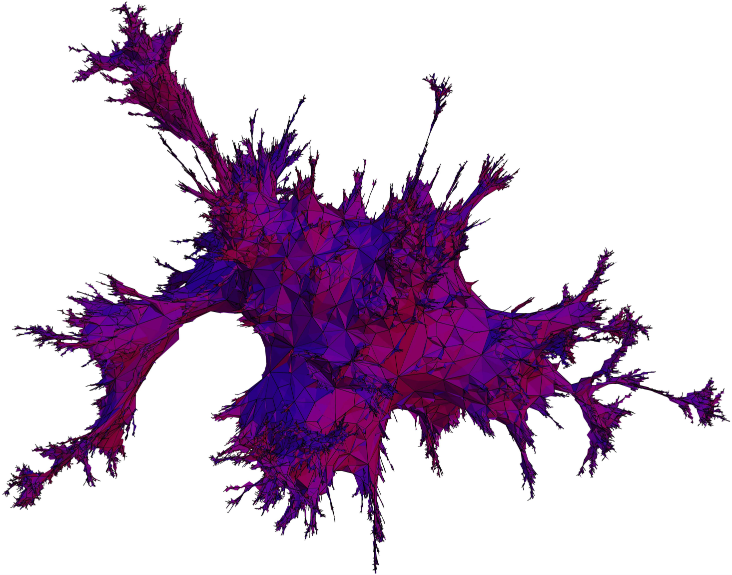
- This theorem has been proven independently by two different approaches by Miermont and by Le Gall.

Uniform plane quadrangulation with 50 000 faces



Earlier results

- Chassaing–Schaeffer ’04
  - the scaling factor is $n^{1/4}$
  - scaling limit of functionals of random uniform quadrangulations (radius, profile)
- Marckert–Mokkadem ’06
  - introduction of the Brownian map
- Le Gall ’07
  - the sequence of rescaled quadrangulations is relatively compact
  - any subsequential limit has the topology of the Brownian map
  - any subsequential limit has Hausdorff dimension 4
- Le Gall–Paulin ’08 & Miermont ’08
  - the topology of any subsequential limit is that of the two-sphere
- Bouttier–Guitter ’08
  - limiting joint distribution between three uniformly chosen vertices

Universality of the Brownian map

Many other natural models of plane maps converge to the Brownian map (up to a model-dependent scale constant): for well-chosen maps $\mathfrak{m}_n$,

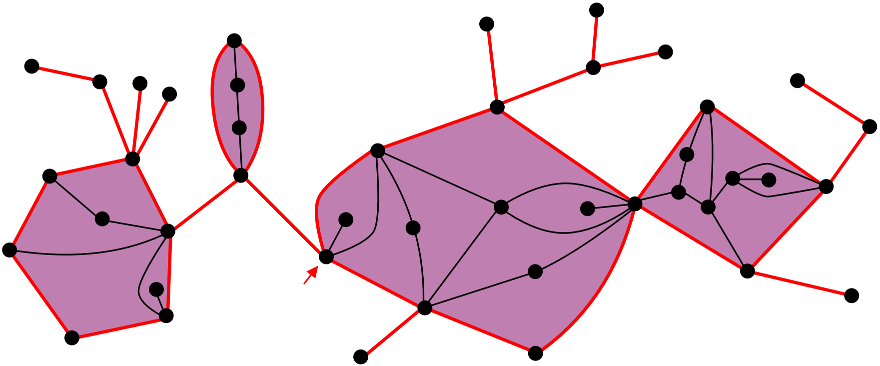
$$c\, n^{-1/4}\, \mathfrak{m}_n \xrightarrow[n\to\infty]{} \text{Brownian map.}$$

- Le Gall '11: uniform p-angulations for $p \in \{3, 4, 6, 8, 10, \dots\}$ and Boltzmann bipartite maps with fixed number of vertices

Using Le Gall's method, many generalizations:

- Beltran and Le Gall '12: quadrangulations with no pendant edges
- Addario-Berry–Albenque '13: simple triangulations and simple quadrangulations
- B.–Jacob–Miermont '14: general maps with fixed number of edges
- Abraham '14: bipartite maps with fixed number of edges
- Marzouk '17: bipartite maps with prescribed degree sequence
- Albenque (in prep.): p-angulations for odd $p \geq 5$

Plane quadrangulations with a boundary



plane quadrangulations with a boundary: plane map whose faces have degree 4, except maybe the root face

the boundary is not in general a simple curve

Scaling limit: generic case

- $\mathfrak{q}_n$ uniform among quadrangulations with a boundary having n internal faces and an external face of degree $2l_n$
- $l_n/\sqrt{2n} \to L \in (0, \infty)$

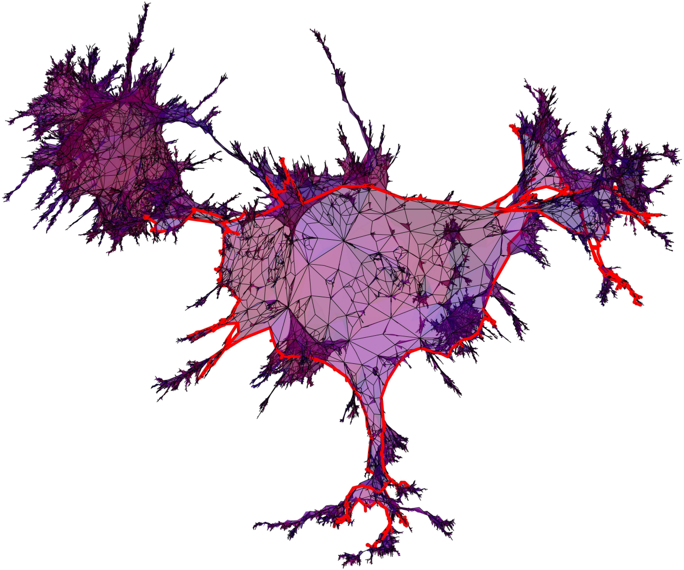
Theorem (B.–Miermont '15)

The sequence $\left((8n/9)^{-1/4}\,\mathfrak{q}_n\right)_{n\geq 1}$ *converges weakly in the sense of the Gromov–Hausdorff topology toward a random compact metric space* BD_L *called the Brownian disk of perimeter L.*

Theorem (B. '11)

Let $L > 0$ *be fixed. Almost surely, the space* BD_L *is homeomorphic to the closed unit disk of* $\mathbb{R}^2$*. Moreover, almost surely, the Hausdorff dimension of* BD_L *is 4, while that of its boundary* $\partial\,\mathrm{BD}_L$ *is 2.*

40 000 faces and boundary length 1 000



Scaling limit: degenerate cases

- $\mathfrak{q}_n$ uniform among quadrangulations with a boundary having n internal faces and an external face of degree $2l_n$
- $l_n/\sqrt{2n} \to 0$

Theorem (B. '11)

The sequence $\left((8n/9)^{-1/4}\,\mathfrak{q}_n\right)_{n\geq 1}$ converges weakly in the sense of the Gromov–Hausdorff topology toward the Brownian map.

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Theorem (B. '11)

The sequence $\left((2\sigma_n)^{-1/2}\mathfrak{q}_n\right)_{n\geq 1}$ converges weakly in the sense of the Gromov–Hausdorff topology toward the Brownian Continuum Random Tree (universal scaling limit of models of random trees).

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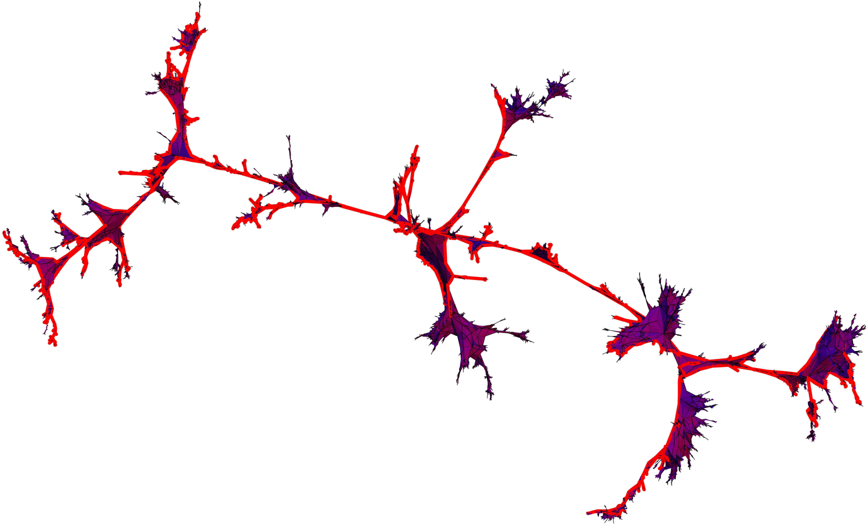
to be compared with Bouttier–Guitter '09

10 000 faces and boundary length 2 000



Universality

Theorem (B.–Miermont '15)

Let $L \in (0,\infty)$ be fixed, $(l_n, n \geq 1)$ be a sequence of integers such that $l_n \sim L\sqrt{p(p-1)n}$ as $n \to \infty$, and $\mathfrak{m}_n$ be uniformly distributed over the set of $2p$-angulations with n internal faces and perimeter $2l_n$. Then $\left(\left(4p(p-1)n/9\right)^{-1/4}\mathfrak{m}_n\right)_{n\geq 1}$ converges weakly in the sense of the Gromov–Hausdorff topology toward BD_L.

