About the arrow **can**: Tensor product of series $\longrightarrow$ Double series, its usage in C.S. and Combinatorics II.

(MO: 200442 & 201753), Schützenberger's calculus (towards continuation of polylogarithms).

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Collaboration at various stages of the work and in the framework of the Project *Evolution Equations in Combinatorics and Physics* : C. Bui, Q.H. Ngô, Vu Dinh, S. Goodenough.

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Goal of this talk

The goal of this talk is threefold

A

Dualization of laws and co-laws of a bialgebra with the conc-bialgebras to begin with.

B

Pursue with the generality. How about a general bialgebra.

C

MRS factorisation(s): Local systems of coordinates for Hausdorff groups.

Transpose of a laws and dual laws

Original Problem

Let $\mathcal{B} = (\mathcal{B}, \mu, 1_{\mathcal{B}}, \Delta, \epsilon)$ be a bialgebra. We now will examine the dualization of it, i.e. ideally the existence of another bialgebra

$$\mathcal{B}_1 = (\mathcal{B}_1, \mu_1, 1_{\mathcal{B}_1}, \Delta_1, \epsilon_1)$$

and a pairing $\langle \, . \mid . \, \rangle : \ \mathcal{B}_1 \otimes \mathcal{B} \to \mathbf{k}$ such that, identically

$$\langle x \mid \mu(y \otimes z) \rangle = \langle \Delta_1(x) \mid y \otimes z \rangle^{\otimes 2} \tag{1}$$

$$\langle \mu_1(x \otimes y) \mid z \rangle = \langle x \otimes y \mid \Delta(z) \rangle^{\otimes 2} \tag{2}$$

$$\epsilon(x) = \langle 1_{\mathcal{B}_1} \mid x \rangle \ ; \ \epsilon_1(x) = \langle x \mid 1_{\mathcal{B}} \rangle \tag{3}$$

In addition, we require that (through $\langle \, . \mid . \, \rangle$) we get an embedding ($\mathcal{B}_1 \hookrightarrow \mathcal{B}^{\vee}$) i.e. $\mathcal{B}^{\perp} = \{0\}$ and that there are sufficiently many elements in $\mathcal{B}_1$ to separate elements of $\mathcal{B}$ i.e. $\mathcal{B}_1^{\perp} = \{0\}$. We say that this pair is *in separating dualty* (see discussion after MO question 179214).

Examples

The identities (1)-(3) mean that there is a correspondence between the elements of $\mathcal{B}$ and $\mathcal{B}_1$.

1. $\mathcal{B} = (\mathbf{k}\langle X\rangle, \mathrm{conc}, 1_{X^*}, \Delta_{⧢}, \epsilon)$; $\mathcal{B}_1 = (\mathbf{k}\langle X\rangle, ⧢, 1_{X^*}, \Delta_{\mathrm{conc}}, \epsilon)$

   $\Delta_{⧢}(w) = \sum_{I+J=[1\cdots|w|]} w[I] \otimes w[J]$; $\Delta_{\mathrm{conc}}(w) = \sum_{uv=w} u \otimes v$

2. φ-**shuffle**. φ : $\mathbf{k}.X \otimes \mathbf{k}.X \to \mathbf{k}.X$ (associative without unit)

$$\begin{aligned} &\text{for } a, b \in X,\ u, v \in X^* \\ &u ⧢_\varphi 1_{X^*} = 1_{X^*} ⧢_\varphi u = u \\ &a.u ⧢_\varphi b.v = a.(u ⧢_\varphi b.v) + b.(a.u ⧢_\varphi v) + \underbrace{\varphi(a,b).(u ⧢_\varphi v)}_{\text{perturbation}} \end{aligned}$$

With this law $\mathcal{B}_\varphi = (\mathbf{k}\langle X\rangle, ⧢_\varphi, 1_{X^*}, \Delta_{\mathrm{conc}}, \epsilon)$

is a Hopf algebra. The possibility of dualization of $\mathcal{B}_\varphi$ depends crucially on what φ is.

