On universal differential equations

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Outline

INTRODUCTION

Picard-Vessiot theory of ordinary differential equation

$(\mathbf{k}, \partial)$ a commutative differential ring without zero divisors.
$\mathrm{Const}(\mathbf{k}) = \{c \in \mathbf{k} | \partial c = 0\}$ is supposed to be a field.

$$(ODE) \quad (a_n\partial^n + a_{n-1}\partial^{n-1} + \ldots + a_0)y = 0, \quad a_0, \ldots, a_{n-1}, a_n \in \mathbf{k}.$$

a_n^{-1} is supposed to exist.

Definition 1

1. Let $y_1, \ldots, y_n$ be $\mathrm{Const}(\mathbf{k})$-linearly independent solutions of (ODE). Then $\{y_1, \ldots, y_n\}$ is called a fundamental set of solutions of (ODE) and it generates a $\mathrm{Const}(\mathbf{k})$-vector subspace of dimension at most n.
2. If[1] $M = \mathbf{k}\{y_1, \ldots, y_n\}$ and $\mathrm{Const}(M) = \mathrm{Const}(\mathbf{k})$ then M is called a Picard-Vessiot extension related to (ODE)
3. Let $\mathbf{k} \subset \mathbb{K}_1$ and $\mathbf{k} \subset \mathbb{K}_2$ be differential rings. An isomorphism of rings $\sigma : \mathbb{K}_1 \to \mathbb{K}_2$ is a differential $\mathbf{k}$-isomorphism if
$$\forall a \in \mathbb{K}_1, \quad \partial(\sigma(a)) = \sigma(\partial a) \text{ and, if } a \in \mathbf{k},\ \sigma(a) = a.$$
If $\mathbb{K}_1 = \mathbb{K}_2 = \mathbb{K}$, the differential galois group of $\mathbb{K}$ over $\mathbf{k}$ is by
$$\mathrm{Gal}_{\mathbf{k}}(\mathbb{K}) = \{\sigma | \sigma \text{ is a differential } \mathbf{k}\text{-automorphism of } \mathbb{K}\}.$$

1. Let R_1, R_2 be differential rings s.t. $R_1 \subset R_2$. Let S be a subset of R_2. $R_1\{S\}$ denotes the smallest differential subring of R_2 containing R_1. $R_1\{S\}$ is the ring (over R_1) generated by S and their derivatives of all orders.

Linear differential equations and Dyson series

Let $a_0, \ldots, a_n \in \mathbb{C}(z)$, $\quad a_n(z)\partial^n y(z) + \ldots + a_1(z)\partial y(z) + a_0(z)y(z) = 0.$

$$(ED) \quad \left\{ \begin{array}{rcll} \partial q(z) & = & A(z)q(z), & A(z) \in \mathcal{M}_{n,n}(\mathbb{C}(z)), \\ q(z_0) & = & \eta, & \lambda \in \mathcal{M}_{1,n}(\mathbb{C}), \\ y(z) & = & \lambda q(z), & \eta \in \mathcal{M}_{n,1}(\mathbb{C}). \end{array} \right.$$

By successive Picard iterations, with the initial point $q(z_0) = \eta$, we get [2] $y(z) = \lambda U(z_0; z)\eta$, where $U(z_0; z)$ is the following functional expansion

$$U(z_0; z) = \sum_{k \geq 0} \int_{z_0}^{z} A(z_1)dz_1 \int_{z_0}^{z_1} A(z_2)dz_2 \ldots \int_{z_0}^{z_{k-1}} A(z_k)dz_k, \text{(Dyson series)}$$

and $(z_0, z_1 \ldots, z_k, z)$ is a subdivision of the path of integration $z_0 \rightsquigarrow z$.
In order to find the matrix $\Omega(z_0; z)$ s.t.

$$U(z_0; z) = \exp[\Omega(z_0; z)] = \top \exp \int_{z_0}^{z} A(s)ds, \qquad \text{(Feynman's notation)}$$

Magnus computed $\Omega(z_0; z)$ as limit of the following Lie-integral-functionals

$$\begin{aligned} \Omega_1(z_0; z) &= \int_{z_0}^{z} A(z)ds, \\ \Omega_k(z_0; z) &= \int_{z_0}^{z} [A(z) + [A(z), \Omega_{k-1}(z_0; s)]/2 \\ &\quad + [[A(z), \Omega_{k-1}(z_0; s)], \Omega_{k-1}(z_0; s)]/12 + \ldots)ds. \end{aligned}$$

2. Subject to convergence.

Fuchsian linear differential equations

Let Ω be a simply connected domain and $\mathcal{H}(\Omega)$ be the ring of holomorphic functions over Ω (with $1_{\mathcal{H}(\Omega)}$ as neutral element). Let us consider, here, $\sigma = \{s_i\}_{i=0,..,m}$, $m \geq 1$, as set of simple poles of (ED) and $\Omega = \widetilde{\mathbb{C} \setminus \sigma}$.

$$A(z) = \sum_{i=0}^{m} M_i u_i(z), \quad \text{where} \quad \begin{cases} \hfill M_i & \in \mathcal{M}_{n,n}(\mathbb{C}), \\ u_i(z) = (z - s_i)^{-1} & \in \mathbb{C}(z). \end{cases}$$

$$(ED) \quad \begin{cases} \partial q(z) & = & \left(\sum_{i=0}^{m} M_i u_i(z) \right) q(z), \\ q(z_0) & = & \eta, \\ y(z) & = & \lambda q(z). \end{cases}$$

Let X^* be the set of words over $X = \{x_0, \ldots, x_m\}$ and

$$\alpha_{z_0}^{z} \otimes \mathcal{M} : \mathbb{C}\langle X \rangle \otimes \mathbb{C}\langle X \rangle \to \mathcal{M}_{n,n}(\mathcal{H}(\Omega))$$

($z_0 \rightsquigarrow z$ is the path of integration previously introduced) s.t.

$$\mathcal{M}(1_{X^*}) = \mathtt{Id}_n \quad \text{and} \quad \mathcal{M}(x_{i_1} \cdots x_{i_k}) = M_{i_1} \ldots M_{i_k},$$

$$\alpha_{z_0}^{z}(1_{X^*}) = 1_{\mathcal{H}(\Omega)} \quad \text{and} \quad \alpha_{z_0}^{z}(x_{i_1} \cdots x_{i_k}) = \int_{z_0}^{z} \frac{dz_1}{z_1 - s_{i_1}} \ldots \int_{z_0}^{z_{k-1}} \frac{dz_k}{z_k - s_{i_k}}.$$

Then [3] $y(z) = \lambda U(z_0; z)\eta$ with

$$U(z_0; z) = \sum_{w \in X^*} \mathcal{M}(w) \alpha_{z_0}^{z}(w) = (\mathcal{M} \otimes \alpha_{z_0}) \sum_{w \in X^*} w \otimes w.$$

3. Subject to convergence.

Examples of linear dynamical systems

Example 2 (Hypergeometric equation)

Let t_0, t_1, t_2 be parameters and

$$z(1-z)\ddot{y}(z) + [t_2 - (t_0 + t_1 + 1)z]\dot{y}(z) - t_0 t_1 y(z) = 0.$$

Let $q_1(z) = -y(z)$ and $q_2(z) = (1-z)\dot{y}(z)$. Hence, one has

$$y(z) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} q_1(z) \\ q_2(z) \end{pmatrix}$$

and

$$\begin{aligned} \begin{pmatrix} \dot{q}_1(z) \\ \dot{q}_2(z) \end{pmatrix} &= \left(\frac{M_0}{z} + \frac{M_1}{1-z} \right) \begin{pmatrix} q_1(z) \\ q_2(z) \end{pmatrix} \\ &= (u_0(z)M_0 + u_1(z)M_1) \begin{pmatrix} q_1(z) \\ q_2(z) \end{pmatrix}, \end{aligned}$$

where $u_0(z) = z^{-1}, u_1(z) = (1-z)^{-1}$ and

$$M_0 = -\begin{pmatrix} 0 & 0 \\ t_0 t_1 & t_2 \end{pmatrix} \quad \text{and} \quad M_1 = -\begin{pmatrix} 0 & 1 \\ 0 & t_2 - t_0 - t_1 \end{pmatrix}.$$

Nonlinear differential equations

$$(NED) \quad \begin{cases} \partial q(z) &= \left(\sum_{i=0}^{m} T_i(q) u_i(z)\right)(q), \\ q(z_0) &= q_0, \\ y(z) &= f(q(z)), \end{cases}$$

where

- $u_i \in (\mathbf{k}, \partial)$,
- the state $q = (q_1, \ldots, q_n)$ belongs the complex analytic manifold Q of dimension n and q_0 is the initial state,
- the observation $f \in \mathcal{O}$, with $\mathcal{O}$ the ring of analytic functions over Q,
- for $i = 0..1$, $T_i = (T_i^1(q)\partial/\partial q_1 + \cdots + T_i^m(q)\partial/\partial q_m)$ is an analytic vector field over $Q,$ with $T_i^j(q) \in \mathcal{O},$ for $j = 1, \ldots, n$.