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Theorem (B.–Miermont '15)

Let $\mathfrak{m}_n$ be a uniform random bipartite map with n edges and with perimeter $2l_n$, where $l_n \sim 3L\sqrt{n/2}$ for some $L > 0$. Then $\left((2n)^{-1/4}\,\mathfrak{m}_n\right)_{n\geq 1}$ converges weakly in the sense of the Gromov–Hausdorff topology toward BD_L.

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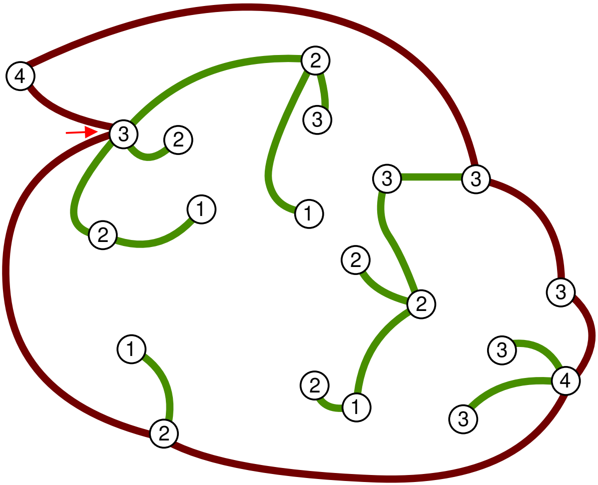
- More universality results for bipartite Boltzmann maps conditionned on their number of vertices, faces or edges.

The encoding bijection



- Take a labeled forest.

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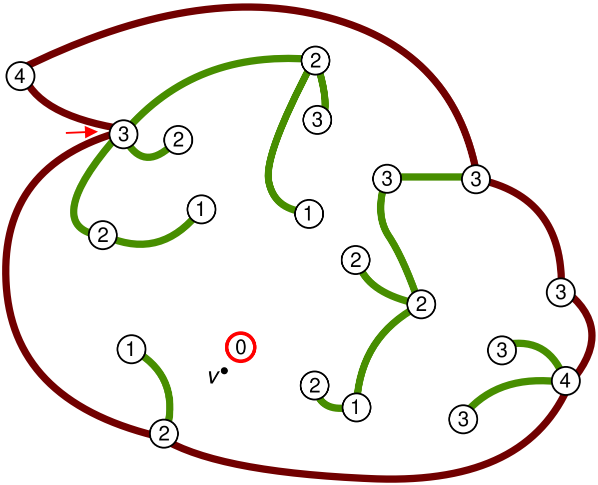

- Take a labeled forest.
- Add a vertex $v^\bullet$ inside the unique face.

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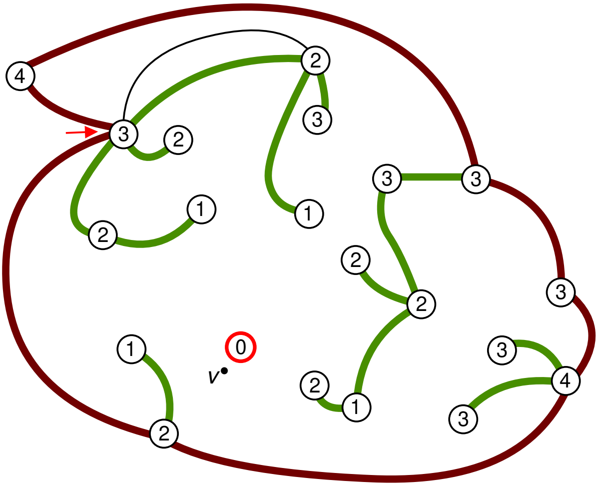

- Take a labeled forest.
- Add a vertex $v^\bullet$ inside the unique face.
- Link every corner to the first subsequent corner having a strictly smaller label.

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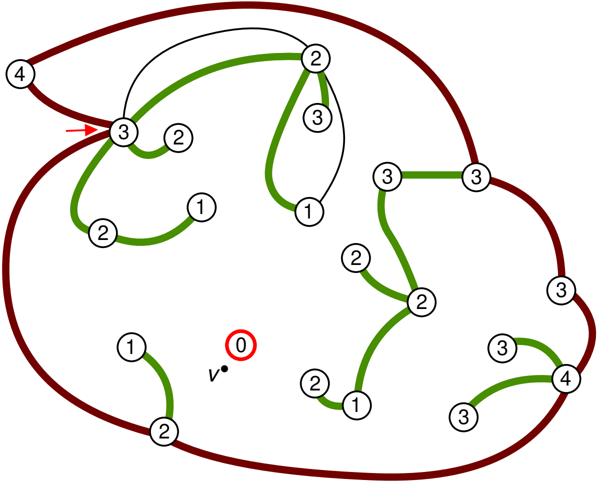

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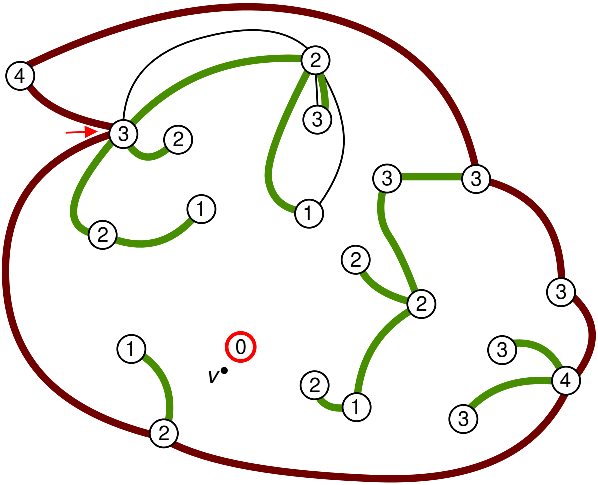

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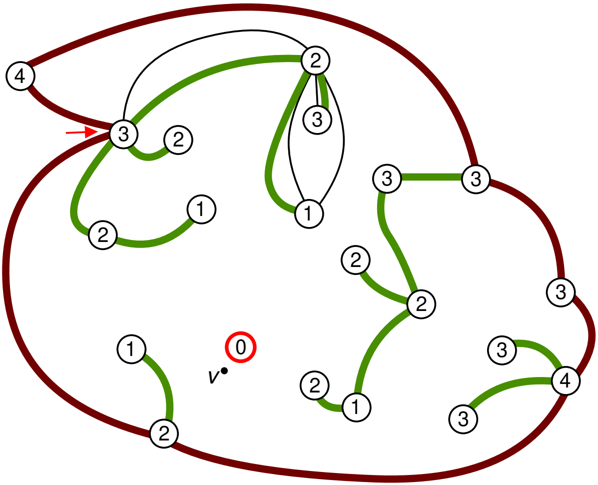

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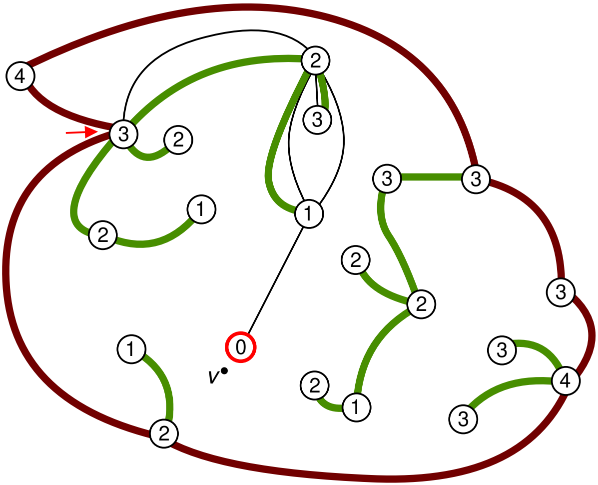

