Distances in Noncommutative Geometry

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Metric aspect of noncommutative geometry

$$'' \; ds = D^{-1} \; ''$$

Distance between states of an algebra $\mathcal{A}$. Not so much studied but many interesting links with other distances:

- distance on graph ($\mathcal{A}$ finite dimensional) (Lizzi & al; Dimakis, Müller-Hosen; Iochum, Krajewski, P.M.),
- horizontal distance in subriemannian geometry ($\mathcal{A} = C_0^\infty(\mathcal{M}) \otimes M_n(\mathbb{C})$) (P.M.),
- Wasserstein distance in optimal transport theory (commutative $\mathcal{A}$) (D'Andrea, P.M.),
- distance in some model of quantum spacetime ($\mathcal{A} = \mathcal{K} = (\mathcal{S}, \star)$) (Cagnache, D'Andrea, P.M., Wallet);

also yields a metric interpretation of the Higgs field in Connes description of the standard model (Wulkenhaar, P.M.).

Topological aspect mostly studied by Rieffel, Latrémolière and a recent paper of Bélissard, Marcolli and Reihani.

Outline:

1. Distance in noncommutative geometry
2. The commutative case and the Wassertein distance in optimal transport
3. Product of geometries and the horizontal distance in sub-Riemannian geometry
4. Moyal plane

1. Distance in noncommutative geometry

$$\begin{array}{ccc} \text{commutative algebra} & \longrightarrow & \text{non-commutative algebra} \\ \updownarrow & & \downarrow \\ \text{differential geometry} & & \text{non-commutative geometry} \end{array}$$

How to define the distance in purely algebraic terms, so that to export this definition to the noncommutative framework ?

The distance formula

- Let $(\mathcal{X}, d)$ be a locally compact complete metric space.

$$d(x,y) = \sup_{f \in C_0(\mathcal{X})} \{ |f(x) - f(y)| \, ; \, \|f\|_{\text{Lip}} \leq 1 \}.$$

- Gelfand duality: let $\mathcal{P}(\mathcal{A})$ denote the pure states of a C^*-algebra $\mathcal{A}$ (extremal points of the set of normalized positive linear maps $\mathcal{A} \to \mathbb{C}$).

$$\mathcal{P}(C_0(\mathcal{X}) \simeq \mathcal{X} : \omega_x(f) = f(x).$$

- $(\mathcal{M}, d_{\text{geo}})$ with $\mathcal{M}$ a Riemannian (spin) manifold:

$$\|f\|_{\text{Lip}} = \left\| [d + d^\dagger, \pi_1(f)]^2 \right\|_{\text{op}} = \frac{1}{2} \left\| [\Delta, \pi_2(f)], \pi_2(f)] \right\|_{\text{op}} = \| [\partial\!\!\!/, \pi(f)] \|_{\text{op}}^2$$

where $d + d^\dagger$ is the signature operator, $\Delta = dd^\dagger + d^\dagger d$, $\partial\!\!\!/ = -i \sum_{\mu=1}^{\dim \mathcal{M}} \gamma^\mu \partial_\mu$, π_1, π_2, π are representations of $C_0^\infty(\mathcal{M})$ on $L^2(\mathcal{M}, \wedge), L^2(\mathcal{M}), L^2(\mathcal{M}, S)$.

$$d_{geo}(x,y) = d(\omega_x, \omega_y) = \sup_{f \in C_0^\infty(\mathcal{M})} \{ |\omega_x(f) - \omega_y(f)| \, / \, \|[\partial\!\!\!/, f]\| \leq 1 \}.$$