Name	Formula (recursion)	φ	Reference
Shuffle	$au \sqcup\!\sqcup bv = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v)$	$\varphi \equiv 0$	Ree
Stuffle	$x_i u \uplus x_j v = x_i(u \uplus x_j v) + x_j(x_i u \uplus v) + x_{i+j}(u \uplus v)$	$\varphi(x_i, x_j) = x_{i+j}$	Hoffman
Min-stuffle	$x_i u \sqcup x_j v = x_i(u \sqcup x_j v) + x_j(x_i u \sqcup v) - x_{i+j}(u \sqcup v)$	$\varphi(x_i, x_j) = -x_{i+j}$	Costermans
Muffle	$x_i u \odot x_j v = x_i(u \odot x_j v) + x_j(x_i u \odot v) + x_{i\times j}(u \odot v)$	$\varphi(x_i, x_j) = x_{i\times j}$	Enjalbert,HNM
q-shuffle	$x_i u \uplus_q x_j v = x_i(u \uplus_q x_j v) + x_j(x_i u \uplus_q v) + q x_{i+j}(u \uplus_q v)$	$\varphi(x_i, x_j) = q x_{i+j}$	Bui
q-shuffle$_2$	$x_i u \uplus_q x_j v = x_i(u \uplus_q x_j v) + x_j(x_i u \uplus_q v) + q^{i.j} x_{i+j}(u \uplus_q v)$	$\varphi(x_i, x_j) = q^{i.j} x_{i+j}$	Bui
$\mathtt{LDIAG}(1, q_s)$	$au \sqcup\!\sqcup bv = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v) + q_s^{\lvert a\rvert \lvert b\rvert} a.b(u \sqcup\!\sqcup v)$	$\varphi(a, b) = q_s^{\lvert a\rvert \lvert b\rvert}(a.b)$	GD,Koshevoy,Penson,Tollu
q-Infiltration	$au \uparrow bv = a(u \uparrow bv) + b(au \uparrow v) + q\delta_{a,b} a(u \uparrow v)$	$\varphi(a, b) = q\delta_{a,b} a$	Chen-Fox-Lyndon
AC-stuffle	$au \sqcup\!\sqcup_\varphi bv = a(u \sqcup\!\sqcup_\varphi bv) + b(au \sqcup\!\sqcup_\varphi v) + \varphi(a, b)(u \sqcup\!\sqcup_\varphi v)$	$\varphi(a, b) = \varphi(b, a)$ $\varphi(\varphi(a, b), c) = \varphi(a, \varphi(b, c))$	Enjalbert,HNM
Semigroup- -stuffle	$x_t u \sqcup\!\sqcup_\perp x_s v = x_t(u \sqcup\!\sqcup_\perp x_s v) + x_s(x_t u \sqcup\!\sqcup_\perp v) + x_{t\perp s}(u \sqcup\!\sqcup_\perp v)$	$\varphi(x_t, x_s) = x_{t\perp s}$	Deneufchâtel
φ-shuffle	$au \sqcup\!\sqcup_\varphi bv = a(u \sqcup\!\sqcup_\varphi bv) + b(au \sqcup\!\sqcup_\varphi v) + \varphi(a, b)(u \sqcup\!\sqcup_\varphi v)$	$\varphi(a, b)$ law of AAU	Manchon, Paycha

Common pattern

$$w \sqcup\!\sqcup_\varphi 1_{X^*} = 1_{X^*} \sqcup\!\sqcup_\varphi w = w \text{ and}$$

$$au \sqcup\!\sqcup_\varphi bv = a(u \sqcup\!\sqcup_\varphi bv) + b(au \sqcup\!\sqcup_\varphi v) + \varphi(a, b)(u \sqcup\!\sqcup_\varphi v)$$

Dualizable laws in conc-shuffle bialgebras/1

We can exploit the basis of words as follows

1. Any bilinear law (shuffle, stuffle or any) $\mu : A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle \to A\langle\mathcal{X}\rangle$ can be described through its structure constants w.r.t. to the basis of words, *i.e.* for $u, v, w \in \mathcal{X}^*$, $\Gamma_{u,v}^{w} := \langle \mu(u \otimes v) \mid w\rangle$ so that
$$\mu(u \otimes v) = \sum_{w\in\mathcal{X}^*} \Gamma_{u,v}^{w} w.$$
2. In the case when $\Gamma_{u,v}^{w}$ is locally finite in w, we say that the given law is dualizable, the arrow ${}^t\mu$ restricts nicely to $A\langle\mathcal{X}\rangle \hookrightarrow A\langle\langle\mathcal{X}\rangle\rangle$ and one can define on the polynomials a comultiplication by the finite sum
$$\Delta_\mu(w) := \sum_{u,v\in\mathcal{X}^*} \Gamma_{u,v}^{w} u \otimes v\ .$$
3. When the law μ is dualizable, we have