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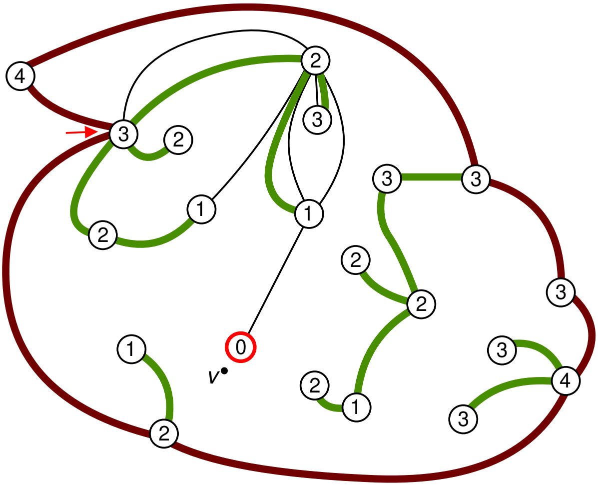

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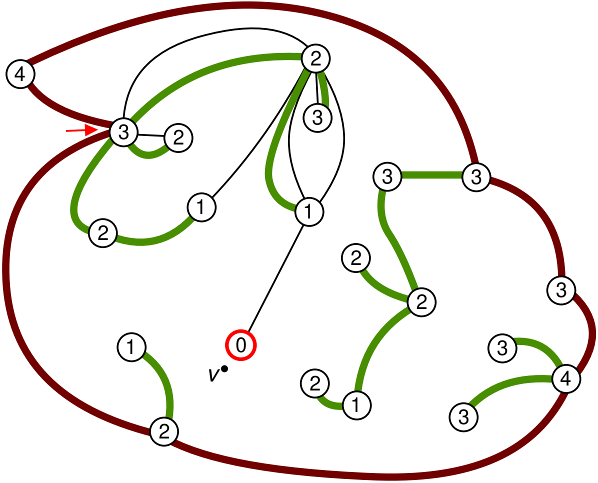

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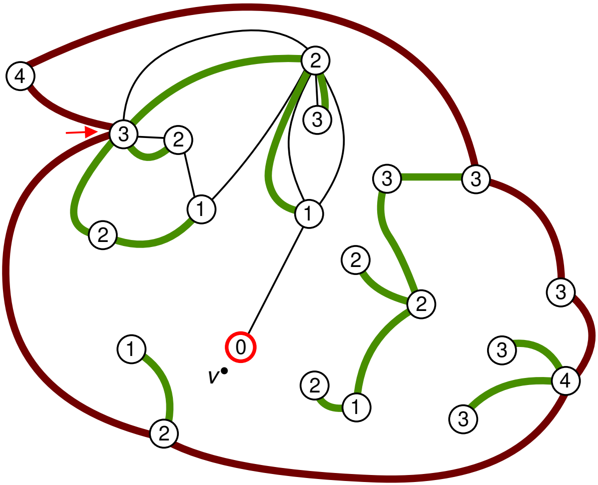

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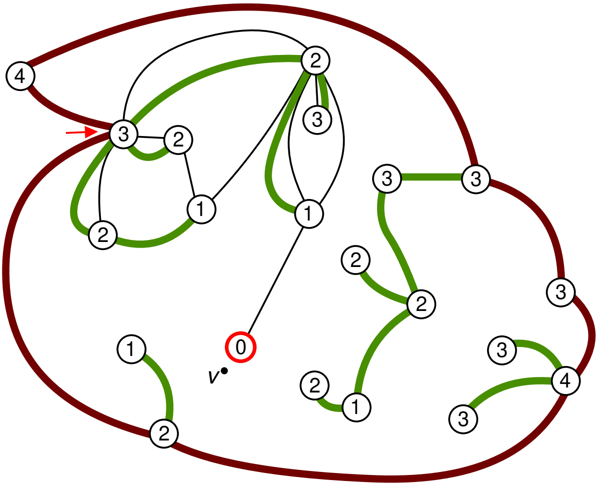

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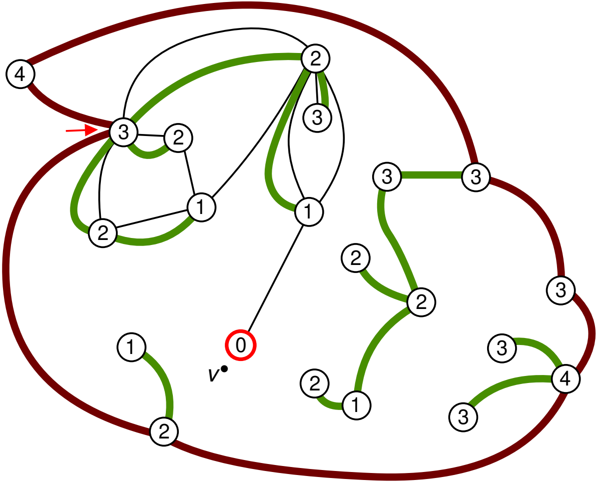

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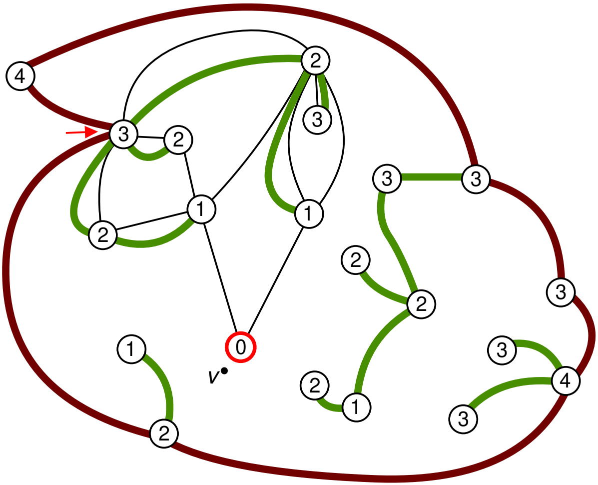

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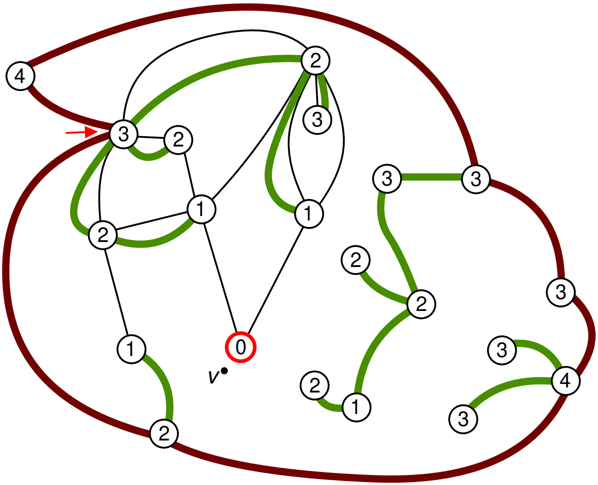

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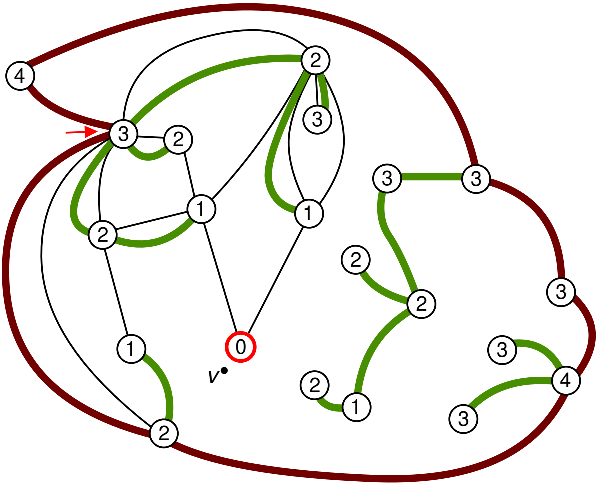

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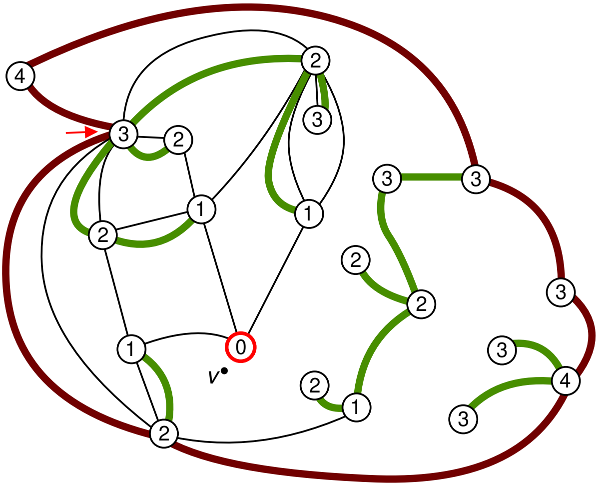

