Renormalizability in (noncommutative) field theories

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LIPN

in collaboration with:
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Plan

- Introduction - quantum field theory (QFT)
- QFT and Feynman graphs
- Renormalizability in QFT
- Connes-Kreimer approach for renormalizability in QFT
- Noncommutative QFT (NCQFT) and renormalizability
- Connes-Kreimer approach for NCQFT
- Perspectives

Introduction - QFT

QFT - quantum description of particles and interactions, compatible with Einstein's special relativity

$\hookrightarrow$ elementary particle physics (high energy physics)
(Standard Model of Elementary Particle Physics)

greatest experimental success

QFT formalism applies also to:
statistical mechanics, condensed matter *etc.*

"QFT has remained throughout the years one of the most important tools in understanding the microscopic world."
C. Itzykson and J.-B. Zuber, "QFT"

Scalar field theory and Feynman graphs

$\Phi : \mathbb{R}^4 \to \mathbb{K}$ - a scalar field

$\mathbb{R}^4$ - the 4−dimensional space(time), Euclidean metric

the action (functional in the field)

$$S[\Phi(x)] = \int d^4x \left[\frac{1}{2} \sum_{\mu=1}^{4} \left(\frac{\partial}{\partial x_\mu} \Phi(x) \right)^2 + \frac{1}{2} m^2 \Phi^2(x) + \frac{\lambda}{4!} \Phi^4(x) \right]$$

m - the mass of the particle,
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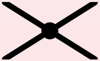
- quadratic part - propagation - edges
- non-quadratic part - interaction potential $V[\Phi(x)] = \frac{\lambda}{4!}\Phi^4(x)$ - vertices:



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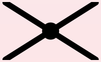
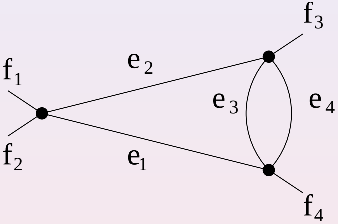


$\Longrightarrow$ (Feynman) graphs of valence 4 - perturbation theory (in λ)

example of a Feynman graph:



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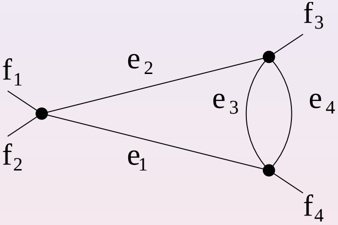

Fourier transform: position space (x) $\to$ momentum space (p)

$$S[\Phi] = \int d^4 p \left[\frac{1}{2} \sum_{\mu=1}^{4} (p_\mu \Phi)^2 + \frac{1}{2} m^2 \Phi^2 + \tilde{V}_{\rm int} \right]$$

- propagation - $\frac{1}{p^2+m^2}$
- interaction - $\lambda\, \delta$(sum of incoming/outgoing momentae)

$\Longrightarrow$ Feynman amplitude (in momentum space)

Renormalizability

first computations in QFT end in infinite results

a cure for these infinities (such that the theoretical results can be compared with experiments) - renormalization.

huge experimental success!

renormalizable theories - building block of mathematical physics

Main ingredients of renormalizability

1. power counting theorem: indicates which Feynman graphs are primitively divergent
   superficial degree of divergence ω -
   should not depend on the internal structure
   exemple: the ϕ^4 model
   $$\omega = N - 4.$$
   N - number of external legs of the graph
   primitively divergent graphs: $2-$ and $4-$point graphs

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2. locality

$\hookrightarrow$ Bogoliubov subtraction operator R
(defined as a *sum over forests*)

subtraction of divergences

The physical principle of locality (Feynman graph level)

connected graphs can be reduced to points

graph made of internal propagators of
high energy (or short distance) (ultraviolet (UV) regime) - local

example:

subtraction of local counterterms -
i. e. counterterms have the same form as the terms of the action

via Taylor expansion

$\Longrightarrow$ renormalized amplitude $\mathcal{A}_R$: finite!