Spectral triple

An involutive algebra $\mathcal{A}$, a faithful representation π on $\mathcal{H}$, an operator D on $\mathcal{H}$ such that $[D, \pi(a)]$ is bounded for any $a \in \mathcal{A}$ and $\pi(a)[D - \lambda\mathbb{I}]^{-1}$ is compact for any $\lambda \notin$ Sp D; together with a set of necessary and sufficient conditions guaranteeing that

i. For $\mathcal{M}$ a compact Riemannian spin manifold, $(C^\infty(\mathcal{M}), L^2(\mathcal{M}, S), \partial\!\!\!/)$ is a spectral triple;

ii. $(\mathcal{A}, \mathcal{H}, D)$ a spectral triple with $\mathcal{A}$ unital commutative, then there exists a compact spin manifold $\mathcal{M}$ such that $\mathcal{A} = C^\infty(\mathcal{M})$.

$$\boxed{d_D(\varphi_1, \varphi_2) \doteq \sup_{a \in \mathcal{A}} \{|\varphi_1(a) - \varphi_2(a)| \,/\, \|[D, a]\| \leq 1\}}$$

is a distance (possibly infinite) on the state space of $\overline{\mathcal{A}}$ which:

- makes sense whether $\mathcal{A}$ is commutative or not;
- is coherent with the commutative case: $d_D = d_{\text{geo}}$ between pure states;
- does not involve notion ill-defined at the quantum level, but only spectral properties of $\mathcal{A}$ and D: *spectral distance*.

2. The commutative case and the Wassertein distance in optimal transport

Transportation map and Wassertein distance

$\mathcal{X}$ is a locally compact separable metric space. A state $\varphi \in S(C_0(\mathcal{X}))$ is a probability measure μ on $\mathcal{X}$,

$$\varphi(f) \doteq \int_{\mathcal{X}} f \mathrm{d}\mu \quad \forall f \in \mathcal{A}.$$

Let $c(x, y)$ be a positive real function — the "cost function" — representing the work needed to move from x to y.

Minimal work W required to move the configuration φ_1 to the configuration φ_2,

$$W(\varphi_1, \varphi_2) \doteq \inf_{\pi} \int_{\mathcal{X}\times\mathcal{X}} c(x, y)\, \mathrm{d}\pi \qquad (1)$$

where the infimum is over all measures π on $\mathcal{X} \times \mathcal{X}$ with marginals μ_1, μ_2, i.e.

$$\left.\begin{array}{rc} \mathbb{X}, \mathbb{Y}: & \mathcal{X} \times \mathcal{X} \to \mathcal{X}, \\ & \mathbb{X}(x, y) \doteq x, \\ & \mathbb{Y}(x, y) \doteq y, \end{array}\right\} \mathbb{X}_*(\pi) = \mu_1,\ \mathbb{Y}_*(\pi) = \mu_2.$$

Finding the optimal transportation plan (i.e. which minimizes W) is a non-trivial question known as the Monge-Kantorovich problem.

When the cost function c is a distance d,

$$W(\varphi_1, \varphi_2) \doteq \inf_{\pi} \int_{\mathcal{X} \times \mathcal{X}} d(x, y) \, \mathrm{d}\pi$$

is a distance on the space of states (possibly infinite), called the Kantorovich-Rubinstein distance, or the *Wasserstein distance of order* 1.

Proposition 1: Rieffel 99, puis D'Andrea, P.M. 2009

Let $\mathcal{M}$ be a complete, Riemannian, finite dimensional, connected, without boundary, spin manifold. For any $\varphi_1, \varphi_2 \in S(C_0(\mathcal{M}))$,

$$W(\varphi_1, \varphi_2) = d_D(\varphi_1, \varphi_2)$$

where W is the Wasserstein distance associated to the cost d_{geo}.