$$\begin{array}{ccc}
A\langle\langle\mathcal{X}\rangle\rangle & \xrightarrow{\ {}^t\mu\ } & A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle \\
{\scriptstyle can}\uparrow & & \uparrow{\scriptstyle \Phi|_{A\langle\mathcal{X}\rangle\otimes_A A\langle\mathcal{X}\rangle}} \\
A\langle\mathcal{X}\rangle & \xrightarrow{\ \Delta_\mu\ } & A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle
\end{array}$$

(still when μ is dualizable), the arrow Δ_μ is unique to be able to close the rectangle and $\Delta_\mu(P)$ is defined as above.

4. Now, let us give a family of (counter) examples. We start with $\mathcal{X} = Y_S$ where $S \subset \mathbb{C}$ is an additive subsemigroup (intances of S are $\mathbb{N}, \mathbb{N}_+, \mathbb{R}_+, [2, +\infty[, \mathbb{Z}$ and the upper-quater plane $\mathcal{P} = \mathbb{N} \oplus i\mathbb{N}$).
Building, with $Y_S = \{y_s\}_{s\in S}$ and $\varphi(y_s, y_t) := y_{(s+t)}$, the φ-shuffle $ш_\varphi$ as above (slide 5).
With the above, we get the following table

S	$\mathbb{N}$	$\mathbb{N}_+$	$\mathbb{R}_+$	$\mathbb{Z}$	$\mathcal{P}$
Dualizable ?	Y	Y	N	N	Y

the test is simple: $ш_\varphi$ is dualizable iff S iff M satisfies condition (D) in [Bourba89, Ch. III, §2.10]) which is

$$(\forall r \in S)(\{(s,t) \in S^2 | r = st\} \text{ is finite}) \tag{4}$$

Dualizable laws in conc-shuffle bialgebras/2

5. Our last example will be the bialgebra of a monoid M. It is with $\Delta_{\odot}$, the Hadamard (pointwise) coproduct $\mathcal{B} = (\mathbf{k}[M], \mu_M, 1_M, \Delta_{\odot}, \epsilon)$ where μ is the standard product in the algebra $\mathbf{k}[M]$, $\Delta_{\odot}(m) = m \otimes m$ and $\epsilon(f) = \sum_{m \in M} \langle f \mid m \rangle$.
   Then, if M satifies condition (D) of Bourbaki, $\mathcal{B}$ is dualizable with $\Delta_1(m) = \sum_{pq=m} p \otimes q$ and $f \odot g := \sum_{m \in m} \langle f \mid m \rangle \langle g \mid m \rangle\, m$ (pointwise product), and $\epsilon_1(f) = \langle f \mid 1_M \rangle$, we have

$$\mathcal{B}_1 = (\mathbf{k}[M], \odot, \chi_M, \Delta_1, \epsilon_1)$$

   where χ_M is the characteristic function of M $(m \mapsto 1_{\mathbf{k}},\ \forall m \in M)$.
6. To end with, we remark that the dualization of a comultiplication is always possible, the difficulty being to dualize a product. We now consider separately this problem.

Transpose of a laws/1

7. We start with a $\mathbf{k}$ a field and $\mathcal{A}$ a $\mathbf{k}-\mathbf{AAU}$

   Let $\mu: \ \mathcal{A}\otimes_{\mathbf{k}}\mathcal{A}\to\mathcal{A}$, we have

$$\begin{array}{ccc}
\mathcal{A}^{\vee} & \xrightarrow{\ {}^{t}\mu\ } & (\mathcal{A}\otimes_{\mathbf{k}}\mathcal{A})^{\vee} \\
\uparrow{\scriptstyle can} & & \uparrow{\scriptstyle \Phi} \\
? & \xrightarrow{\ \Delta_{\mu}\ } & \mathcal{A}^{\vee}\otimes_{\mathbf{k}}\mathcal{A}^{\vee}
\end{array}$$

   where $\Phi(S\otimes T)$ is the linear form such that

$$\langle \Phi(S\otimes T) \mid u\otimes v\rangle := \langle S\mid u\rangle\langle T\mid v\rangle \qquad (5)$$

   Due to the fact that $\mathbf{k}$ is a field, the arrow Φ is into.