Connes-Kreimer Hopf algebra

A. Connes and D. Kreimer, *Commun. Math. Phys.*, '00

$\hookrightarrow$ definition of a coproduct Δ

$\mathcal{H}$ - the algebra generated by Feynman graphs
multiplication: disjoint union of graphs

$$\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}, \;\; \Delta(G) = G \otimes 1 + 1 \otimes G + \sum_{\gamma \in \underline{G}} \gamma \otimes G/\gamma,$$

$\underline{G}$ – primitively divergent subgraphs of G

renormalization as a factorization issue

$$\varepsilon : \mathcal{H} \to \mathbb{K}, \;\; \varepsilon(1) = 1, \; \varepsilon(G) = 0, \; \forall G \neq 1,$$

$$S : \mathcal{H} \to \mathcal{H},$$

$$S(1_{\mathcal{H}}) = 1_{\mathcal{H}}, \; G \mapsto -G - \sum_{\gamma \in \underline{G}} S(\gamma) G/\gamma.$$

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Theorem: *$(\mathcal{H}, \Delta, \varepsilon, S)$ is a Hopf algebra.*

Algebraic framework for renormalization

R - the map which given a formal integral returns it evaluated at the subtraction point
$R\mathcal{A}(G)$ - the singular part of the Feynman amplitude $\mathcal{A}(G)$

twisted antipode (recursive definition)

$$S_R^{\mathcal{A}}(1_{\mathcal{H}}) = 1,$$
$$S_R^{\mathcal{A}}(G) = -R(\mathcal{A}(G)) - \sum_{\gamma \in \underline{G}} S_R^{\mathcal{A}}(\gamma) R(\mathcal{A}(G/\gamma)).$$

the renormalized amplitude of the graph

$$\mathcal{A}_R = S_R^{\mathcal{A}} \star \mathcal{A}.$$

Connes-Kreimer Hopf algebra structure -
the combinatorial backbone of renormalization

NCQFT - glimpse of the mathematical setup

the Moyal space

The *Moyal algebra* is the linear space of smooth and rapidly decreasing functions $\mathcal{S}(\mathbb{R}^{\mathcal{D}})$ equipped with the *Moyal product*:

$$(f \star g)(x) = \int \frac{d^D k}{(2\pi)^D} d^D y \, f(x + \frac{1}{2}\Theta \cdot k) g(x+y) e^{ik\cdot y}.$$

$\star$ - Moyal product
(non-local, noncommutative, associative product)

$$\Theta = \begin{pmatrix} \Theta_2 & 0 \\ 0 & \Theta_2 \end{pmatrix}, \quad \Theta_2 = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}.$$

Field theory on Moyal space

Φ^4 model:

$$\mathcal{S} = \int d^4x \, [\frac{1}{2}\partial_\mu \Phi \star \partial^\mu \Phi + \frac{1}{2} m^2 \Phi \star \Phi + \frac{\lambda}{4!} \Phi \star \Phi \star \Phi \star \Phi],$$

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$\int d^4x\, (\Phi \star \Phi)(x) = \int d^4x\, \Phi(x)\, \Phi(x)$
(same propagation as in the commutative theory)

Implications of the use of the Moyal product in QFT

$$\int d^4 x \Phi^{\star 4}(x) \propto \int \prod_{i=1}^{4} d^4 x_i \Phi(x_i) \delta(x_1 - x_2 + x_3 - x_4)$$
$$e^{2i(x_1 - x_2)\Theta^{-1}(x_3 - x_4)}$$

oscillation $\propto$ area of parallelogram

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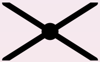
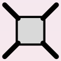
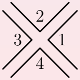


$\rightarrow$



$\hookrightarrow$ non-locality

$\hookrightarrow$ restricted invariance: only under cyclic permutation



$\rightarrow$ ribbon graphs

$\rightarrow$ clear distinction between planar and non-planar graphs

Feynman graphs in NCQFT

n - number of vertices,
L - number of internal lignes,
F - number of faces,

$$2 - 2g = n - L + F$$

$g \in \mathbb{N}$ - genus

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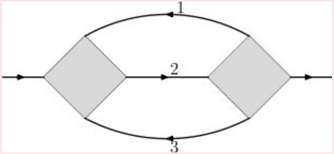
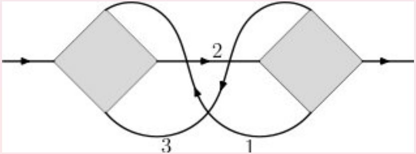
$g \in \mathbb{N}$ - genus

$g = 0$ - planar graph
example:



$n = 2,\ L = 3,\ F = 3,\ g = 0$

$g \geq 1$ - non-planar graph
example:



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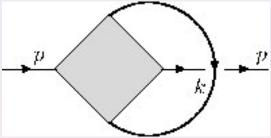
Renormalization on the Moyal space

UV/IR mixing (S. Minwalla *et. al.*, *JHEP*, '00)



$$B = 2$$

B - number of faces broken by external lignes

$B > 1$, *planar irregular graph*

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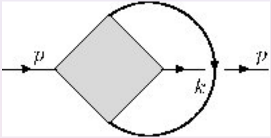

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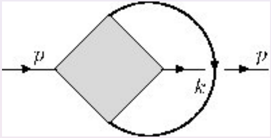

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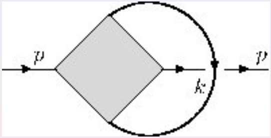

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$$\lambda \int d^4k \frac{e^{ik_\mu \Theta^{\mu\nu} p_\nu}}{k^2 + m^2} \to_{|p| \to 0} \frac{1}{\theta^2 p^2}$$

same type of behavior at any order in perturbation theory

J. Magnen, V. Rivasseau and A. T., *Europhys. Lett.* '09

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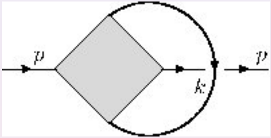

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$\to$ non-renormalizability!