i. Kantorovich duality: $W(\varphi_1, \varphi_2) = \sup_{\|f\|_{\text{Lip}} \leq 1} \left(\int_{\mathcal{X}} f \mathrm{d}\mu_1 - \int_{\mathcal{X}} f \mathrm{d}\mu_2 \right)$. The supremum is **on all** real 1-Lipschitz. functions f on $\mathcal{X}$,

$$|f(x) - f(y)| \leq d_{\text{geo}}(x, y) \text{ for all } x, y \in \mathcal{X}.$$

ii. $$||[D = \partial\!\!\!/, f]||_{\text{op}} = \|f\|_{\text{Lip}}$$

iii. $\mathcal{M}$ is locally compact non compact: get rid of the vanishing at infinity. For any 1-Lip. f, consider the sequence of functions vanishing at infinity

$$f_n(x) \doteq f(x) e^{-d(x_0, x)/n} \quad n \in \mathbb{N},\ x_0 \text{ is any fixed point.} \tag{2}$$

Then $\lim_{n \to +\infty} (\varphi_1 - \varphi_2)(f_n) = (\varphi_1 - \varphi_2)(f)$ and $||f_n||_{\text{Lip}} \leq 1$.

- (2) requires $\mathcal{M}$ to be (geodesically) complete (Hopf-Rinow theorem).

On the importance of being complete

$\mathcal{N}$ compact, $\mathcal{M} = \mathcal{N} \setminus \{x_0\} \implies W = d_{\text{geo}}$ on both $\mathcal{M}$ and $\mathcal{N}$.

$$\left.\begin{array}{l}\mathcal{N} = S^1 = [0,1] \\ \mathcal{M} = (0,1)\end{array}\right\} W_{\mathcal{M}}(x,y) = |x-y| \neq W_{\mathcal{N}}(x,y) = \min\{|x-y|, 1-|x-y|\}.$$

$\mathcal{N} = S^2$, $\mathcal{M} = S^2 \setminus \{x_0\}$ then $W_{\mathcal{N}} = W_{\mathcal{M}}$.

- Removing a point from a complete compact manifold may change or not W.
- It does not modify the spectral distance: $C^\infty(\mathcal{N}) = C(\mathcal{N})$ has a unit so

$$\begin{aligned} d_D^{\mathcal{N}}(\varphi_1, \varphi_2) &= \sup_{f \in C(\mathcal{N})} \left\{ |\varphi_1(f) - \varphi_2(f)|;\ ||f||_{\mathrm{Lip}} \leq 1 \right\} \\ &= \sup_{f \in C(\mathcal{N}), f(x_0)=0} \left\{ |\varphi_1(f) - \varphi_2(f)|;\ ||f||_{\mathrm{Lip}} \leq 1 \right\} = d_D^{\mathcal{M}}(\varphi_1, \varphi_2) \end{aligned}$$

since $(C(\mathcal{N}),$ vanishing at $x_0) = C_0(\mathcal{M})$.

$$\mathcal{N} = S^1, \mathcal{M} = (0,1) : d_D^{\mathcal{M}} = d_D^{\mathcal{N}} = W_{\mathcal{N}} = d_{S^1} \neq W_{\mathcal{M}}.$$

$$\mathcal{N} = S^2, \mathcal{M} = S^2 \setminus \{x_0\} : d_D^{\mathcal{M}} = d_{S^2} = W_{\mathcal{M}}.$$

Connected components

Proposition 2: For any $x \in \mathcal{M}$ and any state φ of $C_0^\infty(\mathcal{M})$,

$$d_D(\varphi, \delta_x) = \mathbb{E}\big(d(x, \circ); \mu\big) = \int_{\mathcal{M}} d_{\text{geo}}(x, y) d\mu(y) \; .$$

In particular for two pure states δ_x, δ_y,

$$d_D(\delta_x, \delta_y) = d_{\text{geo}}(x, y).$$

Let $S_1(C_0^\infty(\mathcal{M})) \doteq \{\varphi \text{ such that } \mathbb{E}\big(d(x, \circ); \mu\big) < \infty\}$.

Corollary 3: $\varphi \in S_1(C_0^\infty(\mathcal{M}))$ if and only if φ is at finite spectral distance from any pure state.