With X and $\alpha_{z_0}^z$ given as previously, let the morphism τ be defined by $\tau(1_{X^*}) = \mathtt{Id}$ and $\tau(x_{i_1} \cdots x_{i_k}) = T_{i_1} \ldots T_{i_k}$. Then [4] $y(z) = \mathcal{T} \circ f_{|_{q_0}}$ with

$$\mathcal{T} = \sum_{w \in X^*} \tau(w) \alpha_{z_0}^z(w) = (\tau \otimes \alpha_{z_0}) \sum_{w \in X^*} w \otimes w.$$

4. Subject to convergence.

Examples of nonlinear dynamical systems (1/2)

Example 3 (Harmonic oscillator)

Let k_1, k_2 be parameters and $\partial^2 y(z) + k_1 y(z) + k_2 y^2(z) = u_1(z)$ which can be represented by the following state equations (with $n = 1$)

$$\begin{aligned} y(z) &= q(z), \\ \partial q(z) &= A_0(q)u_0(z) + A_1(q)u_1(z), \end{aligned}$$

$$\text{where} \quad A_0 = -(k_1 q + k_2 q^2)\frac{\partial}{\partial q} \quad \text{and} \quad A_1 = \frac{\partial}{\partial q}.$$

Example 4 (Duffing equation)

Let a, b, c be parameters and $\partial^2 y(z) + a\partial y(z) + by(z) + cy^3(z) = u_1(z)$ which can be represented by the following state equations (with $n = 2$)

$$\begin{aligned} y(z) &= q_1(z), \\ \begin{pmatrix} \partial q_1(z) \\ \partial q_2(z) \end{pmatrix} &= \begin{pmatrix} q_2 \\ -(aq_2 + b^2 q_1 + cq_1^3) \end{pmatrix} u_0(z) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_1(z) \\ &= A_0(q)u_0(z) + A_1(q)u_1(z), \end{aligned}$$

$$\text{where} \quad A_0 = -(aq_2 + b^2 q_1 + cq_1^3)\frac{\partial}{\partial q_2} + q_2 \frac{\partial}{\partial q_1} \quad \text{and} \quad A_1 = \frac{\partial}{\partial q_2}.$$

Examples of nonlinear dynamical systems (2/2)

Example 5 (Van der Pol oscillator)

Let γ, g be parameters and

$$\partial^2 x(z) - \gamma[1 + x(z)^2]\partial x(z) + x(z) = g\cos(\omega z)$$

which can be tranformed into (with C is some constant of integration)

$$\partial x(z) = \gamma[1 + x(z)^2/3]x(z) - \int_{z_0}^{z} x(s)ds + \frac{g}{\omega}\sin(\omega z) + C.$$

Supposing $x = \partial y$ and $u_1(z) = g\sin(\omega z)/\omega + C$, it leads then to

$$\partial^2 y(z) = \gamma[\partial y(z) + (\partial y(z))^3/3] + y(z) + u_1(z)$$

which can be represented by the following state equations (with $n = 2$)

$$\begin{aligned}
y(z) &= q_1(z), \\
\begin{pmatrix} \partial q_1(z) \\ \partial q_2(z) \end{pmatrix} &= \begin{pmatrix} q_2 \\ \gamma(q_2 + q_2^3/3) + q_1 \end{pmatrix} u_0(z) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_1(z) \\
&= A_0(q)u_0(z) + A_1(q)u_1(z),
\end{aligned}$$

where $A_0 = [\gamma(q_2 + q_2^3/3) + q_1]\dfrac{\partial}{\partial q_2} + q_2\dfrac{\partial}{\partial q_1}$ and $A_1 = \dfrac{\partial}{\partial q_2}$.

DUAL LAWS AND REPRESENTATIVE SERIES

Dual law in bialgebra

Startting with a $\mathbf{k}-$**AAU** ($\mathbf{k}$ is a ring) $\mathcal{A}$. Dualizing $\mu : \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \to \mathcal{A}$, we get the transpose ${}^t\mu : \mathcal{A}^\vee \to (\mathcal{A} \otimes_{\mathbf{k}} \mathcal{A})^\vee$ so that we do not get a co-multiplication in general.

- Remark that when $\mathbf{k}$ is a field, the following arrow is into (due to the fact that $\mathcal{A}^\vee \otimes_{\mathbf{k}} \mathcal{A}^\vee$ is torsionfree)
$$\Phi : \mathcal{A}^\vee \otimes_{\mathbf{k}} \mathcal{A}^\vee \to (\mathcal{A} \otimes_{\mathbf{k}} \mathcal{A})^\vee.$$
- One restricts the codomain of ${}^t\mu$ to $\mathcal{A}^\vee \otimes_{\mathbf{k}} \mathcal{A}^\vee$ and then the domain to $({}^t\mu)^{-1}\Phi(\mathcal{A}^\vee \otimes_{\mathbf{k}} \mathcal{A}^\vee) =: \mathcal{A}^\circ$.

$$\begin{tikzcd}
\mathcal{A}^\vee \arrow[r, "{}^t\mu"] & (\mathcal{A} \otimes_{\mathbf{k}} \mathcal{A})^\vee \\
\mathcal{A}^\circ \arrow[u, hook, "can"] \arrow[r, "\Delta_\mu"] & \mathcal{A}^\vee \otimes_{\mathbf{k}} \mathcal{A}^\vee \arrow[u, hook, "\Phi"'] \\
\mathcal{A}^{\circ\circ} \arrow[u, hook, "can"] \arrow[r, "\Delta_\mu"] & \mathcal{A}^\circ \otimes_{\mathbf{k}} \mathcal{A}^\circ \arrow[u, hook, "j\otimes j"']
\end{tikzcd}$$

The descent can stop at first step for a field $\mathbf{k}$ and then $\mathcal{A}^{\circ\circ} = \mathcal{A}^\circ$. The coalgebra $(\mathcal{A}^\circ, \Delta_\mu)$ is called the Sweedler's dual of $(\mathcal{A}, \mu)$.

Case of algebras noncommutative series

- Denoting the (ordered) alphabets $Y := \{y_k\}_{k\geq 1}$ (with $y_1 \succ y_2 \succ \ldots$) or $X := \{x_0, x_1\}$ (with $x_1 \succ x_0$) by $\mathcal{X}$, we use the correspondence among words of the free monoid $(\mathcal{X}^*, \mathtt{conc}, 1_{\mathcal{X}^*})$:
$$(s_1, \ldots, s_r) \in \mathbb{N}_+^r \leftrightarrow y_{s_1} \ldots y_{s_r} \in Y^* \underset{\pi_Y}{\overset{\pi_X}{\rightleftharpoons}} x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \in X^*x_1.$$
- Let $\mathcal{L}yn\mathcal{X}$ denote the set of Lyndon words generated by $\mathcal{X}$.
- Let $(\mathcal{L}ie_A\langle\!\langle\mathcal{X}\rangle\!\rangle, [.])$ and $(A\langle\!\langle\mathcal{X}\rangle\!\rangle, \mathtt{conc})$ (resp. $\mathcal{L}ie_A\langle\mathcal{X}\rangle, [.])$ and $(A\langle\mathcal{X}\rangle, \mathtt{conc})$) denote the algebras of (Lie) series (resp. polynomials) with coefficients in the ring A, over $\mathcal{X}$.
- $\mathcal{H}_{\sqcup\!\sqcup}(\mathcal{X}) := (A\langle\mathcal{X}\rangle, \mathtt{conc}, 1_{\mathcal{X}^*}, \Delta_{\sqcup\!\sqcup}, \mathrm{e})$ and $\mathcal{H}_{⧢}(Y) := (A\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{⧢}, \mathrm{e})$ with [5]
$$\forall x \in \mathcal{X}, \quad \Delta_{\sqcup\!\sqcup}\, x = x \otimes 1_{\mathcal{X}^*} + 1_{\mathcal{X}^*} \otimes x,$$
$$\forall y_i \in Y, \quad \Delta_{⧢}\, y_i = y_i \otimes 1_{Y^*} + 1_{Y^*} \otimes y_i + \sum_{k+l=i} y_k \otimes y_l.$$
- The dual law associated to conc is defined by
$$\forall w \in \mathcal{X}^*, \quad \Delta_{\mathtt{conc}}(w) = \sum_{u,v\in\mathcal{X}^*, uv=w} u \otimes v.$$

5. Or equivalently, for $x, y \in \mathcal{X}, y_i, y_j \in Y$ and $u, v \in \mathcal{X}^*$ (resp. Y^*), $u \sqcup\!\sqcup 1_{\mathcal{X}^*} = 1_{\mathcal{X}^*} \sqcup\!\sqcup u = u$ and $xu \sqcup\!\sqcup yv = x(u \sqcup\!\sqcup yv) + y(xu \sqcup\!\sqcup v)$, $u ⧢ 1_{Y^*} = 1_{Y^*} ⧢ u = u$ and $x_i u ⧢ y_j v = y_i(u ⧢ y_j v) + y_j(y_i u ⧢ v) + y_{i+j}(u ⧢ v)$.