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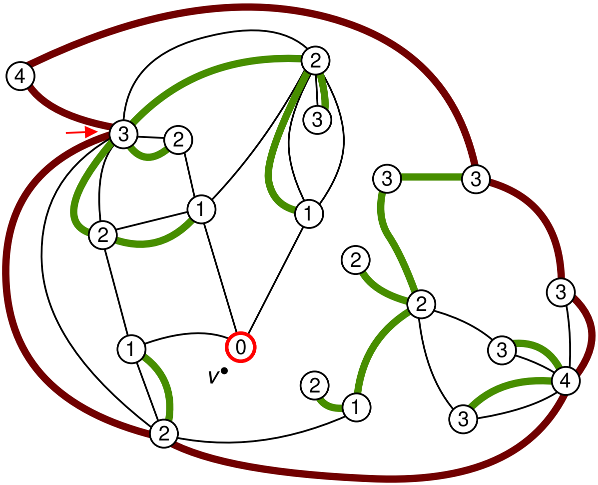

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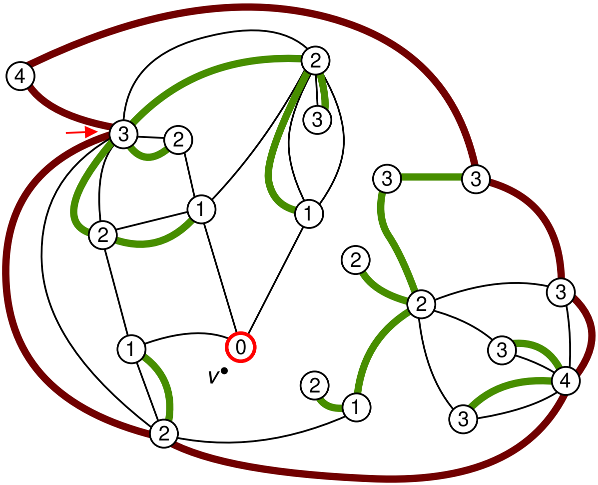

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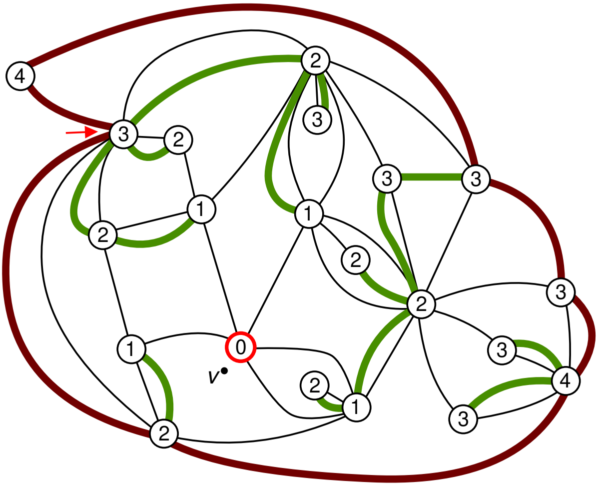

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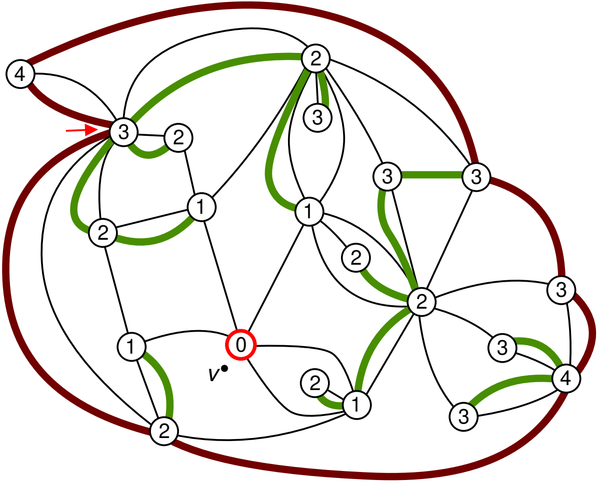

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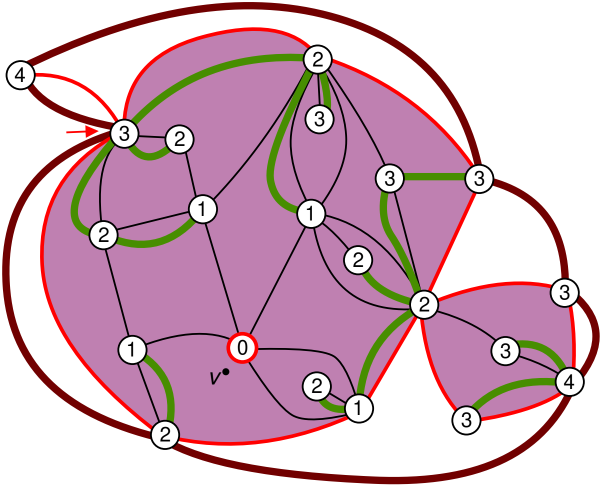

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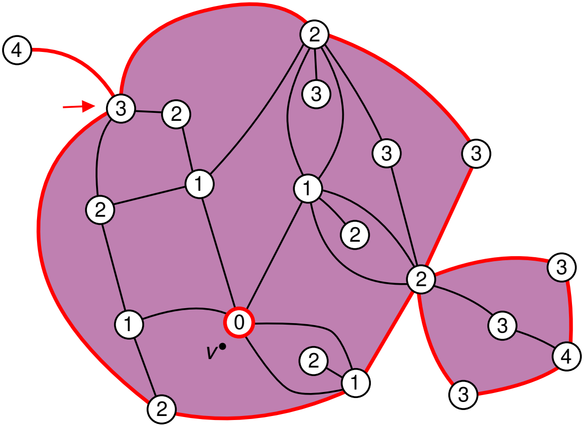


- Take a labeled forest.
- Add a vertex $v^\bullet$ inside the unique face.
- Link every corner to the first subsequent corner having a strictly smaller label.
- Remove the initial edges.

Key facts

Theorem (Bouttier–Di Francesco–Guitter (generalization of Cori–Vauquelin–Schaeffer))

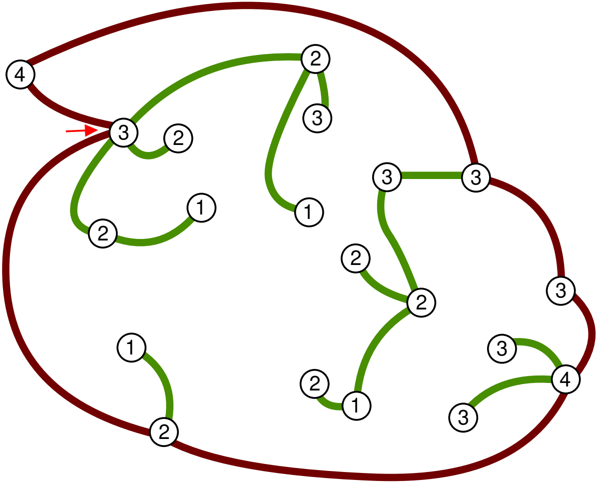
The previous construction yields a bijection between the following:

- *labeled forests with n edges and l trees;*
- *pointed quadrangulations with a boundary having n internal faces and boundary length $2l$ such that the root vertex is farther away from the distinguished vertex than the previous vertex in clockwise order around the boundary.*

Lemma

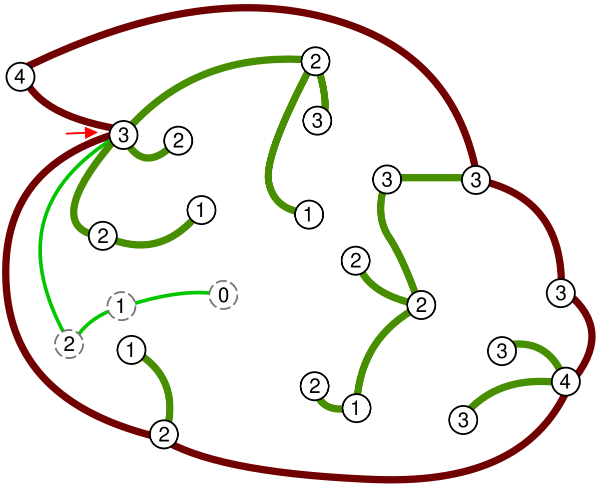
The labels of the forest become the distances in the map to the distinguished vertex $v^\bullet$.