8. The set ? is the set of elements $f\in\mathcal{A}^{\vee}$ such that ${}^{t}\mu(f)\in Im(\Phi)$. One has a very simple criterium to characterize them.

Transpose of a laws/2

7. We have the following (proof is left as an exercise).

Theorem A (Sweedler, Abe)

Let $\mathbf{k}$ be a field and $\mathcal{A}$ be a $\mathbf{k}$-AAU $(\mathcal{A}, \mu, 1_{\mathcal{A}})$, we use also infix notation $\mu(u \otimes v) = u * v$ and define the left-right-shifts by $\langle x \triangleright f \triangleleft y \mid z \rangle := \langle f \mid yzx \rangle$ (one-sided shifts are derived by $x, y = 1_{\mathcal{A}}$). Then TFAE

1. ${}^t\mu(f) \in Im(\Phi)$
2. There exists a double (finite) sequence $(g_i, h_i)_{1 \leq i \leq n}$ such that for all $x, y \in \mathcal{A}$,
$$\langle f \mid x * y \rangle = \sum_{i=1}^{n} \langle g_i \mid x \rangle \langle h_i \mid y \rangle$$
3. The left-shifts $(x \triangleright f)_{x \in \mathcal{A}}$ form a family of finite rank.
4. The right-shifts $(f \triangleleft x)_{x \in \mathcal{A}}$ form a family of finite rank.
5. The bi-shifts $(x \triangleright f \triangleleft y)_{x,y \in \mathcal{A}})$ form a family of finite rank.

Transpose of a laws/3

7. End of the theorem

Theorem A (Sweedler, Abe), cont'd

6. It exists a matrix representation $\mu : \mathcal{A} \to \mathbf{k}^{n\times n}$ and vectors $\lambda \in \mathbf{k}^{1\times n}$, $\tau \in \mathbf{k}^{n\times 1}$ such that, for all $a \in \mathcal{A}$

$$\langle f \mid a \rangle = \lambda\mu(a)\tau \tag{6}$$

Remarks

i) Condition 2 in "Theorem A (Sweedler, Abe)" is exactly ${}^t\mu(f) \in Im(\Phi)$ so that, equivalence 1 $\Longleftrightarrow$ 2 is just a reformulation.
ii) Property 6 allows to prove that, if ${}^t\mu(f) \in Im(\Phi)$, in fact $\Delta_\mu(f) \in \mathcal{A}^\circ \otimes \mathcal{A}^\circ$. So the commutative square in slide 9 give rise to a ladder which stops at the first step.

Transpose of a laws/4

We start with a **k** − **AAU** (**k** a field) $\mathcal{A}$, dualizing
$\mu : \ \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \to \mathcal{A}$, we have

$$\begin{array}{ccc} \mathcal{A}^{\vee} & \xrightarrow{\ {}^t\mu\ } & (\mathcal{A} \otimes_{\mathbf{k}} \mathcal{A})^{\vee} \\ {\scriptstyle can}\uparrow & & \uparrow{\scriptstyle \Phi} \\ \mathcal{A}^{\circ} & \xrightarrow{\ \Delta_\mu\ } & \mathcal{A}^{\vee} \otimes_{\mathbf{k}} \mathcal{A}^{\vee} \\ {\scriptstyle can}\uparrow & & \uparrow{\scriptstyle j\otimes j} \\ \mathcal{A}^{\circ\circ} & \xrightarrow{\ \Delta_\mu\ } & \mathcal{A}^{\circ} \otimes_{\mathbf{k}} \mathcal{A}^{\circ} \end{array}$$

In fact, as said in the remarks above (slide 11), one sees that the "descent" stops at first step

and then $\mathcal{A}^{\circ\circ} = \mathcal{A}^{\circ}$ this space will be defined as Sweedler's dual of $\mathcal{A}$.