Dualizable laws in conc-shuffle bialgebras (1/2)

We can exploit the basis of words as follows

1. Any bilinear law (shuffle, stuffle or any) $\mu : A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle \to A\langle\mathcal{X}\rangle$ can be decribed through its structure constants wrt to the basis of words, *i.e.* for $u, v, w \in \mathcal{X}^*$, $\Gamma_{u,v}^w := \langle \mu(u \otimes v) | w \rangle$ so that
$$\mu(u \otimes v) = \sum_{w \in \mathcal{X}^*} \Gamma_{u,v}^w w.$$
2. In the case when $\Gamma_{u,v}^w$ is locally finite in w, we say that the given law is dualizable, the arrow ${}^t\mu$ restricts nicely to $A\langle\mathcal{X}\rangle \hookrightarrow A\langle\langle\mathcal{X}\rangle\rangle$ and one can define on the polynomials a comultiplication by
$$\Delta_\mu(w) := \sum_{u,v \in \mathcal{X}^*} \Gamma_{u,v}^w u \otimes v.$$
3. When the law μ is dualizable, we have

$$\begin{tikzcd} A\langle\langle\mathcal{X}\rangle\rangle \arrow[r, "{}^t\mu"] & A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle \\ A\langle\mathcal{X}\rangle \arrow[u, hook, "can"] \arrow[r, "\Delta_\mu"] & A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle \arrow[u, hook, "\Phi|_{A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle}"'] \end{tikzcd}$$

The arrow Δ_μ is unique to be able to close the rectangle and $\Delta_\mu(P)$ is defined as above.

Dualizable laws in conc-shuffle bialgebras (2/2)

4. Proof that the arrow $A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle \longrightarrow A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle$ is into :
   Let $T = \sum_{i=1}^n P_i \otimes_A Q_i$ such that $\Phi(T) = 0$. Rewriting T as a finitely supported sum $T = \sum_{u,v\in\mathcal{X}^*} c_{u,v} u \otimes v$ (this is indeed the iso between $A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle$ and $A[\mathcal{X}^* \times \mathcal{X}^*]$), $\Phi(T)$ is by definition of Φ the double series (here a polynomial) s.t. $\langle\Phi(T)|u \otimes v\rangle = c_{u,v}$. If $\Phi(T) = 0$, then for all $(u, v) \in \mathcal{X}^* \times \mathcal{X}^*$, $c_{u,v} = 0$ entailing $T = 0$.

We extend by linearity and infinite sums, for $S \in A\langle\langle Y\rangle\rangle$ (resp. $A\langle\langle\mathcal{X}\rangle\rangle$), by

$$\begin{aligned}
\Delta_{\uplus} S &= \sum_{w\in Y^*} \langle S|w\rangle \Delta_{\uplus} w &\in A\langle\langle Y^* \otimes Y^*\rangle\rangle,\\
\Delta_{\mathtt{conc}} S &= \sum_{w\in \mathcal{X}^*} \langle S|w\rangle \Delta_{\mathtt{conc}} w &\in A\langle\langle \mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle,\\
\Delta_{\sqcup\!\sqcup} S &= \sum_{w\in \mathcal{X}^*} \langle S|w\rangle \Delta_{\sqcup\!\sqcup} w &\in A\langle\langle \mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle.
\end{aligned}$$

$A\langle\langle\mathcal{X}\rangle\rangle \otimes A\langle\langle\mathcal{X}\rangle\rangle$ embeds injectively in [6] $A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle \cong [A\langle\langle\mathcal{X}\rangle\rangle]\langle\langle\mathcal{X}\rangle\rangle$.

6. $A\langle\langle\mathcal{X}\rangle\rangle \otimes A\langle\langle\mathcal{X}\rangle\rangle$ contains the elements of the form $\sum_{i\in I \text{ finite}} G_i \otimes D_i$, for $(G_i, D_i) \in A\langle\langle\mathcal{X}\rangle\rangle \times A\langle\langle\mathcal{X}\rangle\rangle$. But since elements of $M \otimes N$ are finite combination of $m_i \otimes n_i, m_i \in M, n_i \in N$ then $\sum_{i\geq 0} u^i \otimes v^i$ belongs to $A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle$ and does not belong to $A\langle\langle\mathcal{X}\rangle\rangle \otimes A\langle\langle\mathcal{X}\rangle\rangle$, for $u, v \in \mathcal{X}^{\geq 1}$.

Extended Ree's theorem

Let $S \in A\langle\langle Y \rangle\rangle$ (resp. $A\langle\langle \mathcal{X} \rangle\rangle$), A is a commutative ring containing $\mathbb{Q}$. The series S is said to be

1. a ⧢ (resp. $\mathtt{conc}$, ⧢)-character iff, for any $w, v \in Y^*$ (resp. $\mathcal{X}^*$), $\langle S|w\rangle\langle S|v\rangle = \langle S|w \,\text{⧢}\, v\rangle$ (resp. $\langle S|wv\rangle, \langle S|w \,\text{⧢}\, v\rangle$) and $\langle S|1\rangle = 1$.
2. an infinitesimal ⧢ (resp. $\mathtt{conc}$, ⧢)-character iff, for any $w, v \in Y^*$ (resp. $\mathcal{X}^*$), $\langle S|w \,\text{⧢}\, v\rangle = \langle S|w\rangle\langle v|1_{Y^*}\rangle + \langle w|1_{Y^*}\rangle\langle S|v\rangle$ (resp. $\langle S|wv\rangle = \langle S|w\rangle\langle v|1_{\mathcal{X}^*}\rangle + \langle w|1_{\mathcal{X}^*}\rangle\langle S|v\rangle$, $\langle S|w \,\text{⧢}\, v\rangle = \langle S|w\rangle\langle v|1_{\mathcal{X}^*}\rangle + \langle w|1_{\mathcal{X}^*}\rangle\langle S|v\rangle$).
3. a group-like series iff $\langle S|1_{\mathcal{X}^*}\rangle = 1$ and $\Delta_{\text{⧢}} S = \Phi(S \otimes S)$ (resp. $\Delta_{\mathtt{conc}} S = \Phi(S \otimes S), \Delta_{\text{⧢}} S = \Phi(S \otimes S)$).
4. a primitive series iff $\Delta_{\text{⧢}} S = 1_{Y^*} \otimes S + S \otimes 1_{Y^*}$ (resp. $\Delta_{\mathtt{conc}} S = 1_{\mathcal{X}^*} \otimes S + S \otimes 1_{\mathcal{X}^*}, \Delta_{\text{⧢}} S = 1_{\mathcal{X}^*} \otimes S + S \otimes 1_{\mathcal{X}^*}$).

Then the following assertions are equivalent

1. S is a ⧢ (resp. $\mathtt{conc}$ and ⧢)-character.
2. $\log S$ an infinitesimal ⧢ (resp. $\mathtt{conc}$ and ⧢)-character.
3. S is group-like, for $\Delta_{\text{⧢}}$ (resp. $\Delta_{\mathtt{conc}}$ and $\Delta_{\text{⧢}}$).
4. $\log S$ is primitive, for $\Delta_{\text{⧢}}$ (resp. $\Delta_{\mathtt{conc}}$ and $\Delta_{\text{⧢}}$).

Extension by continuity (infinite sums)

Now, suppose that the ring A (containing $\mathbb{Q}$) is a field $\mathbf{k}$. Then

$$\Delta_{\sqcup\!\sqcup} : \mathbf{k}\langle\mathcal{X}\rangle \to \mathbf{k}\langle\mathcal{X}\rangle \otimes \mathbf{k}\langle\mathcal{X}\rangle \text{ and } \Delta_{\uplus\!\!\sqcup} : \mathbf{k}\langle Y\rangle \to \mathbf{k}\langle Y\rangle \otimes \mathbf{k}\langle Y\rangle$$

are graded for the multidegree. Then $\Delta_{\uplus\!\!\sqcup}$ is graded for the length. Their extension to the completions (*i.e.* $\mathbf{k}\langle\!\langle\mathcal{X}\rangle\!\rangle$ and $\mathbf{k}\langle\!\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\!\rangle$) are continuous and then, when exist, commute with infinite sums. Hence[7, 8],

$$\forall c \in \mathbf{k}, \quad \Delta_{\sqcup\!\sqcup}(cx)^* = \sum_{n\geq 0} c^n \Delta_{\sqcup\!\sqcup} x^n = \sum_{n\geq 0} c^n \sum_{j=0}^{n} \binom{n}{j} x^j \otimes x^{n-j}.$$

For $c \in \mathbb{N}_{\geq 2}$ which is neither a field nor a ring (containing $\mathbb{Q}$), we also get

$$(cx)^* = (c-1)^{-1} \sum_{a,b\in\mathbb{N}_{\geq 1}, a+b=c} (ax)^* \sqcup\!\sqcup (bx)^* \quad \in \mathbb{N}_{\geq 2}\langle\!\langle\mathcal{X}\rangle\!\rangle,$$

$$\Delta_{\sqcup\!\sqcup}(cx)^* \neq (c-1)^{-1} \sum_{a,b\in\mathbb{N}_{\geq 1}, a+b=c} (ax)^* \otimes (bx)^* \quad \in \mathbb{Q}\langle\!\langle\mathcal{X}\rangle\!\rangle \otimes \mathbb{Q}\langle\!\langle\mathcal{X}\rangle\!\rangle,$$

because

$$\langle \mathrm{LHS} | x \otimes 1_{\mathcal{X}^*}\rangle = c \quad \text{and} \quad \langle \mathrm{RHS} | x \otimes 1_{\mathcal{X}^*}\rangle = (c-1)^{-1} \sum_{a=1}^{c-1} a = \frac{c}{2}.$$

For $c \in \mathbb{Z}$ (or even $\mathbb{Q}, \mathbb{R}, \mathbb{C}$), the such decomposition is not finite.