Slices



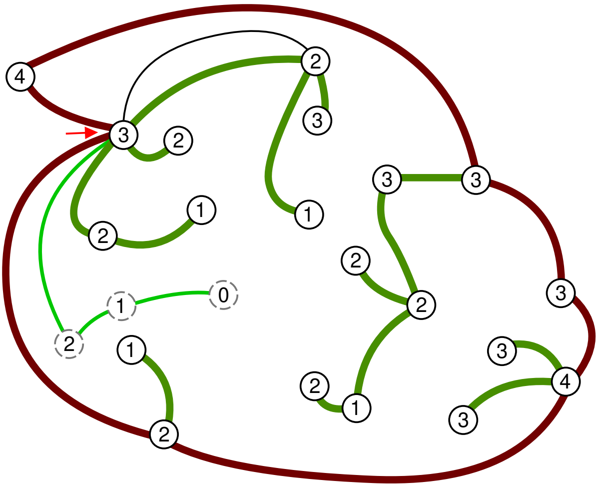
- Proceed tree by tree.

Slices



- Proceed tree by tree.
- Add a chain of vertices linking the root to a vertex with label the minimum of the tree minus 1.

Slices



- Proceed tree by tree.
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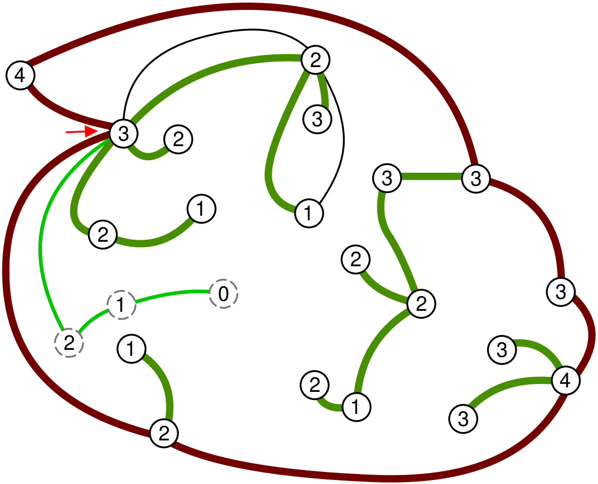

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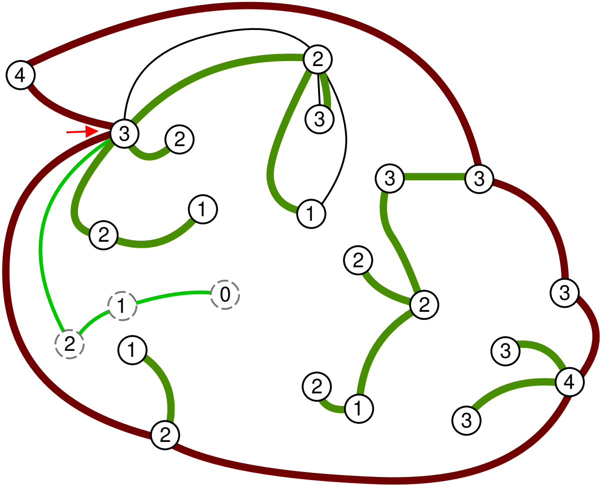

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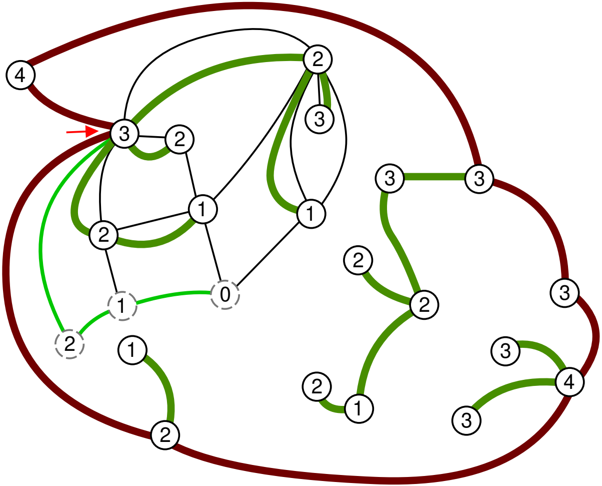

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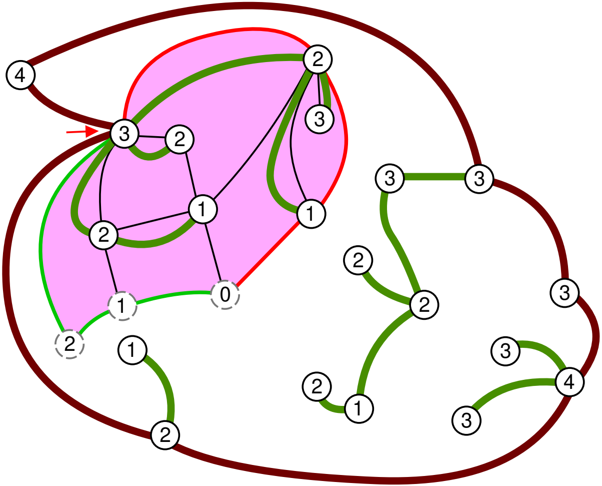

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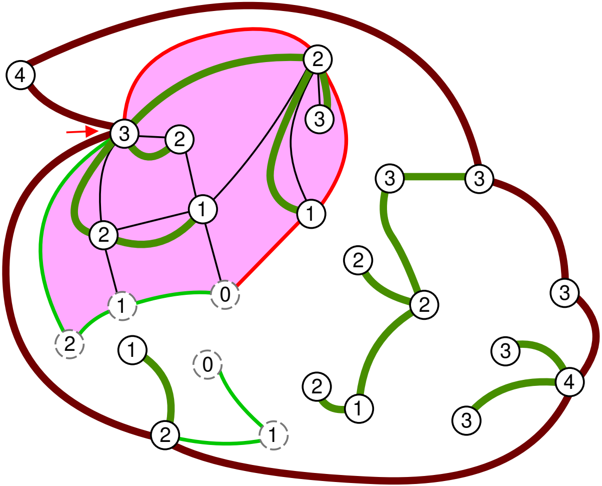

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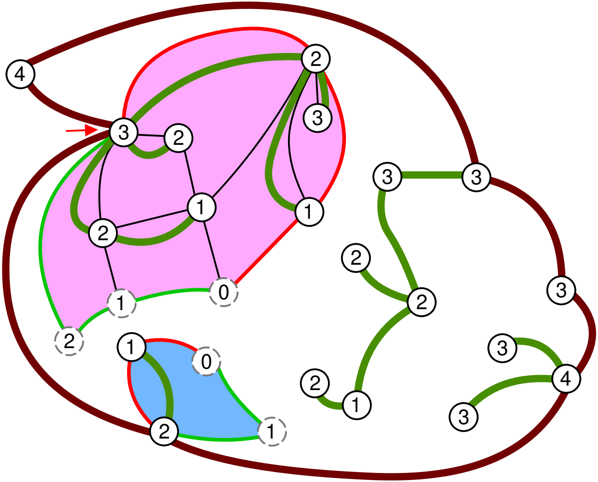

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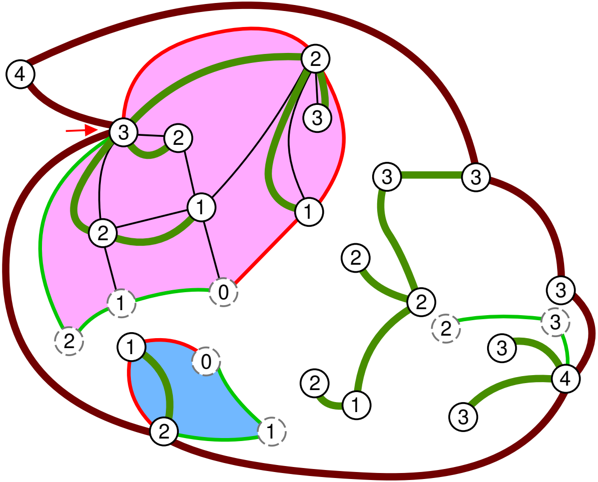