Let $\mathrm{Con}(\varphi) \doteq \{\varphi' \in \mathcal{S}(C_0^\infty(\mathcal{M})) \text{ such that } d_D(\varphi, \varphi') \leq \infty\}$.

Corollary 4: For any $\varphi \in S_1(C_0^\infty(\mathcal{M}))$, $\mathrm{Con}(\varphi) = S_1(C_0^\infty(\mathcal{M}))$.

- Two states not in $S_1(C_0^\infty(\mathcal{M}))$ may be at finite distance from one another.

3. Product of geometries: Higgs, sub-Riemannian distance

Connection

finite projective $C^\infty(\mathcal{M})$-module $\Gamma^\infty(E)$ $\rightarrow$ finite projective $\mathcal{A}$-module $\mathcal{E}$

$\updownarrow$ $\downarrow$

vector bundle E over $\mathcal{M}$ "noncommutative vector bundle"

$$\nabla\colon \Gamma^\infty(E) \to \Gamma^\infty(E) \otimes \Omega_1(\mathcal{M}) \quad \rightarrow \quad \nabla\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega_1(\mathcal{A}) \doteq \left\{ \sum_i a^i [D, b_i] \right\}$$

$\updownarrow$ $\downarrow$

connection on E connection on the "non commutative vector bundle"

Leibniz rule: $\nabla(sa) = (\nabla s)a + s \otimes [D, a] \quad \forall a \in \mathcal{A}, s \in \mathcal{E}$

Hermitian connection: $(s|\nabla r) - (\nabla s|r) = [D, (s|r)]$ where $(.|.) : \mathcal{E} \otimes \mathcal{E} \to \mathcal{A}$.

- traduction of Levi-Civita condition $g(\nabla X, Y) + g(X, \nabla Y) = d(g(X, Y))$.

Covariant Dirac operator

given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and an hermitian connection on a finite-projective C^*-module $\mathcal{E}$, define

$$\tilde{\mathcal{A}} \doteq End_A(\mathcal{E}), \quad \tilde{\mathcal{H}} \doteq \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}, \quad \tilde{D}(s \otimes \psi) \doteq (\nabla s)\psi + s \otimes D\psi$$

where $End_A(\mathcal{E})$ are the endomorphisms of $\mathcal{E}$ with adjoint (for $\alpha \in End_A(\mathcal{E})$, there exists α^* such that $(r|\alpha s) = (\alpha^* r|s)$). Then

$$(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D}) \text{ is a spectral triple.}$$

Taking $\mathcal{E} = \mathcal{A}$, one builds a new geometry $(\mathcal{A}, \mathcal{H}, D_A)$ where

$$\boxed{D_A = D + A, \quad A = \sum_i a^i [D, b_i] = A^*.}$$

Product of the continuum by the discrete

$$\text{pure state:}(x, \omega_I) \Longleftarrow \quad \begin{array}{rcl} \mathcal{A} & = & C^\infty(\mathcal{M}) \otimes \mathcal{A}_I \\ \mathcal{H} & = & L_2(\mathcal{M}, S) \otimes \mathcal{H}_I \\ D & = & \partial\!\!\!/ \otimes \mathbb{I}_I + \gamma^5 \otimes D_I \end{array} \quad \Longrightarrow A = H - i\gamma^\mu A_\mu$$

- H: scalar field on $\mathcal{M}$ with value in $\mathcal{A}_I$ $\quad\rightarrow$ Higgs.
- A_μ: 1-form field with value in $Lie(U(\mathcal{A}_I))$ $\quad\rightarrow$ gauge field.