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7. For $S \in A\langle\!\langle\mathcal{X}\rangle\!\rangle$ s.t. $\langle S | 1_{\mathcal{X}^*}\rangle = 0$, $S^* = \sum_{n\geq 0} S^n$ is called Kleene star of S.
8. $\Delta_{\sqcup\!\sqcup} x^n = (\Delta_{\sqcup\!\sqcup} x)^n = (1_{\mathcal{X}^*} \otimes x + x \otimes 1_{\mathcal{X}^*})^n = \sum_{j=0}^{n} \binom{n}{j} x^j \otimes x^{n-j}$.

Case of rational series and of $\Delta_{\mathtt{conc}}$

$A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ denotes the algebraic closure by[9] $\{\mathtt{conc}, +, *\}$ of $\widehat{A.\mathcal{X}}$ in $A\langle\langle\mathcal{X}\rangle\rangle$.

$$\begin{tikzcd}
A\langle\langle\mathcal{X}\rangle\rangle \arrow[rr, "{}^t\mathtt{conc}"] & & A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle \\
A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \arrow[u, hook, "can"] \arrow[rr, dashed] & & A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \otimes_A A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \arrow[u, "\Phi|_{A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \otimes_A A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle}"']
\end{tikzcd}$$

The dashed arrow may not exist in general, but for any $R \in A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ admitting (λ, μ, η) as linear representation of dimension n, we can get

$$ {}^t\mathtt{conc}(R) = \Phi(\textstyle\sum_{i=1}^n G_i \otimes D_i).$$

Indeed, since $\langle R|xy\rangle = \lambda\mu(xy)\eta = \lambda\mu(x)\mu(y)\eta$ $(x, y \in \mathcal{X})$ then, letting e_i is the vector such that ${}^t e_i = \begin{pmatrix} 0 & \ldots & 0 & 1 & 0 & \ldots & 0\end{pmatrix}$, one has

$$\langle R|xy\rangle = \sum_{i=1}^n \lambda\mu(x)e_i {}^t e_i\mu(y)\eta = \sum_{i=1}^n \langle G_i|x\rangle\langle D_i|y\rangle = \sum_{i=1}^n \langle G_i \otimes D_i|x \otimes y\rangle.$$

G_i (resp. D_i) admits then (λ, μ, e_i) (resp. $({}^t e_i, \mu, \eta)$) as linear representation. If $A = \mathbf{k}$ being a field then, due to the injectivity of Φ, all expressions of the type $\sum_{i=1}^n G_i \otimes D_i$, of course, coincide. Hence, the dashed arrow (a restriction of $\Delta_{\mathtt{conc}}$) in the above diagram is well-defined.

9. $A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ is closed under $\sqcup\!\sqcup$. $A^{\mathrm{rat}}\langle\langle Y\rangle\rangle$ is also closed under $\uplus$.

Representative series and Sweedler's dual

Theorem 6 (representative series)

Let $S \in A\langle\langle\mathcal{X}\rangle\rangle$. The following assertions are equivalent

1. *The series S belongs to $A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$.*
2. *There exists a linear representation (ν, μ, η), of rank n, for S with $\nu \in M_{1,n}(A), \eta \in M_{n,1}(A)$ and a morphism of monoids $\mu : \mathcal{X}^* \to M_{n,n}(A)$ s.t., for any $w \in \mathcal{X}^*$, $\langle S|w\rangle = \nu\mu(w)\eta$.*
3. *The shifts[10] $\{S \triangleleft w\}_{w\in\mathcal{X}^*}$ (resp. $\{w \triangleright S\}_{w\in\mathcal{X}^*}$) lie within a finitely generated shift-invariant A-module.*

Moreover, if A is a field $\mathbf{k}$, the previous assertions are equivalent to

4. *There exist $(G_i, D_i)_{i\in F\, finite}$ s.t. $\Delta_{\mathrm{conc}}(S) = \sum_{i\in F\, finite} G_i \otimes D_i$.*

Hence, $\mathcal{H}^{\circ}_{\sqcup\!\sqcup}(\mathcal{X}) = (\mathbf{k}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, \sqcup\!\sqcup, 1_{\mathcal{X}^*}, \Delta_{\mathrm{conc}}, \mathrm{e})$ and $\mathcal{H}^{\circ}_{\uplus}(Y) = (\mathbf{k}^{\mathrm{rat}}\langle\langle Y\rangle\rangle, \uplus, 1_{\mathcal{X}^*}, \Delta_{\mathrm{conc}}, \mathrm{e})$.

Now, let $A_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$ (resp. $A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$) be the set of exchangeable[11] series (resp. series admitting a linear representation with commuting matrices).

10. The *left* (resp. *right*) shift of S by P is $P \triangleright S$ (resp. $S \triangleleft P$) defined by, for $w \in \mathcal{X}^*$, $\langle P \triangleright S|w\rangle = \langle S|wP\rangle$ (resp. $\langle S \triangleleft P|w\rangle = \langle S|Pw\rangle$).

11. *i.e.* if $S \in A_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$ then $(\forall u, v \in \mathcal{X}^*)((\forall x \in \mathcal{X})(|u|_x = |v|_x) \Rightarrow \langle S|u\rangle = \langle S|v\rangle)$.

Kleene stars of the plane and conc-characters

For any $S \in A\langle\langle\mathcal{X}\rangle\rangle$, let ∇S denotes $S - 1_{\mathcal{X}^*}$.

Theorem 7 (rational exchangeable series)

1. $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \subset A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \cap A_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$. *If A is a field then the equality holds and* $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle X\rangle\rangle = A^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \sqcup\!\sqcup A^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$ *and, for the algebra of series over subalphabets* $A_{\mathrm{fin}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle := \cup_{F \subset_{finite} Y} A^{\mathrm{rat}}\langle\langle F\rangle\rangle$, *we get*[12] $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle \cap A_{\mathrm{fin}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle = \cup_{k\geq 0} A^{\mathrm{rat}}\langle\langle y_1\rangle\rangle \sqcup\!\sqcup \ldots \sqcup\!\sqcup A^{\mathrm{rat}}\langle\langle y_k\rangle\rangle \subsetneq A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle$.
2. $\forall x \in \mathcal{X}, A^{\mathrm{rat}}\langle\langle x\rangle\rangle = \{P(1-xQ)^{-1}\}_{P,Q\in A[x]}$. *If $\mathbf{k}$ is an algebraically closed field then* $\mathbf{k}^{\mathrm{rat}}\langle\langle x\rangle\rangle = \mathrm{span}_{\mathbf{k}}\{(ax)^* \sqcup\!\sqcup \mathbf{k}\langle x\rangle | a \in K\}$.
3. *If A is a $\mathbb{Q}$-algebra without zero divisors, $\{x^*\}_{x\in\mathcal{X}}$ (resp. $\{y^*\}_{y\in Y}$) are conc-character and algebraically independent over $(A\langle\mathcal{X}\rangle, \sqcup\!\sqcup)$ (resp. $(A\langle Y\rangle, \uplus)$) within $(A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, \sqcup\!\sqcup)$ (resp. $(A^{\mathrm{rat}}\langle\langle Y\rangle\rangle, \sqcup\!\sqcup)$).*
4. *Let $S \in A\langle\langle\mathcal{X}\rangle\rangle$. If $A = \mathbf{k}$, a field, then t.f.a.e.*
   a) *S is groupe-like, for Δ_{conc}.*
   b) *There exists* $M := \sum_{x\in\mathcal{X}} c_x x \in \widehat{\mathbf{k}.\mathcal{X}}$ *s.t.* $S = M^*$.
   c) *There exists* $M := \sum_{x\in\mathcal{X}} c_x x \in \widehat{\mathbf{k}.\mathcal{X}}$ *s.t.* $\nabla S = MS = SM$.