A first solution to this problem - the Grosse-Wulkenhaar model

additional harmonic term

(H. Grosse and R. Wulkenhaar, *Comm. Math. Phys.*, '05)

$$s[\phi(x)] = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \star \partial^\mu \phi + \frac{\Omega^2}{2} (\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right),$$

$$\tilde{x}_\mu = 2(\Theta^{-1})_{\mu\nu} x^\nu.$$

modification of the propagator - the model becomes renormalizable

- most of the techniques of QFT extend to Grosse-Wulkenhaar-like models:
  - the parametric representation

    (R. Gurău and V. Rivasseau, *Commun. Math. Phys.*, '07, A. T. and V. Rivasseau, *Commun. Math. Phys.*, '08, A. T., *J. Phys. Conf. Series*, '08, A. T., solicited by de *Modern Encyclopedia Math. Phys.*)

    (algebraic geometric properties P. Aluffi and M. Marcolli, 0807.1690[math-ph])
  - the Mellin representation

    (R. Gurău, A. Malbouisson, V. Rivasseau and A. T., *Lett. Math. Phys.*, '07)
  - dimensional regularization

    (R. Gurău and A. T., *Annales H. Poincaré*, '08)
- study of vacuum configurations (A. de Goursac, A. T. and J-C. Wallet, *EPJ* **C**, 2008)
- gauge model propositions

  $\hookrightarrow$ non-trivial vacuum state

  (A. de Goursac, J-C. Wallet and R. Wulkenhaar *EPJ* **C**, 2007,2008, H. Grosse and M. Wohelegant *EPJ* **C**, 2007)

Translation-invariant renormalizable scalar model

(R. Gurău, J. Magen, V. Rivasseau and A. T., *Commun. Math. Phys.* 2009)

the Grosse-Wulkenhaar model breaks translation-invariance !

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the complete propagator:

$$C(p, m, \theta) = \frac{1}{p^2 + a\frac{1}{\theta^2 p^2} + m^2}$$

arbitrary planar irregular 2-point function: same type of $\frac{1}{p^2}$ behavior !

J. Magnen, V. Rivasseau and A. T., *Europhys. Lett.* 2009

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$\hookrightarrow$ other modification of the action:

$$S = \int d^4 p\, [\frac{1}{2} p_\mu \phi \star p^\mu \phi + \frac{1}{2} a \frac{1}{\theta^2 p^2} \phi \star \phi + \frac{1}{2} m^2 \phi \star \phi + \frac{\lambda}{4!} V^\star[\phi]]. \quad (1)$$

renormalizability at any order in perturbation theory !

Glimpse of renormalizability proof - BPHZ scheme

power counting theorem:

$\hookrightarrow$ 2− and 4−point planar functions *(primitively divergent)*

- planar regular 2−point function: wave function and mass renormalization
- planar regular 4−point function: coupling constant renormalization
- planar irregular 2−point function: *finite* renormalization of the constant *a*
- planar irregular 4−point function: convergent

Comparison between noncommutative models

	the "naive" model		*GW model*		*model* (1)	
	2P	4P	2P	4P	2P	4P
planar regular	ren.	ren.	ren.	ren.	ren.	ren.
planar irregular	UV/IR	log UV/IR	conv.	conv.	finite ren.	conv.
non-planar	IR div.	IR div.	conv.	conv.	conv.	conv.