The standard model:

$$\begin{array}{lcl} \mathcal{A}_I & = & \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \\ \mathcal{H}_I & = & \mathbb{C}^{96} \\ D_I & \text{is a} & 96 \times 96 \text{ matrix with the fermions masses, the CKM matrix} \\ & & \text{ and the neutrinos mixing angles.} \end{array}$$

Fluctuations of the metric

The replacement $D \to D_A$ yields a *fluctuation of the metric* since

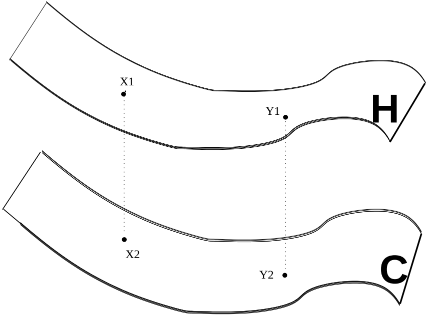
$$[D_A, a] = [D + H - i\gamma^\mu A_\mu, a] \neq [D, a].$$

"Fluctuated distance" on the set $\mathcal{P}(\mathcal{A})$ of (pure) states of $\mathcal{A}$,

$$\boxed{d_{D_A}(\omega_1, \omega_2) \doteq \sup_{a \in \mathcal{A}} \{ |\omega_1(a) - \omega_2(a)| \, ; \, \|[D_A, a]\| \leq 1 \}}$$

Scalar fluctuation: $A_\mu = 0, H \neq 0$ *(Wulkenhaar, P.M. 2001)*

$\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_I$ with $\mathcal{A}_I = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \Longrightarrow \mathcal{P}(\mathcal{A})$ is a two-sheet model



Proposition 5: The spectral distance d_{D_A} coincides with the geodesic distance in $\mathcal{M} \times [0,1]$ given by

$$\begin{pmatrix} g^{\mu\nu} & 0 \\ 0 & \left(|1+h_1|^2 + |h_2|^2\right) m_{\text{top}}^2 \end{pmatrix} \text{ where } \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \text{ is the Higgs doublet.}$$

P.M., R. Wulkenhaar 2001

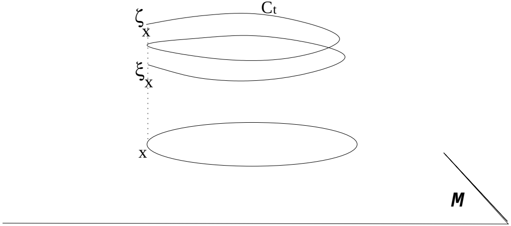
Gauge fluctuation: $A_\mu \neq 0, H = 0$

$\mathcal{A}_I = C^\infty(\mathcal{M}) \otimes M_n(\mathbb{C})$. Pure states: $P \xrightarrow{\pi} \mathcal{M}$ with fibre $\mathbb{C}P^{n-1}$.

The distance is fully encoded with the covariant Dirac operator

$$D_A = -i\gamma^\mu(\partial_\mu + A_\mu)$$

$$A_\mu \Rightarrow \begin{cases} \text{distance spectrale } d_{D_A} \\ \text{horizontal distance } d_H \end{cases} \Rightarrow d_{D_A} = d_H \text{ ?} \quad \text{(Connes 96)}$$



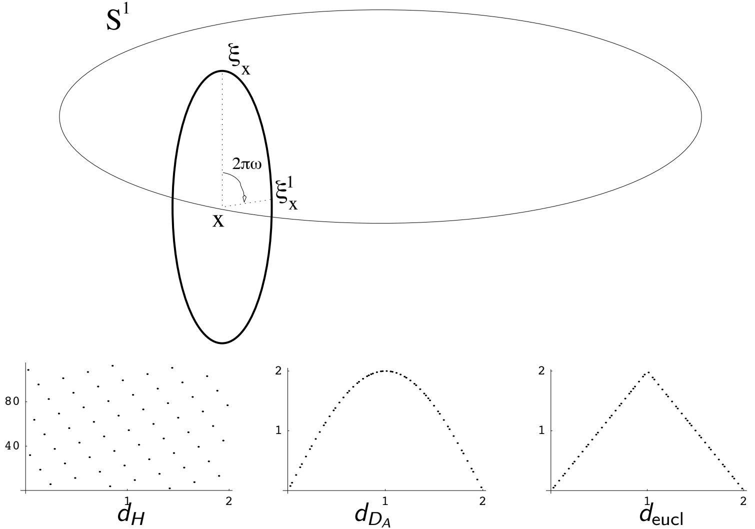
$\mathbb{R}^3$ with $\sum_\mu A_\mu dx^\mu = (x^2 dx^1 - x^1 dx^2) \otimes \theta\partial_3 \Longrightarrow d_H(\xi_x, \zeta_x) = 4\pi$