---

12. The following identity lives in $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle$ but not in $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle \cap A_{\mathrm{fin}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle$,
$(y_1 + \ldots)^* = \lim_{k\to+\infty}(y_1 + \ldots + y_k)^* = \lim_{k\to+\infty} y_1^* \sqcup\!\sqcup \ldots \sqcup\!\sqcup y_k^* = \sqcup\!\sqcup_{k\geq 1} y_k$.

CONTINUITY OVER CHEN SERIES

Continuity, indiscernability and growth condition

For $i = 0, 2$, let $(\mathbf{k}_i, \|.\|_i)$ be a semi-normed space and $g_i \in \mathbb{Z}$.

Definition 8

1. Let $\mathcal{C}l$ be a class of $\mathbf{k}_1\langle\langle\mathcal{X}\rangle\rangle$. Let $S \in \mathbf{k}_2\langle\langle\mathcal{X}\rangle\rangle$ and it is said to be
   a) *continuous* over $\mathcal{C}l$ if, for $\Phi \in \mathcal{C}l$, the following sum is convergent
   $$\sum_{w\in\mathcal{X}^*} \|\langle S|w\rangle\|_2 \|\langle\Phi|w\rangle\|_1.$$
   We will denote $\langle S\|\Phi\rangle$ the sum $\sum_{w\in\mathcal{X}^*}\langle S|w\rangle\langle\Phi|w\rangle$ and $\mathbf{k}_2\langle\langle\mathcal{X}\rangle\rangle^{\mathrm{cont}}$ the set of continuous power series over $\mathcal{C}l$.
   b) *indiscernable* over $\mathcal{C}l$ iff, for any $\Phi \in \mathcal{C}l$, $\langle S\|\Phi\rangle = 0$.
2. Let χ_1 and χ_2 be real positive functions over $\mathcal{X}^*$. Let $S \in \mathbf{k}_1\langle\langle\mathcal{X}\rangle\rangle$.
   a) S satisfies the χ_1-*growth condition* of order g_1 if it satisfies
   $$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in \mathcal{X}^{\geq n}, \quad \|\langle S|w\rangle\|_1 \leq K\chi_1(w)\, |w|!^{g_1}.$$
   We denote by $\mathbf{k}_1^{(\chi_1,g_1)}\langle\langle\mathcal{X}\rangle\rangle$ the set of formal power series in $\mathbf{k}_1\langle\langle\mathcal{X}\rangle\rangle$ satisfying the χ_1-growth condition of order g_1.
   b) If S is continuous over $\mathbf{k}_2^{(\chi_2,g_2)}\langle\langle\mathcal{X}\rangle\rangle$ then it will be said to be (χ_2, g_2)-*continuous*. The set of formal power series which are (χ_2, g_2)-continuous is denoted by $\mathbf{k}_2^{(\chi_2,g_2)}\langle\langle\mathcal{X}\rangle\rangle^{\mathrm{cont}}$.

Convergence condition

Proposition 1

Let χ_1 and χ_2 be real positive functions over $\mathcal{X}^*$.
Let g_1 and $g_2 \in \mathbb{Z}$ such that $g_1 + g_2 \leq 0$.

1. *Let $\mathbf{k}_1^{(\chi_1,g_1)}\langle\langle\mathcal{X}\rangle\rangle$ and let $P \in \mathbf{k}_1\langle\mathcal{X}\rangle$.*
   The right residual of S by P belongs to $\mathbf{k}_1^{(\chi_1,g_1)}\langle\langle\mathcal{X}\rangle\rangle$.
2. *Let $R \in \mathbf{k}_2^{(\chi_2,g_2)}\langle\langle\mathcal{X}\rangle\rangle$ and let $Q \in \mathbf{k}_2\langle\mathcal{X}\rangle$.*
   The concatenation QR belongs to $\mathbf{k}_2^{(\chi_2,g_2)}\langle\langle\mathcal{X}\rangle\rangle$.
3. *χ_1, χ_2 are morphisms over $\mathcal{X}^*$ satisfying $\sum_{x\in X} \chi_1(x)\chi_2(x) < 1$.*
   If $F_1 \in \mathbf{k}_1^{(\chi_1,g_1)}\langle\langle\mathcal{X}\rangle\rangle$ (resp. $F_2 \in \mathbf{k}_2^{(\chi_2,g_2)}\langle\langle\mathcal{X}\rangle\rangle$) then F_1 (resp. F_2) is continuous over $\mathbf{k}_2^{(\chi_2,g_2)}\langle\langle\mathcal{X}\rangle\rangle$ (resp. $\mathbf{k}_1^{(\chi_1,g_1)}\langle\langle\mathcal{X}\rangle\rangle$).

Proposition 2

Let $\mathcal{C}l \subset \mathbf{k}_1\langle\langle\mathcal{X}\rangle\rangle$ be a monoid containing $\{e^{tx}\}_{x\in\mathcal{X}}^{t\in\mathbf{k}_1}$. Let $S \in \mathbf{k}_2\langle\langle\mathcal{X}\rangle\rangle^{cont}$.

1. *If S is indiscernable over $\mathcal{C}l$ then for any $x \in \mathcal{X}$, $x \triangleleft S$ and $S \triangleright x$ belong to $\mathbf{k}_2\langle\langle\mathcal{X}\rangle\rangle^{cont}$ and they are indiscernable over $\mathcal{C}l$.*
2. *S is indiscernable over $\mathcal{C}l$ iff $S = 0$.*

Iterated integrals over $\omega_i(z) = u_{x_i}(z)dz$ and along $z_0 \rightsquigarrow z$

Recall that Ω is a simply connected domain with $1_{\mathcal{H}(\Omega)}$ as neutral element, $\mathcal{A} := \mathcal{H}(\Omega)$ and $\mathcal{C}_0$ is a differential subring of $\mathcal{A}$ ($\partial(\mathcal{C}_0) \subset \mathcal{C}_0$).
$\mathbb{C}\{\{(g_i)_{i\in I}\}\}$ denotes the differential subalgebra of $\mathcal{A}$ generated by $(g_i)_{i\in I}$, *i.e.* the $\mathbb{C}$-algebra generated by g_i's and their derivatives

$\{u_x\}_{x\in\mathcal{X}}$: elements in $\mathcal{C}_0 \cap \mathcal{A}^{-1}$ in correspondence with $\{\theta_x\}_{x\in\mathcal{X}}$ ($\theta_x = u_x^{-1}\partial$).
Let Θ be defined by $\Theta(w) = \theta_x\Theta(u)$, for $w = xu \in \mathcal{X}\mathcal{X}^*$, and $\Theta(1_{\mathcal{X}^*}) = \mathrm{Id}$.

The iterated integral over $\omega_i(z) = u_{x_i}(z)dz$ and along $z_0 \rightsquigarrow z$ is defined by

$$\begin{aligned}
\alpha_{z_0}^z(1_{\mathcal{X}^*}) &= 1_\Omega, \\
\alpha_{z_0}^z(x_{i_1}\ldots x_{i_k}) &= \int_{z_0}^z \omega_{i_1}(z_1)\ldots\int_{z_0}^{z_{k-1}} \omega_{i_k}(z_k). \\
\partial\alpha_{z_0}^z(x_{i_1}\ldots x_{i_k}) &= u_{x_{i_1}}(z)\int_{z_0}^z \omega_{i_2}(z_2)\ldots\int_{z_0}^{z_{k-1}} \omega_{i_k}(z_k).
\end{aligned}$$

$$\forall w \in \mathcal{X}^*, \quad \Theta(\tilde{w})\alpha_{z_0}^z(w) = 1_\Omega.$$

$$\begin{aligned}
\mathrm{span}_{\mathbb{C}}\{\partial^l\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*,l\geq 0} &\subset \mathrm{span}_{\mathbb{C}\{\{(u_x)_{x\in\mathcal{X}}\}\}}\{\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*} \\
&\subset \mathrm{span}_{\mathbb{C}\{\{(u_x^{\pm 1})_{x\in\mathcal{X}}\}\}}\{\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*} \\
&\cong \mathbb{C}\{\{(u_x^{\pm 1})_{x\in\mathcal{X}}\}\} \otimes_{\mathbb{C}} \mathrm{span}_{\mathbb{C}}\{\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*}\,?
\end{aligned}$$

Examples of linear differential equation

Let us consider the following examples, with $\mathbf{k} = \mathbb{C}(z)$

$$(\partial - z)y = y' - zy = 0. \tag{1}$$

1. $e^{z^2/2}$ is solution of (1).
2. $ce^{z^2/2} = e^{z^2/2}e^{\log c}$ is an other solution ($c \in \mathbb{R} \setminus \{0\}$).
3. $\{e^{z^2/2}\}$ is a fundamental set of solutions of (1).
4. $\mathbf{k}\{e^{z^2/2}\}$ is a Picard-Vessiot extension related to (1).