Scales - noncommutative renormalization group

definition of the RG scales:

- locus where $C^{-1}(p)$ is big
- locus where $C^{-1}(p)$ is low

$$C^{-1}_{\rm comm}(p) = p^2$$

$$C^{-1}_{GW} = p^2 + \Omega^2 x^2$$

$$C^{-1}(p) = p^2 + \frac{a}{\theta^2 p^2}$$

mixing of the UV and IR scales - key of the renormalization

Other translation-invariant field theoretical techniques

- parametric representation (A. T., *J. Phys.* **A** 2009)
  - power counting dependence on the graph genus
- relation with Bollobás-Riordan topologic ribbon graph polynomial

  (T. Krajewski, V. Rivasseau, A. T. and Z. Wang, *J. Noncomm. Geom.* (2010)

  trees $\rightarrow$ $\star-$trees (quasi-trees)
- renormalization group flow

  (J. Ben Geloun and A. T., *Lett. Math. Phys.* 2008)

$$\beta_\lambda \propto \beta_\lambda^{comm}, \ \beta_a = 0$$

- commutative limit

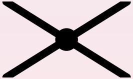
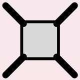
  (J. Magnen, V. Rivasseau and A. T., *Lett. Math. Phys.* 2008)
- field theories with other products (*ex.:* the Wick-Voros product)

  $\hookrightarrow$ A. T. and P. Vitale, *Phys. Rev.* **D** (2010)

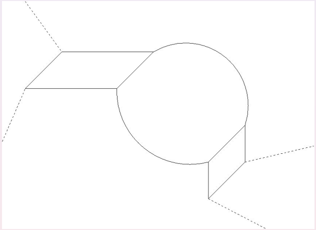
Renormalizability of NCQFT: locality → "Moyality"

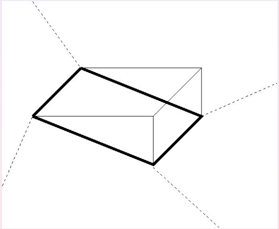
QFT → NCQFT

locality → "Moyality"

 → 

The principle of "Moyality" - ribbon Feynman graph level





valid iff the graph is planar

renormalization necessary only for the planar sector !

Hopf algebra for renormalizable NCQFTs

A. T. and F. Vignes-Tourneret, *J. Noncomm. Geom.*, 2008

A. T. and D. Kreimer, arXiv: 0907.2182, submitted

$\hookrightarrow$ definition of a coproduct Δ

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$$S : \mathcal{H} \to \mathcal{H}, \quad G \mapsto -G - \sum_{\gamma \in \underline{G}} S(\gamma) G/\gamma.$$

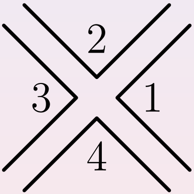
Theorem: $(\mathcal{H}, \Delta, \varepsilon, S)$ *is a Hopf algebra.*

$\hookrightarrow$ 2$-$ and 4$-$point graphs (in commutative ϕ^4)
$\rightarrow$ 2$-$ and 4$-$point planar regular graph

this Hopf algebra structure - the combinatorial backbone of noncommutative renormalization

(pre-)Lie algebra structures

Hochschild cohomology - combinatorial Dyson-Schwinger equation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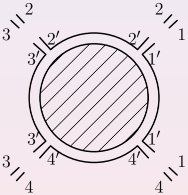
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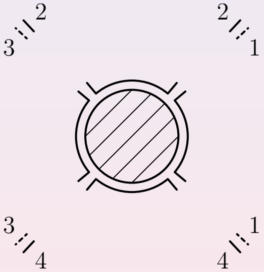

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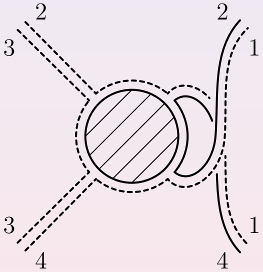

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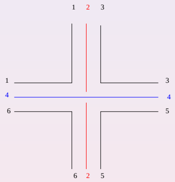
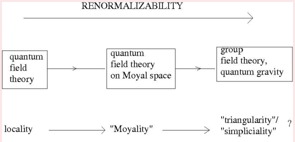
Perspectives - can things be (even) more complicated?

Perspectives - can things be (even) more complicated?

- generalization to tensor models (appearing in recent approaches for a theory of quantum gravity)



various non-trivial combinatorial structures:
  - topological graph polynomial (A. T., work in progress)
  - combinatorial Hopf algebras
- applications of these techniques for the renormalizability study of quantum gravity models (A. T., *Class. Quant. Grav.* 2010)



Thank you for your attention!

"The amount of theoretical work one has to cover before being able to solve
problems of real practical value is rather large, but this circumstance is [...] likely to become more pronounced in the theoretical physics of the future."
P.A.M. Dirac, *"The principles of Quantum Mechanics"*, 1930

renormalization conditions

$$\Gamma^4(0,0,0,0) = -\lambda_r, \;\; G^2(0,0) = \frac{1}{m^2}, \;\; \frac{\partial}{\partial p^2} G^2(p,-p)|_{p=0} = -\frac{1}{m^4}. \quad (2)$$

where Γ^4 and G^2 are the connected functions and

$0 \to p_m$ (the minimum of $p^2 + \frac{a}{\theta^2 p^2}$)