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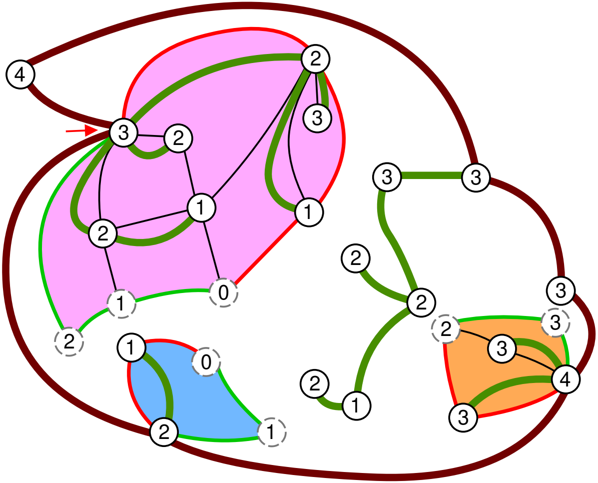

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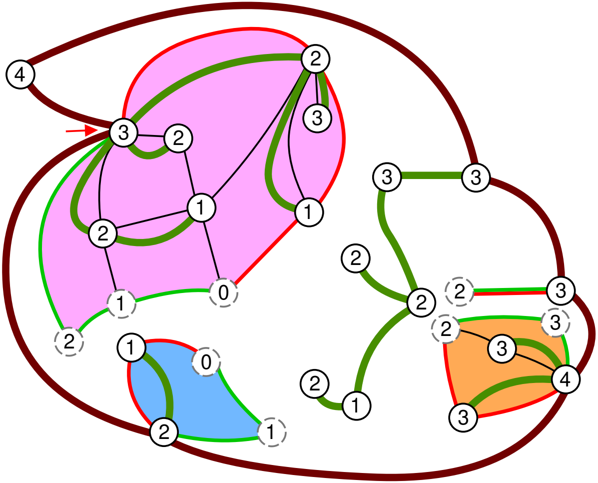

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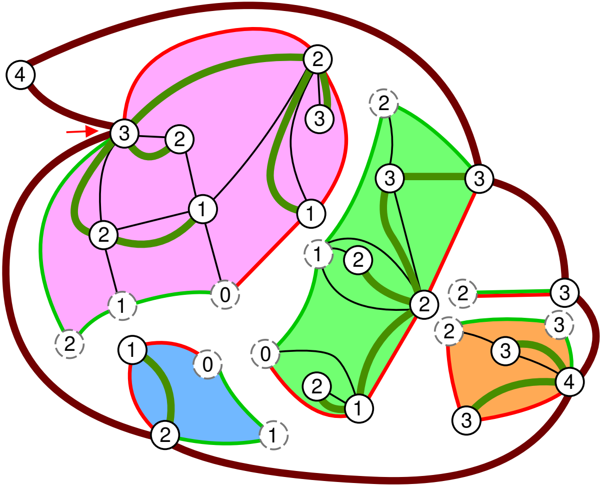

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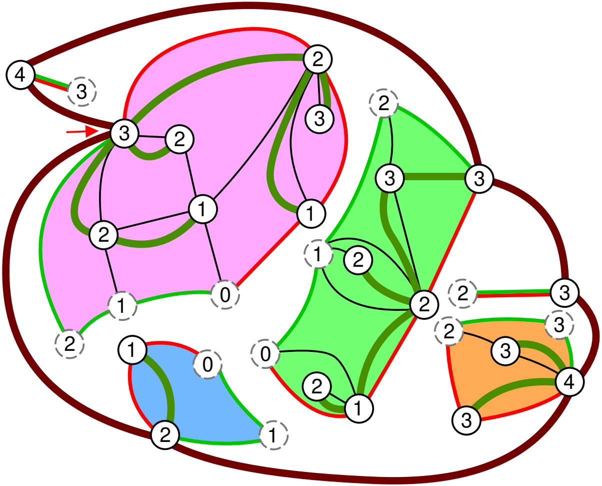

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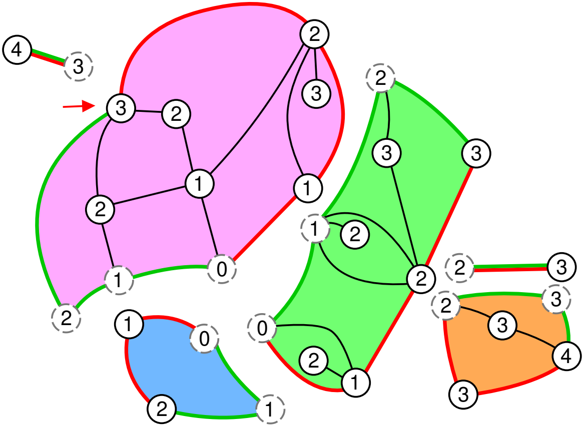

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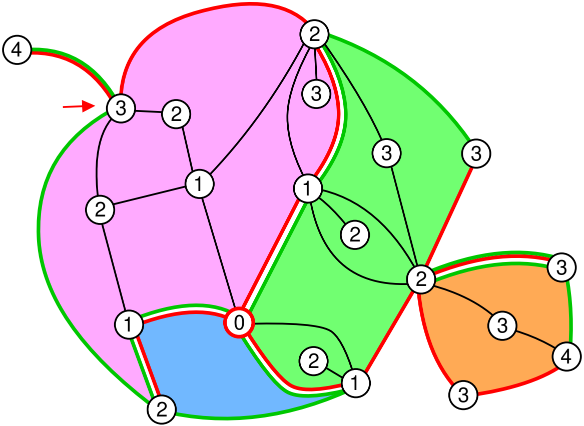

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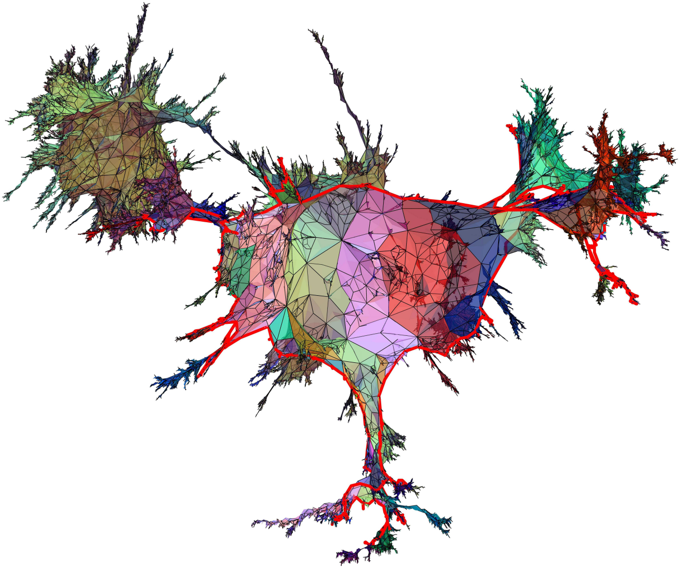
Slices of the previous computer simulation



Case of the Brownian map ($l = 1$)

- Distinguishing a uniformly chosen vertex in a uniform quadrangulation gives a uniform pointed quadrangulation.
- A uniform pointed quadrangulation corresponds via the previous bijection to a uniform labeled tree.
- Relax the positivity constraints on the label by shifting them in such a way that the root vertex gets label 0.

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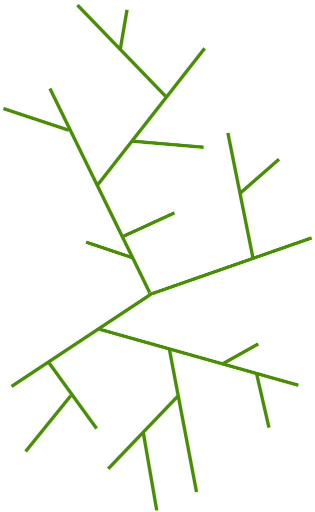
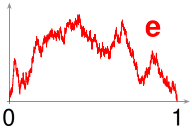
- A uniform pointed quadrangulation corresponds via the previous bijection to a uniform labeled tree.
- Relax the positivity constraints on the label by shifting them in such a way that the root vertex gets label 0.
- After proper rescaling ($\sqrt{n}$ for tree length and $n^{1/4}$ for labels), the resulting labeled tree converges in a natural sense (encoding by contour and label functions) to $(\mathcal{T}_{\mathbf{e}}, Z)$, where
  - $\mathcal{T}_{\mathbf{e}}$ is Aldous's Brownian Continuum Random Tree (universal scaling limit of random tree models);
  - Z is a Brownian motion indexed by $\mathcal{T}$.

Construction of the Brownian map



- Consider the CRT $\mathcal{T}_{\mathbf{e}}$, that is, the random real tree encoded by the normalized Brownian excursion.