For $\theta_0 = z\partial, \theta_1 = (1-z)\partial$, since $\partial\theta_1\theta_0 \in \mathbf{k}[\partial]$ then let us consider

$$(\partial\theta_1\theta_0)y = (z(1-z)\partial^3 + (2-3z)\partial^2 - 1)y \tag{2}$$

$$= z(1-z)y^{(3)} + (2-3z)y'' - y' = 0. \tag{3}$$

1. $(\partial\theta_1\theta_0)\,\mathrm{Li}_2 = 0$ meaning that Li_2 is solution of (3).
2. $c\,\mathrm{Li}_2 = \mathrm{Li}_2\, e^{\log c}$ is an other solution ($c \in \mathbb{R} \setminus \{0\}$) but it is not independent to Li_2.
3. $\{\mathrm{Li}_2, \log, 1_\Omega\}$ is a fundamental set of solutions of (3).
4. $\mathbf{k}\{\mathrm{Li}_2, \log, 1_\Omega\}$ is a Picard-Vessiot extension [13] related to (3).

13. $\mathbf{k}\{\mathrm{Li}_2(z)\} = \mathbf{k} \otimes \mathbb{C}[\mathrm{Li}_2(z), \log(1-z), \log(z)]$.

Chen series of $\{\omega_i\}_{i\geq 1}$ and along $z_0 \rightsquigarrow z$

Since iterated integrals satisfy the Chen's lemma (or *Friedrichs criterion*), *i.e.* $\alpha_{z_0}^z(u \sqcup\!\sqcup v) = \alpha_{z_0}^z(u)\alpha_{z_0}^z(v)$ $(u, v \in \mathcal{X}^*)$, then the Chen series is given by [14]

$$C_{z_0 \rightsquigarrow z} := \sum_{w \in \mathcal{X}^*} \alpha_{z_0}^z(w) w = (\alpha_{z_0}^z \otimes \mathtt{Id})\mathcal{D}_{\mathcal{X}} = \prod_{l \in \mathcal{L}yn\mathcal{X}}^{\searrow} e^{\alpha_{z_0}^z(S_l)P_l} \in \mathcal{H}(\Omega)\langle\langle \mathcal{X} \rangle\rangle.$$

Theorem 9

If $R \in \mathbb{C}^{\mathrm{rat}}\langle\langle X \rangle\rangle$ with minimal representation of dimension n then [15] *$y(z_0, z) = \alpha_{z_0}^z(R) = \langle R \| C_{z_0 \rightsquigarrow z} \rangle$ and there exists $l = 0, .., n-1$ s.t. $\{\partial^k y\}_{0 \leq k \leq l}$ is $\mathcal{C}_0$-linearly independent and $a_l, \ldots, a_1, a_0 \in \mathcal{C}_0$ s.t.*

$$(a_l \partial^l + a_{l-1}\partial^{l-1} + \ldots + a_1 \partial + a_0)y = 0.$$

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14. A is supposed contain $\mathbb{Q}$. On $\mathcal{H}_{\sqcup\!\sqcup}(\mathcal{X})$ and $\mathcal{H}_{\uplus}(Y)$, we also get

$$\mathcal{D}_{\mathcal{X}} := \sum_{w \in \mathcal{X}^*} w \otimes w = \sum_{w \in \mathcal{X}^*} S_w \otimes P_w = \prod_{l \in \mathcal{L}yn\mathcal{X}}^{\searrow} e^{S_l \otimes P_l},$$

$$\mathcal{D}_{Y} := \sum_{w \in Y^*} w \otimes w = \sum_{w \in Y^*} \Sigma_w \otimes \Pi_w = \prod_{l \in \mathcal{L}ynY}^{\searrow} e^{\Sigma_l \otimes \Pi_l},$$

where $\{P_l\}_{l \in \mathcal{L}yn\mathcal{X}}$ (resp. $\{\Pi_l\}_{l \in \mathcal{L}ynY}$) is a basis of Lie algebra of primitive elements and $\{S_l\}_{l \in \mathcal{L}yn\mathcal{X}}$ (resp. $\{\Sigma_l\}_{l \in \mathcal{L}ynY}$) is a transcendence basis of $(A\langle \mathcal{X} \rangle, \sqcup\!\sqcup, 1_{\mathcal{X}^*})$ (resp. $(A\langle Y \rangle, \uplus, 1_{Y^*})$).

15. Subject to convergence.

Chen series and differential equations

For any $n \geq 0$, one has[16] $\mathbf{d}^n C_{z_0 \rightsquigarrow z} = p_n C_{z_0 \rightsquigarrow z}$ with

$$p_n = \sum_{\mathrm{wgt}\,\mathbf{r}=n} \sum_{w \in \mathcal{X}^n} \prod_{i=1}^{\deg \mathbf{r}} \binom{\sum_{j=1}^{i} r_j + j - 1}{r_i} \tau_{\mathbf{r}}(w) \in \mathcal{C}_0\langle\mathcal{X}\rangle,$$

where, for any word $w = x_{i_1} \ldots x_{i_k} \in \mathcal{X}^*$ associated to the derivation multi-index $\mathbf{r} = (r_1, \ldots, r_k) \in \mathbb{N}^k$ of degree $\deg \mathbf{r} = |w|$ and of weight $\mathrm{wgt}\,\mathbf{r} = |w| + \sum_{i=1}^{k} r_i$, $\tau_{\mathbf{r}}(w) := \tau_{r_1}(x_{i_1}) \ldots \tau_{r_k}(x_{i_k})$.

Proposition 3

Let K be a compact on Ω. There is $c_K \in \mathbb{R}_{\geq 0}$ and a morphism M_K s.t.

$$\forall w \in \mathcal{X}^*, \qquad \|\langle C_{z_0 \rightsquigarrow z}|w\rangle\|_K \leq c_K M_K(w)\,|w|!^{-1}.$$

Let $R \in \mathbb{C}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ s.t. $\langle R \| C_{z_0 \rightsquigarrow z}\rangle$ exists and $\alpha_{z_0}^{z}(R) = \langle R \| C_{z_0 \rightsquigarrow z}\rangle$. Thus,

$$\forall y \in \mathcal{X}, \qquad \theta_y \alpha_{z_0}^{z}(R) = \sum_{x \in \mathcal{X}} u_x(z) u_y^{-1}(z) \alpha_{z_0}^{z}(R \triangleright x).$$

The following assertions are equivalent :

1. $\alpha_{z_0}^{z}(R)$ satisfies a ODE with coefficients in $(\mathcal{C}_0, \partial)$.
2. There is $p \in \mathcal{C}_0\langle\mathcal{X}\rangle$ s.t. $\langle R \| p C_{z_0 \rightsquigarrow z}\rangle = \langle R \triangleright p \| C_{z_0 \rightsquigarrow z}\rangle = 0$.

16. Considering $A = (\mathcal{H}(\Omega), \partial)$ as the differential ring of holomorphic functions on Ω, equipped 1_Ω as the neutral element, the differential ring $(\mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle, \mathbf{d})$ is defined, for any $S \in \mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle$, by $\mathbf{d}S = \sum_{w \in \mathcal{X}^*} (\partial\langle S|w\rangle) w \in \mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle$.

More about Chen series

Chen series $C_{z_0 \rightsquigarrow z}$ of $\{\omega_i\}_{i\geq 1}$ satisfies the following Freidrichs criterion

$$\forall u, v \in \mathcal{X}^*, \quad \langle C_{z_0 \rightsquigarrow z} | u \sqcup\!\sqcup\, v\rangle = \langle C_{z_0 \rightsquigarrow z} | u\rangle \langle C_{z_0 \rightsquigarrow z} | v\rangle.$$

On the other hand, for any u and $v \in \mathcal{X}^*$,

$$\begin{aligned} \langle C_{z_0 \rightsquigarrow z} | u\rangle \langle C_{z_0 \rightsquigarrow z} | v\rangle &= \langle C_{z_0 \rightsquigarrow z} \otimes C_{z_0 \rightsquigarrow z} | u \otimes v\rangle, \\ \langle C_{z_0 \rightsquigarrow z} | u \sqcup\!\sqcup\, v\rangle &= \langle \Delta_{\sqcup\!\sqcup}\, C_{z_0 \rightsquigarrow z} | u \otimes v\rangle. \end{aligned}$$

Hence, $\Delta_{\sqcup\!\sqcup}\, C_{z_0 \rightsquigarrow z} = C_{z_0 \rightsquigarrow z} \otimes C_{z_0 \rightsquigarrow z}$ and $\langle C_{z_0 \rightsquigarrow z} | 1_{\mathcal{X}^*}\rangle = 1$.

Note that $C_{z_0 \rightsquigarrow z}$ only depends on the homotopy class of $z_0 \rightsquigarrow z$ and the endpoints z_0, z. One has $C_{z_0 \rightsquigarrow z} C_{z_1 \rightsquigarrow z_0} = C_{z_1 \rightsquigarrow z}$. Or equivalently [17],

$$\forall w \in \mathcal{X}^*, \quad \langle C_{z_1 \rightsquigarrow z} | w\rangle = \sum_{u,v \in \mathcal{X}^*, uv = w} \langle C_{z_0 \rightsquigarrow z} | u\rangle \langle C_{z_1 \rightsquigarrow z_0} | v\rangle.$$

Note also that, for $g \in \mathcal{H}(\Omega)$, one has $C_{g(z_0) \rightsquigarrow g(z)} = g_* C_{z_0 \rightsquigarrow z}$, *i.e.* the Chen series of $\{g^*\omega_i\}_{i\geq 1}$ along the path $g^*(z_0 \rightsquigarrow z)$.