Case of the shuffle algebra/1

With the example of $\mathcal{A} = (\mathbb{C}\langle X \rangle, ш, 1_{X^*})$, the square

$$\begin{array}{ccc} \mathcal{A}^\vee & \xrightarrow{\quad {}^t\mu \quad} & (\mathcal{A} \otimes_{\mathbf{k}} \mathcal{A})^\vee \\ \uparrow{\scriptstyle can} & & \uparrow{\scriptstyle \Phi} \\ \mathcal{A}^\circ & \xrightarrow{\quad \Delta_\mu \quad} & \mathcal{A}^\circ \otimes_{\mathbf{k}} \mathcal{A}^\circ \end{array}$$

remarking that $\mathbb{C}\langle X \rangle \otimes \mathbb{C}\langle X \rangle \simeq \mathbb{C}[X^* \otimes X^*]$ becomes

$$\begin{array}{ccc} \mathbb{C}\langle\langle X \rangle\rangle & \xrightarrow{\quad {}^t ш \quad} & \mathbb{C}[[X^* \otimes X^*]] \\ \uparrow{\scriptstyle can} & & \uparrow{\scriptstyle \Phi} \\ \mathcal{A}^\circ & \xrightarrow{\quad \Delta_{ш} \quad} & \mathcal{A}^\circ \otimes_{\mathbb{C}} \mathcal{A}^\circ \end{array}$$

Case of the shuffle algebra/2

8. So that, in this case [shuffle algebra], condition 2 in Theorem A reads for all $P, Q \in \mathbb{C}\langle X\rangle$ (equivalent, by bilinearity, to "for all $u, v \in X^*$")

$$\langle R \mid u \sqcup\!\sqcup\, v\rangle = \sum_{i=1}^{n} \langle S_i \mid u\rangle\langle T_i \mid v\rangle$$

9. Now considering the identity $(\alpha.x)^* \sqcup\!\sqcup\, (\beta.x)^* = ((\alpha+\beta).x)^*$ with $c = \alpha + \beta \in \mathbb{N}_{\geq 2}$, we get $(c.x)^* = (a.x)^* \sqcup\!\sqcup\, (b.x)^*$ for $a + b = c$ and then identities like

$$(c.x)^* = \frac{1}{c+1} \sum_{\substack{a+b=c \\ a,b\in\mathbb{N}}} (a.x)^* \sqcup\!\sqcup\, (b.x)^*$$

$$(c.x)^* = \frac{1}{c-1} \sum_{\substack{a+b=c \\ a,b\in\mathbb{N}_+}} (a.x)^* \sqcup\!\sqcup\, (b.x)^* \tag{7}$$

Case of the shuffle algebra/3

10. But, in spite of these identities, there is no formula of the type $\Delta_{ш}((c.x)^*) = \sum_{i=1}^{n} S_i \otimes T_i$
11. Let us compute the shifts (defined as in slide 10) $x^k \triangleright^{ш} (c.x)^*$. We have

$$x^k \triangleright^{ш} (c.x)^* = \sum_{n\geq 0} \langle (c.x)^* \mid x^n \,ш\, x^k \rangle \, x^n = \sum_{n\geq 0} \langle (c.x)^* \mid \binom{n+k}{k} x^{n+k} \rangle \, x^n = \frac{c^k}{k!} \sum_{n\geq 0} Q_k(n)\, c^n.x^n \qquad (8)$$

with $Q_k \in \mathbb{C}[x]$ is of degree k (exactly). This proves that the shifts $x^k \triangleright^{ш} (c.x)^*$ are all $\mathbb{C}$-linearly independent.

12. This shows that there is no hope that identities (7) could be dualized.

Computation of $\Delta_{ш}((c.x)^*)/1$

13. As said above $\Delta_{ш}((c.x)^*) \in \mathbb{C}\langle\langle X^* \otimes X^* \rangle\rangle$ but, as was proved $\Delta_{ш}((c.x)^*) \notin \mathbb{C}\langle\langle X^* \rangle\rangle \otimes \mathbb{C}\langle\langle X^* \rangle\rangle$, so

$$\Delta_{ш}((c.x)^*) = \sum_{u,v \in X^*} c(u,v) u \otimes v \tag{9}$$

14. Firstly, we remark that $\mathbb{C}\langle\langle X^* \otimes X^* \rangle\rangle$ (the algebra of functions on $X^* \otimes X^*$, the total algebra of the monoid $X^* \otimes X^*$, see "total algebra of a monoid" in [Bourba89, Ch. III, §2.10]) comes with a filtration due to the gradation of $X^* \otimes X^*$ as follows
    1. If M is a $\mathbb{N}$-graded monoid[a]
    2. So is the "double" of M ($M \times M \simeq M \otimes_{\mathbf{k}} M \subset \mathbf{k}\langle M \rangle \otimes \mathbf{k}\langle M \rangle$) with

$$(M \otimes_{\mathbf{k}} M)_n := \sqcup_{p+q=n} M_p \otimes M_q \tag{10}$$

[a] That is $M = \sqcup_{n \geq 0} M_n$ with $M_p.M_q \subset M_{p+q}$.