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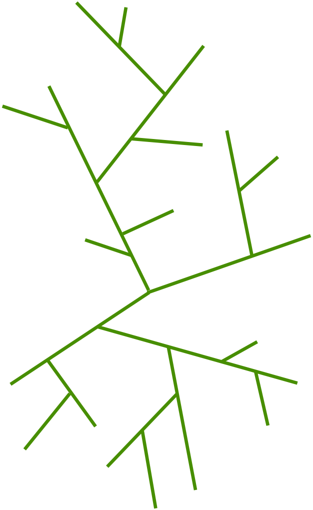

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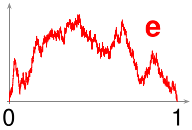

- Put Brownian labels Z on $\mathcal{T}_{\mathbf{e}}$.

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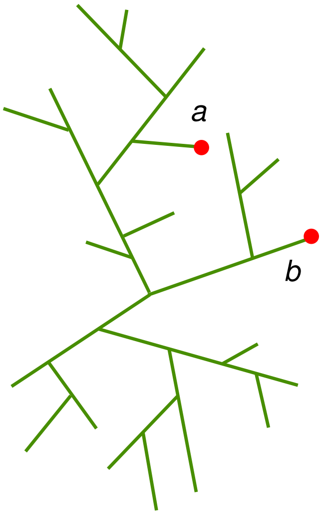

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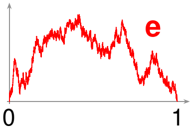
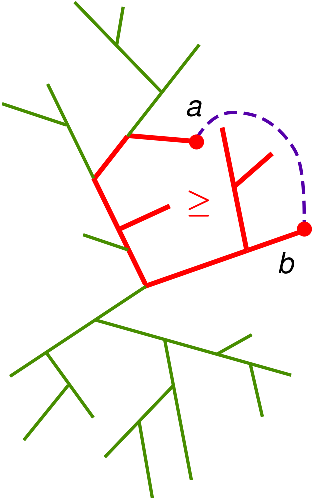


✧ Put Brownian labels Z on $\mathcal{T}_{\mathbf{e}}$.

✧ Identify the points a and b whenever $Z_a = Z_b = \min_{[a,b]} Z$ or $Z_a = Z_b = \min_{[b,a]} Z$.

Construction of the Brownian map



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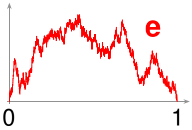

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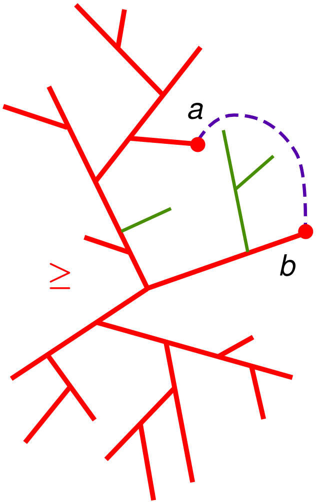

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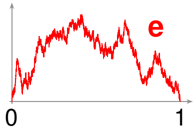

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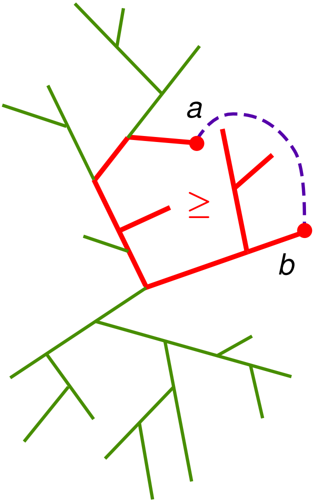
Scaling limit of a uniform slice

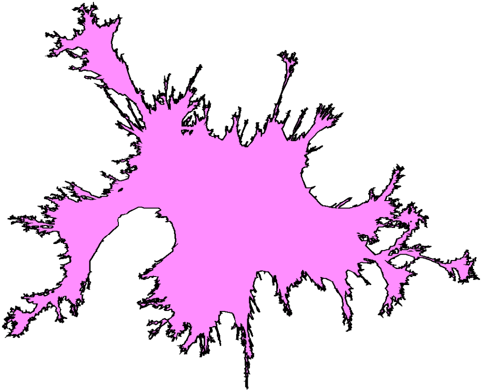


- Same construction as before but only identify points a and b if

$$Z_a = Z_b = \min_{\mathcal{I}} Z$$

where $\mathcal{I}$ is the "interval" among $\{[a, b], [b, a]\}$ that do not contain the root of the tree (equivalence class of 0).

Scaling limit of a uniform slice



- Alternatively, consider the Brownian map.

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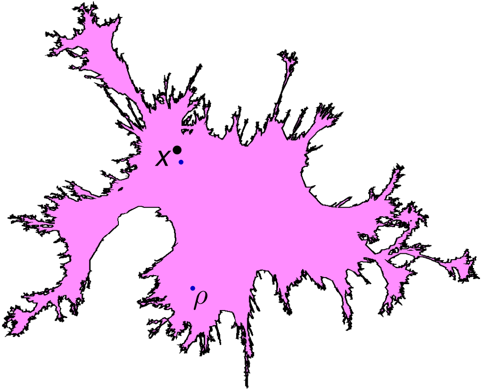

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- Consider its root ρ (the image of the root of the CRT $\mathcal{T}_{\mathbf{e}}$) and the image of the (a.s. unique) point with minimum label $x^\bullet := \operatorname{argmin} Z$.

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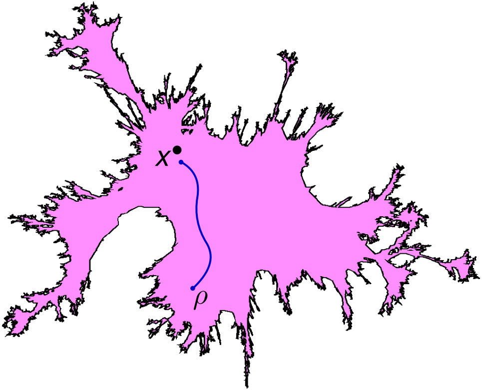

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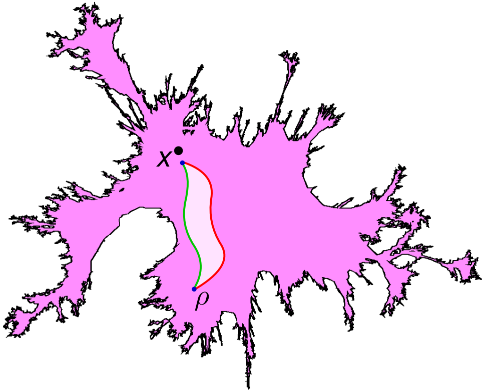

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- Consider the (a.s. unique) geodesic linking them.
- Slit it open.

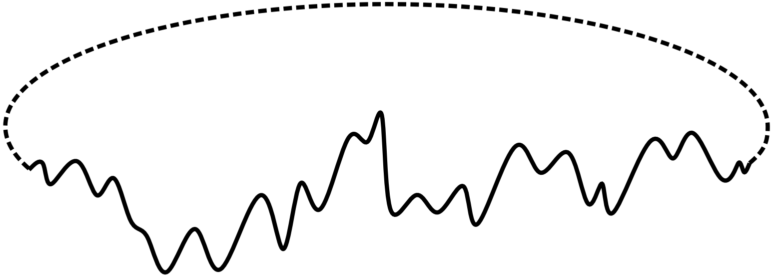
Construction of Brownian disks

- A uniform quadrangulation with a boundary corresponds to a uniform labeled forest.
- The boundary of the quadrangulation corresponds to the floor of the forest (the set of tree roots).
- In the scaling limit,
  - the labels of this floor constitute a Brownian bridge;
  - the labeled trees converge to a Poisson point process of Brownian CRTs with Brownian labels.
- A Brownian disk is obtained by gluing the corresponding slices.

Caveat

There is an infinite number of slices... Fortunately, they accumulate near the boundary and we can show that a geodesic between two typical points stays away from the boundary, thus visits a finite number of slices.