Example 10 (with $\omega_0(z) = z^{-1}dz$ and $\omega_1(z) = (1-z)^{-1}dz$)

$g(z)$	z	z^{-1}	$(z-1)z^{-1}$	$z(z-1)^{-1}$	$(1-z)^{-1}$	$1-z$
$g^*\omega_0$	ω_0	$-\omega_0$	$-\omega_1 - \omega_0$	$\omega_1 + \omega_0$	ω_1	$-\omega_1$
$g^*\omega_1$	ω_1	$\omega_1 + \omega_0$	$-\omega_0$	$-\omega_1$	$-\omega_1 - \omega_0$	$-\omega_0$

17. Although $\Delta_{\mathrm{conc}} w = \sum_{u,v \in \mathcal{X}^*, uv = w} u \otimes v$ but $\Delta_{\mathrm{conc}} C_{z_1 \rightsquigarrow z} \neq C_{z_0 \rightsquigarrow z} \otimes C_{z_1 \rightsquigarrow z_0}$.

NONCOMMUTATIVE PV THEORY
AND INDEPENDENCE VIA WORDS

Noncommutative differential equations

Considering $A = (\mathcal{H}(\Omega), \partial)$ as the differential ring of holomorphic functions on Ω, the differential ring $(\mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle, \mathbf{d})$ is defined, for any $S \in \mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle$, by

$$\mathbf{d}S = \sum_{w\in\mathcal{X}^*} (\partial\langle S|w\rangle)w \in \mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle.$$

The Chen series $C_{z_0\rightsquigarrow z}$ satisfies the following differential equation

$$(NCDE) \quad \mathbf{d}S = MS, \quad \text{with} \quad M = \sum_{x\in\mathcal{X}} u_x x.$$

$$\Delta_{\sqcup\!\sqcup} M = \sum_{x\in\mathcal{X}} u_x(1_{\mathcal{X}^*} \otimes x + x \otimes 1_{\mathcal{X}^*}) = 1_{\mathcal{X}^*} \otimes M + M \otimes 1_{\mathcal{X}^*}.$$

More generally, for any $k \geq 1$, $C_{z_0\rightsquigarrow z}$ satisfies $\mathbf{d}^k S = Q_k S$ with $Q_k \in \mathbb{C}\{\{(u_x^{\pm 1})_{x\in\mathcal{X}}\}\}\langle\mathcal{X}\rangle$ satisfying the recursion

$$Q_0 = 1 \text{ and } Q_k = Q_{k-1}M + \mathbf{d}Q_{k-1}.$$

Q_k can be computed explicitly by (summing over words $w = x_{i_1}\ldots x_{i_k}$ and derivation multiindices $\mathbf{r} = (r_1, \ldots, r_k)$ of degree $\deg \mathbf{r} = |w| = k$ and of weight $\mathtt{wgt}\ \mathbf{r} = k + r_1 + \ldots + r_k$)

$$Q_k = \sum_{\mathtt{wgt}\ \mathbf{r}=k, w\in\mathcal{X}^{\deg \mathbf{r}}} \prod_{j=1}^{\deg \mathbf{r}} \binom{\sum_{j=1}^{j} r_j + j - 1}{r_k} \tau_{\mathbf{r}}(w), \quad \text{where}$$

$$\tau_{\mathbf{r}}(w) = \tau_{r_1}(x_{i_1})\ldots\tau_{r_k}(x_{i_k}) = (\partial^{r_1} u_{x_{i_1}})x_{i_1}\ldots(\partial^{r_k} u_{x_{i_k}})x_{i_k} \in \mathbb{C}\{\{(u_x^{\pm 1})_{x\in\mathcal{X}}\}\}\langle\mathcal{X}\rangle.$$

First step of noncommutative PV theory

1. The space of solutions of
$$(NCDE) \quad \mathbf{d}S = MS, \quad \text{with} \quad M = \sum_{x\in\mathcal{X}} u_x x.$$
is a right free $\mathbb{C}\langle\langle X\rangle\rangle$-module of rank 1.

2. By a theorem of Ree, $C_{z_0\rightsquigarrow z}$ is a $\shuffle$ −group-like solution of $(NCDE)$ and it can be obtained as the limit of a convergent Picard iteration, initialized at $\langle C_{z_0\rightsquigarrow z}|1_{\mathcal{X}^*}\rangle = 1_{\mathcal{H}(\Omega)}1_{\mathcal{X}^*}$, for ultrametric distance.

3. If G and H are $\shuffle$ −group-like solutions $(NCDE)$ there is a constant Lie series C such that $G = He^C$ (and conversely).

From this, it follows that

- the differential Galois group of $(NCDE)$ + $\shuffle$ −group-like is the group [18] $\{e^C\}_{C\in\mathcal{L}ie_{\mathbb{C}.1_\Omega}\langle\langle\mathcal{X}\rangle\rangle}$.

Which leads us to the following definition

- the PV extension related to $(NCDE)$ is $\widehat{\mathcal{C}_0.\mathcal{X}}\{C_{z_0\rightsquigarrow z}\}$.

It, of course, is such that $\mathrm{Const}(\mathcal{C}_0\langle\langle\mathcal{X}\rangle\rangle) = \ker\mathbf{d} = \mathbb{C}.1_\Omega\langle\langle\mathcal{X}\rangle\rangle$.

18. In fact, the Hausdorff group (group of characters) of $\mathcal{H}_{\shuffle}(\mathcal{X})$.

Basic triangular theorem over a differential ring

Suppose that the $\mathbb{C}$-commutative ring $\mathcal{A}$ is without zero divisors and equipped with a differential operator ∂ such that $\mathbb{C} = \ker \partial$.
Let $S \in \mathcal{A}\langle\langle\mathcal{X}\rangle\rangle$ be a group-like solution of $(NCDE)$ in the following form

$$S = \sum_{w \in \mathcal{X}^*} \langle S|w\rangle w = \sum_{w \in \mathcal{X}^*} \langle S|S_w\rangle P_w = \prod_{l \in \mathcal{L}yn\mathcal{X}}^{\searrow} e^{\langle S|S_l\rangle P_l}.$$

Then

1. If $H \in \mathcal{A}\langle\langle\mathcal{X}\rangle\rangle$ is another grouplike solution then there exists $C \in \mathcal{L}ie_{\mathcal{A}}\langle\langle\mathcal{X}\rangle\rangle$ such that $S = He^C$ (and conversely).
2. The following assertions are equivalent
   a) $\{\langle S|w\rangle\}_{w \in \mathcal{X}^*}$ is $\mathcal{C}_0$-linearly independent,
   b) $\{\langle S|l\rangle\}_{l \in \mathcal{L}yn\mathcal{X}}$ is $\mathcal{C}_0$-algebraically independent,
   c) $\{\langle S|x\rangle\}_{x \in \mathcal{X}}$ is $\mathcal{C}_0$-algebraically independent,
   d) $\{\langle S|x\rangle\}_{x \in \mathcal{X} \cup \{1_{\mathcal{X}^*}\}}$ is $\mathcal{C}_0$-linearly independent,
   e) $\{u_x\}_{x \in \mathcal{X}}$ is such that, for $f \in \mathrm{Frac}(\mathcal{C}_0)$ and $(c_x)_{x \in \mathcal{X}} \in \mathbb{C}^{(\mathcal{X})}$,
   $$\sum_{x \in \mathcal{X}} c_x u_x = \partial f \quad \Longrightarrow \quad (\forall x \in \mathcal{X})(c_x = 0).$$
   f) $(u_x)_{x \in \mathcal{X}}$ is free over $\mathbb{C}$ and $\partial \mathrm{Frac}(\mathcal{C}_0) \cap \mathrm{span}_{\mathbb{C}}\{u_x\}_{x \in \mathcal{X}} = \{0\}$.