Proposition 6: $d_{D_A} \leq d_H$ but no equality except if the holonomy is trivial.

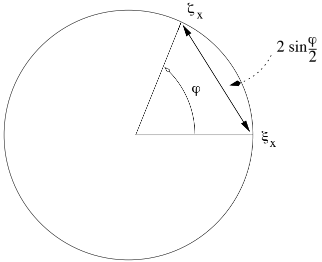
P. M. 2006-08

$\mathcal{A} = C^\infty(S_1) \otimes M_2(\mathbb{C})$.

$$\begin{cases} d_H(\xi_x, \xi_x^k) & = & 2k\pi \\ d_{D_A}(\xi_x, \xi_x^k) & = & C \sin k\pi\omega \text{ where } C \text{ is a constant.} \end{cases}$$



On a fiber



The spectral distance sees the disk through the circle, in the same way it sees between the two sheets of the standard model.

- The pure state space equipped with the spectral distance is *not* a path-metric space, i.e. there is no curve $s \in [0,1] \mapsto \varphi_s$ such that

$$d_D(\varphi_s, \varphi_t) = |t - s| d_D(\varphi_0, \varphi_1).$$

  Seems to be the case as soon as $\mathcal{A}$ is noncommutative.

4. Moyal Plane

a, b Schwartz functions on $\mathbb{R}^2$. Star-product:

$$(a \star b)(x) = \frac{1}{(\pi\theta)^2} \int d^2s \, d^2t \; a(x+s) b(x+t) e^{-i2s\Theta^{-1}t}$$

where

$$s\Theta^{-1}t \equiv s^\mu \Theta^{-1}_{\mu\nu} t^\nu \quad \text{with } \Theta_{\mu\nu} = \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Spectral triple for the Moyal plane

$$\mathcal{A}_\theta = (\mathcal{S}, \star), \quad \mathcal{H} = L^2(\mathbb{R}^2) \otimes \mathbb{C}^2, \quad D = -i \sum_{\mu=1}^{2} \sigma^\mu \partial_\mu .$$

The left regular representation of $a \in \mathcal{A}_\theta$ on $\mathcal{H}$ is

$$\pi(a) = L(a) \otimes \mathbb{I}_2 : \ \pi(f)\psi = \begin{pmatrix} a \star \psi_1 \\ a \star \psi_2 \end{pmatrix} .$$

Defining $\partial = \frac{1}{\sqrt{2}}(\partial_1 - i\partial_2)$, $\bar{\partial} = \frac{1}{\sqrt{2}}(\partial_1 + i\partial_2)$, the Dirac operator writes

$$D = -i\sqrt{2} \begin{pmatrix} 0 & \bar{\partial} \\ \partial & 0 \end{pmatrix} .$$

- Moyal space is non compact $\iff \mathcal{A}_\theta$ has no unit.
  Some axioms of spectral triple, e.g. orientation, require a unitization of $\mathcal{A}_\theta$.
  Not relevant for the distance.

The matrix base

Write $\bar{z} = \frac{1}{\sqrt{2}}(x_1 - ix_2)$, $z = \frac{1}{\sqrt{2}}(x_1 + ix_2)$. Define

$$f_{mn} = \frac{1}{(\theta^{m+n} m!n!)^{1/2}} \bar{z}^{\star m} \star f_{00} \star z^{\star n}, \quad H = \frac{1}{2}(x_1^2 + x_2^2), \quad f_{00} = 2e^{-2H/\theta},$$

the Wigner transitions eigenfunctions of the harmonic oscillator (f_{mm}: Wigner function of the m^{th} energy level of the harmonic oscillator).