Construction of Brownian disks



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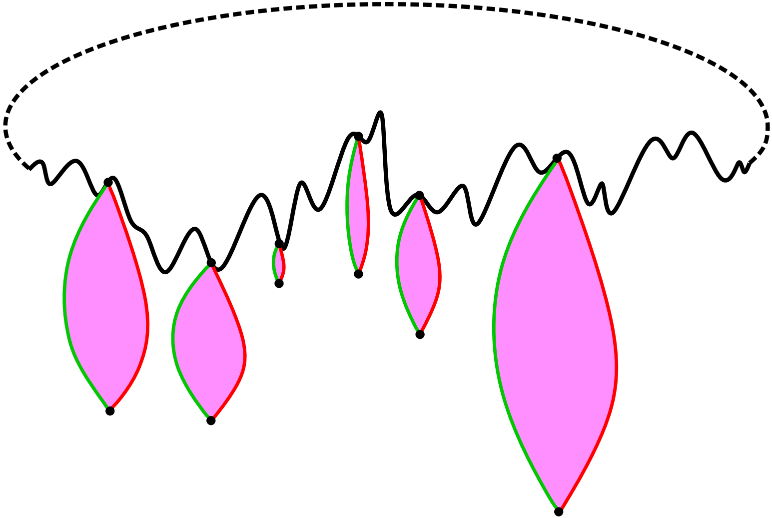

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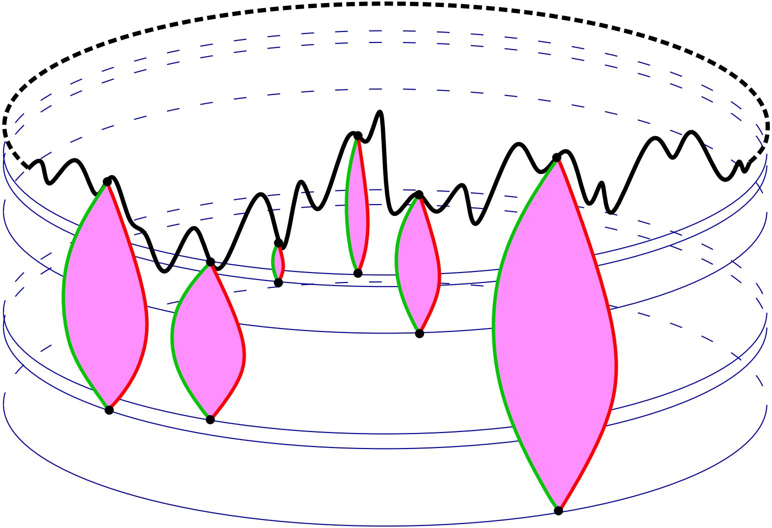

Future work and open questions

- ✧ Orientable compact surfaces with a boundary
  - bijective encoding known (Chapuy–Marcus–Schaeffer '08 & Bouttier–Di Francesco–Guitter '04)
  - subsequential limits of rescaled quadrangulations exist (B. '14)
  - study of the geodesics toward the root (B. '14)
  - uniqueness of the limit (in progress with G. Miermont)
- ✧ Nonorientable compact surfaces
  - bijective encoding recently found (Chapuy–Dołęga '15 & B. '15)
  - subsequential limits of rescaled quadrangulations exist for surfaces without boundary (Chapuy–Dołęga '15)
  - uniqueness of the limit (project with G. Chapuy and M. Dołęga)
- ✧ Universality of the previous objects (different faces, simple boundary components, girth constraints...)
- ✧ Metric gluing of such objects (e.g. two disks along their boundary)
- ✧ Infinite genus: let the number of faces and the genus tend to ∞ in the proper regime


Thank you

Boltzmann random maps

- **B**: set of bipartite plane maps (maps with faces of even degrees)
- $q = (q_1, q_2, \ldots) \neq (0, 0, \ldots)$: sequence of non-negative weights

The Boltzmann measure is defined on **B** by

$$W(\{\mathfrak{m}\}) = \prod_{f \text{ internal face}} q_{\deg(f)/2} \,.$$

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- $\mathbf{B}_l$: set of bipartite plane maps with perimeter (root face degree) $2l$
- $\mathbf{B}^{\mathbf{V}}_{l,n}$: maps of $\mathbf{B}_l$ with $n+1$ vertices
- $\mathbf{B}^{\mathbf{E}}_{l,n}$: maps of $\mathbf{B}_l$ with n edges
- $\mathbf{B}^{\mathbf{F}}_{l,n}$: maps of $\mathbf{B}_l$ with n internal faces

Whenever $0 < W(\mathbf{B}^{\mathbf{S}}_{l,n}) < \infty$, we may define the probability distribution

$$\mathbb{W}^{\mathbf{S}}_{l,n}(\cdot) := W(\cdot \,|\, \mathbf{B}^{\mathbf{S}}_{l,n}) = \frac{W(\cdot \cap \mathbf{B}^{\mathbf{S}}_{l,n})}{W(\mathbf{B}^{\mathbf{S}}_{l,n})} \,.$$

Admissible, regular critical weight sequences

$$f_q(x) := \sum_{k\geq 0} x^k \binom{2k+1}{k} q_{k+1}\,, \qquad x \geq 0\,.$$

- q is admissible if $f_q(z) = 1 - \frac{1}{z}$ admits a solution $z > 1$.
- q is regular critical if moreover the solution z to the above equation satisfies $z^2 f_q'(z) = 1$ and if there exists $\varepsilon > 0$ such that $f_q(z+\varepsilon) < \infty$.

Convergence of Boltzmann maps

Let q be a regular critical weight sequence and **S** denote one of the symbols **V**, **E**, **F**. We define an explicit quantity $\sigma_{\mathbf{S}}$ whose precise expression will not be needed here.

Let $L > 0$ and $(l_k, n_k)_{k\geq 0}$ be a sequence such that $W(\mathbf{B}^{\mathbf{S}}_{l_k,n_k}) > 0$ and l_k, $n_k \to \infty$ with $l_k \sim L\sigma_{\mathbf{S}}\sqrt{n_k}$ as $k \to \infty$. Then $W(\mathbf{B}^{\mathbf{S}}_{l_k,n_k}) < \infty$.

Theorem (B.–Miermont '15)

For $k \geq 0$, denote by $\mathfrak{m}_k$ a random map with distribution $\mathbb{W}^{\mathbf{S}}_{l_k,n_k}$. Then

$$\left(\frac{4\sigma_{\mathbf{S}}^2}{9}\, n_k\right)^{-1/4} \mathfrak{m}_k \xrightarrow[k\to\infty]{(d)} \mathrm{BD}_L$$

in distribution for the Gromov–Hausdorff topology.

Application 1: uniform $2p$-angulations

Let $p \geq 2$. The weight sequence

$$q := \frac{(p-1)^{p-1}}{p^p \binom{2p-1}{p}} \delta_p$$

is regular critical and $\mathbb{W}^{\mathbf{F}}_{l,n}$ is the uniform distribution on the set of $2p$-angulations with n faces and perimeter $2l$.

Corollary

Let $L \in (0, \infty)$ be fixed, $(l_n, n \geq 1)$ be a sequence of integers such that $l_n \sim L\sqrt{p(p-1)n}$ as $n \to \infty$, and $\mathfrak{m}_n$ be uniformly distributed over the set of $2p$-angulations with n internal faces and perimeter $2l_n$. Then

$$\left(\frac{9}{4p(p-1)\,n}\right)^{1/4} \mathfrak{m}_n \xrightarrow[n\to\infty]{(d)} \mathrm{BD}_L$$

in distribution for the Gromov–Hausdorff topology.

Application 2: uniform bipartite maps

Let $q_k = 8^k$, $k \geq 1$. The weight sequence q is regular critical and $\mathbb{W}^{\mathbf{E}}_{l,n}$ is the uniform distribution over bipartite maps with n edges and perimeter $2l$. (Recall that $\sum_{f \text{ face}} \deg(f)/2 =$ number of edges.)

Corollary

Let $\mathfrak{m}_n$ be a uniform random bipartite map with n edges and with perimeter $2l_n$, where $l_n \sim 3L\sqrt{n/2}$ for some $L > 0$. Then

$$(2n)^{-1/4}\mathfrak{m}_n \xrightarrow[n\to\infty]{} \mathrm{BD}_L$$

in distribution for the Gromov–Hausdorff topology.

Free Brownian disk

- $\mathbf{B}_l$: set of bipartite plane maps with perimeter $2l$
- q: regular critical weight sequence (imply that $W(\mathbf{B}_l) < \infty$)

Theorem (B.–Miermont '15)

For $l \in \mathbb{N}$, let $\mathfrak{m}_l$ be distributed according to $W(\cdot \mid \mathbf{B}_l)$. The sequence $\big((2l/3)^{-1/2}\,\mathfrak{m}_l\big)_{l\geq 1}$ converges weakly in the sense of the Gromov–Hausdorff topology toward a random compact metric space called the free Brownian disk.

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- The free Brownian disk is distributed as $\mathcal{A}^{1/4}\ \mathrm{BD}_{\mathcal{A}^{-1/2}}$ where $\mathcal{A}$ has distribution given by

$$\frac{1}{\sqrt{2\pi A^5}} \exp\left(-\frac{1}{2A}\right) dA\, \mathbf{1}_{\{A>0\}}.$$

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- The scaling is universal: it does not involve q whatsoever!