Examples of positive cases over $\mathcal{X} = \{x\}, \mathcal{A} = (\mathcal{H}(\Omega), \partial)$

1. $\Omega = \mathbb{C}, u_x(z) = 1_\Omega, \mathcal{C}_0 = \mathbb{C}\{\{u_x^{\pm 1}\}\} = \mathbb{C}$.
   $\alpha_0^z(x^n) = z^n/n!$, for $n \geq 1$. Thus, $\mathbf{d}S = xS$ and
   $$S = \sum_{n \geq 0} \alpha_0^z(x^n) x^n = \sum_{n \geq 0} \frac{z^n}{n!} x^n = e^{zx}.$$
   Moreover, $\alpha_0^z(x) = z$ which is transcendent over $\mathcal{C}_0$
   and the family $\{\alpha_0^z(x^n)\}_{n \geq 0}$ is $\mathcal{C}_0$-free. Let $f \in \mathcal{C}_0$ then $\partial f = 0$. Thus,
   if $\partial f = c u_x$ then $c = 0$.
2. $\Omega = \mathbb{C} \backslash]-\infty, 0], u_x(z) = z^{-1}, \mathcal{C}_0 = \mathbb{C}\{\{z^{\pm 1}\}\} = \mathbb{C}[z^{\pm 1}] \subset \mathbb{C}(z)$.
   $\alpha_1^z(x^n) = \log^n(z)/n!$, for $n \geq 1$. Thus $\mathbf{d}S = z^{-1} x S$ and
   $$S = \sum_{n \geq 0} \alpha_1^z(x^n) x^n = \sum_{n \geq 0} \frac{\log^n(z)}{n!} x^n = z^x.$$
   Moreover, $\alpha_1^z(x) = \log(z)$ which is transcendent over $\mathbb{C}(z)$ then
   over $\mathbb{C}[z^{\pm 1}]$. The family the family $\{\alpha_1^z(x^n)\}_{n \geq 0}$ is $\mathbb{C}(z)$-free and
   then $\mathcal{C}_0$-free. Let $f \in \mathcal{C}_0$ then $\partial f \in \mathrm{span}_{\mathbb{C}}\{z^{\pm n}\}_{n \neq 1}$. Thus,
   if $\partial f = c u_x$ then $c = 0$.

Examples of negative cases over $\mathcal{X} = \{x\}, \mathcal{A} = (\mathcal{H}(\Omega), \partial)$

1. $\Omega = \mathbb{C}, u_x(z) = e^z, \mathcal{C}_0 = \mathbb{C}\{\{e^{\pm z}\}\} = \mathbb{C}[e^{\pm z}]$.
   $\alpha_0^z(x^n) = (e^z - 1)^n/n!$, for $n \geq 1$. Thus, $\mathbf{d}S = e^z x S$ and
   $$S = \sum_{n\geq 0} \alpha_0^z(x^n)x^n = \sum_{n\geq 0} \frac{(e^z-1)^n}{n!}x^n = e^{(e^z-1)x}.$$
   Moreover, $\alpha_0^z(x) = e^z - 1$ which is not transcendent over $\mathcal{C}_0$ and and $\{\alpha_0^z(x^n)\}_{n\geq 0}$ is not $\mathcal{C}_0$-free. If $f(z) = ce^z \in \mathcal{C}_0$ $(c \neq 0)$ then $\partial f(z) = ce^z = cu_x(z)$.
2. $\Omega = \mathbb{C}\backslash]-\infty, 0], u_x(z) = z^a (a \notin \mathbb{Q})$,
   $\mathcal{C}_0 = \mathbb{C}\{\{z, z^{\pm a}\}\} = \mathrm{span}_{\mathbb{C}}\{z^{ka+l}\}_{k,l\in\mathbb{Z}}$.
   $\alpha_0^z(x^n) = (a+1)^{-n}z^{n(a+1)}/n!$, for $n \geq 1$. Thus, $\mathbf{d}S = z^a x S$ and
   $$S = \sum_{n\geq 0} \alpha_0^z(x^n)x^n = \sum_{n\geq 0} \frac{z^{n(a+1)}}{(a+1)^n n!}x^n = e^{(a+1)^{-1}z^{(a+1)}x}.$$
   Moreover, $\alpha_0^z(x) = z^{a+1}/(a+1)$ which is not transcendent over $\mathcal{C}_0$ and $\{\alpha_0^z(x^n)\}_{n\geq 0}$ is not $\mathcal{C}_0$-free. If $f(z) = cz^{a+1}/(a+1) \in \mathcal{C}_0$ $(c \neq 0)$ then $\partial f(z) = cz^a = cu_x(z)$.

Chen series of $\omega_0(z) = z^{-1}dz$ and $\omega_1(z) = (1-z)^{-1}dz$

Let $\gamma_0(\varepsilon)$ and $\gamma_1(\varepsilon)$ be the circular paths of radius ε encircling 0 and 1 clockwise, respectively. In particular, letting $\beta = \beta_1 - \beta_0$, one considers

$$\begin{array}{rcccl} \gamma_0(\varepsilon,\beta) & = & \varepsilon e^{i\beta_0} \rightsquigarrow \varepsilon e^{i\beta_1} & \subset & \gamma_0(\varepsilon), \\ \gamma_1(\varepsilon,\beta) & = & 1-\varepsilon e^{i\beta_0} \rightsquigarrow 1-\varepsilon e^{i\beta_1} & \subset & \gamma_1(\varepsilon). \end{array}$$

On the one hand, one has, for any $i = 0$ or 1 and $w \in X^+$,

$$|\langle C_{\gamma_i(\varepsilon,\beta)} | w\rangle| \leq \varepsilon^{|w|_{x_i}} \beta^{|w|} |w|!^{-1}.$$

It follows then

$$C_{\gamma_i(\varepsilon,\beta)} = e^{i\beta x_i} + o(\varepsilon) \quad \text{and} \quad C_{\gamma_i(\varepsilon)} = e^{2i\pi x_i} + o(\varepsilon).$$

On the other hand, for $R \in \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ of minimal representation (λ, μ, η) of dimension n, one has, for any $w \in X^*$,

$$|\langle R|w\rangle| \leq \|\lambda\|_\infty^{1,n} \|\mu(w)\|_\infty^{n,n} \|\eta\|_\infty^{n,1}.$$

Hence,

$$\alpha_{z_0}^z(R) := \langle R \parallel C_{z_0 \rightsquigarrow z}\rangle = \lambda((\alpha_{z_0}^z \otimes \mu)\mathcal{D}_X)\eta = \lambda\Bigg(\prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\alpha_{z_0}^z(S_l)\mu(P_l)}\Bigg)\eta.$$

Note that the map $\alpha_{z_0}^z : \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle \to \mathcal{H}(\Omega)$ is not injective. For example,

$$\alpha_{z_0}^z(z_0 x_0^* + (1-z_0)(-x_1)^* - 1_{X^*}) = 0.$$

$\mathrm{Dom}(\mathrm{Li}_\bullet)$

Let $\mathcal{C}_{\mathbb{C}} := \mathbb{C}[z^a, (1-z)^b]_{a,b\in\mathbb{C}}$ be equipped with ∂.

Proposition 4

Let $\mathrm{Dom}(\mathrm{Li}_\bullet)$ be the set of $S = \sum_{n\geq 0} S_n$ with $S_n = \sum_{|w|=n}\langle S|w\rangle w$ s.t. $\sum_{n\geq 0}\mathrm{Li}_{S_n}$ converges uniformly on any compact of Ω. Then $\mathrm{Dom}(\mathrm{Li}_\bullet)$, containing $\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle ⧢ \mathbb{C}\langle X\rangle$, is closed by shuffle. Moreover,

$$\forall S, T \in \mathrm{Dom}(\mathrm{Li}_\bullet),\ \mathrm{Li}_{S ⧢ T} = \mathrm{Li}_S\,\mathrm{Li}_T.$$

For $R \in \mathrm{Dom}(\mathrm{Li}_\bullet)$, let $\rho := \langle R\|\mathrm{L}\rangle$. Then, $\forall n\geq 0, \partial^n\rho = \langle R\|\mathbf{d}^n\mathrm{L}\rangle$ and

$$\mathbf{d}^n\mathrm{L} = p_n\mathrm{L} \text{ with } p_n = \sum_{\mathrm{wgt}\,\mathbf{r}=n}\ \sum_{w\in X^n}\ \prod_{i=1}^{\deg\mathbf{r}}\binom{\sum_{j=1}^{i} r_j + j - 1}{r_i}\tau_{\mathbf{r}}(w)\in\mathcal{C}\langle X\rangle,$$

where, for any word $w = x_{i_1}\ldots x_{i_k}\in X^*$ associated to the derivation multi-index $\mathbf{r} = (r_1,\ldots,r_k)\in\mathbb{N}^k$ of degree $\deg\mathbf{r} = |w|$ and of weight $\mathrm{wgt}\,\mathbf{r} = |w| + \sum_{i=1}^{k} r_i$, $\tau_{\mathbf{r}}(w) := \tau_{r_1}(x_{i_1})\ldots\tau_{r_k}(x_{i_k})$ and, for any $r\geq 0$,

$$\tau_r(x_0) = -r!(-z)^{-(r+1)}x_0 \quad\text{and}\quad \tau_r(x_1) = r!(1-z)^{-(r+1)}x_1.$$

Proposition 5

The following assertions are equivalent :

1. *ρ satisfies a differential equation with coefficients in $(\mathcal{C}_{\mathbb{C}}, \partial)$.*
2. *There exists $P\in\mathcal{C}_{\mathbb{C}}\langle X\rangle$ such that $\langle R\|P\mathrm{L}\rangle = \langle R \triangleright P\|\mathrm{L}\rangle = 0$.*

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