Computation of $\Delta_{ш}((c.x)^*)/2$

15. Computation continued
    3. So comes the algebra

$$\mathbf{k}[M] \otimes \mathbf{k}[M] \simeq \mathbf{k}[M \otimes_{\mathbf{k}} M]$$

with

$$(\mathbf{k}[M] \otimes \mathbf{k}[M])_n = \bigoplus_{n=p+q} (\mathbf{k}[M])_p \otimes (\mathbf{k}[M])_q \tag{11}$$

    4. So it is a general fact that the dual of a graded algebra comes with a natural (decreasing) filtration[a] given by $(\mathbf{k}\langle\langle M \otimes M\rangle\rangle_{\geq n}$ to be the linear forms that have their support in $(M \otimes_{\mathbf{k}} M)_{\geq n}$.

[a] and even the dual of a increasingly filtered algebra, see MO question 310354.

Computation of $\Delta_{⧢}((c.x)^*)/3$

16. Computation continued
    5. In general, a family $(S_i)_{i\in I}$ in $\mathbf{k}^M$ is said summable if, for each $m \in M$, the function $i \mapsto \langle S_i \mid m\rangle$ is finitely supported.
    6. The end of the computation is left as an exercise using the following ingredients
        - $(c.x)^* = \sum_{n\geq 0}(c.x)^n$
        - The family $(\Delta_{⧢}((c.x)^n))_{n\geq 0}$ is summable
        - If a family $(S_n)_{n\geq 0}$ is summable, so is $(\Delta_{⧢}(S_n))_{n\geq 0}$ and $\Delta_{⧢}$ commutes with the infinite sums.

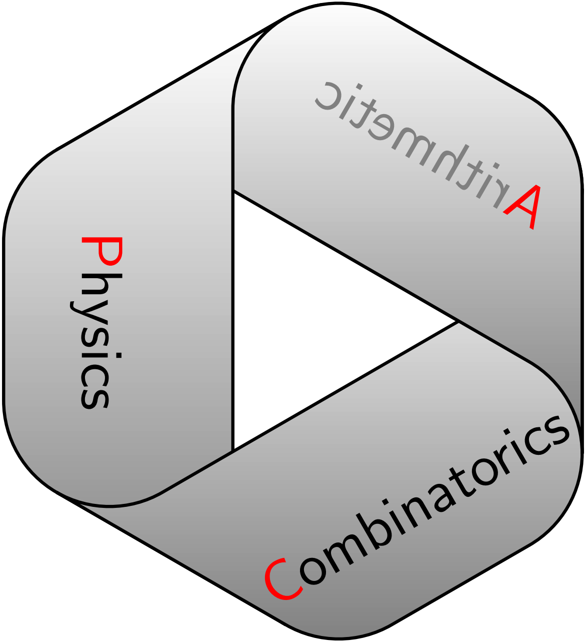
[BeRe88] J. Berstel, C. Reutenauer, *Rational series and their languages*, Springer-Verlag, 1988.

[1] Van Chien Bui, Gérard H.E. Duchamp, Quoc Huan Ngô, Vincel Hoang Ngoc Minh and Christophe Tollu, *(Pure) Transcendence Bases in φ-Deformed Shuffle Bialgebras*, 74 ème Séminaire Lotharingien de Combinatoire (published oct. 2018).

[Bourba89] N. Bourbaki, *Algebra I (Chapters 1-3)*, Springer 1989.

[2] G. H. E. Duchamp and C. Tollu, *Sweedler's duals and Schützenberger's calculus*, In K. Ebrahimi-Fard, M. Marcolli and W. van Suijlekom (eds), Combinatorics and Physics, p. 67 - 78, Amer. Math. Soc. (Contemporary Mathematics, vol. 539), 2011. arXiv:0712.0125v3 [math.CO]

[Schütz61] Marcel-Paul Schützenberger, *On the definition of a family of automata*, Information and Control, **4** (1961), pp. 245–270.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