- $\{f_{mn}\}_{m,n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}^2)$.
- $f_{mn} \star f_{pq} = \delta_{np} f_{mq}$. There is a Frechet algebra isomorphism between $\mathcal{A}_\theta$ and the algebra of fast decreasing sequences $\{a_{mn}\}_{m,n\in\mathbb{N}}$: for any $f \in \mathcal{S}$,

$$a = \sum_{m,n} a_{mn} f_{mn} \quad \text{with } a_{mn} = \int_{\mathbb{R}^2} f(x) f_{mn}(x) d^2x.$$

Pure states

The evaluation at x is not a state of $\mathcal{A}_\theta$ for $(f^* \star f)(x)$ may not be positive.

$\mathcal{A}_\theta$ is a reducible representation of the algebra of compact operators $\mathcal{K}$:

$$\mathcal{H}_p \doteq \overline{\text{span } \{f_{mp},\ m \in \mathbb{N}\}}$$

is invariant for any fixed p.

The set of pure states of $\bar{\mathcal{A}}_\theta$ is the set of vector states

$$\omega_\psi(a) \equiv \langle \psi, L(a)\psi \rangle = 2\pi\theta \sum_{m,n\in\mathbb{N}} \psi_m^* \psi_n a_{mn}$$

where

$$\psi = \sum_{m\in\mathbb{N}} \psi_m f_{mp}, \quad \sum_{m\in\mathbb{N}} |\psi_m|^2 = \frac{1}{2\pi\theta}$$

is a unit vector in $\mathcal{H}_p$.

Spectral distance on the Moyal plane

Proposition 7: The spectral distance on the Moyal plane is not bounded, neither from above nor from below (except by 0).

The eigenstates of the quantum harmonic oscillator,

$$\omega_{f_{m0}}(a) = 2\pi\theta a_{mm} \doteq \omega_m(a).$$

form a 1-dimensional lattice with distance

$$d_D(\omega_m, \omega_n) = \sqrt{\frac{\theta}{2}} \sum_{k=m+1}^{n} \frac{1}{\sqrt{k}}.$$

E. Cagnache, F. D'Andrea, P.M., J.C. Wallet 2009

- Quantum space does not necessarily implies minimum lenght. Compare to DFR model where the distance is the spectrum of $\sqrt{X^2 + Y^2}$.

Conclusion

Spectral distance: viewing $d_{\text{geo}}(x, y)$ as $d_D(\delta_x, \delta_y)$, i.e. as a supremum instead of the length of a minimal curve makes sense in a quantum context.

Kantorovich duality: minimizing a cost (Monge problem)

$$W_-(\mu_1, \mu_2) = \inf_\pi \int_{\mathcal{M}\times\mathcal{M}} d_{\text{geo}}(x, y)\, d\mu$$

is equivalent to maximizing a profit

$$W_+(\mu_1, \mu_2) = \sup_{\|f\|_{\text{Lip}}\leq 1} \left\{ \int_{\mathcal{M}} f\, d\mu_1 - \int_M f\, d\mu_2 \right\}.$$

Transport consortium, looking for the tight price $f(x)$ at wich buy the bread from factories and sell it to bakeries, staying competitive: $|f(x) - f(y)| \leq d_{\text{geo}}(x, y)$.

μ_1 : distribution of bread factories	$\int_M f\, d\mu_1$: total price paid to farmers
μ_2 : distribution of bakeries	$\int_M f\, d\mu_2$: total money got from bakers
W_- : total transportation cost	W_+ : total profit

- What cost does one minimize in a quantum context ? Higgs field as a cost function $c(x, x) \neq 0$? Towards a noncommutative economics ?