Lattice Path Conference
21-25 June 2021
Presentation times for this poster:
Wednesday 1:30-2:30 pm
Thursday 6-7 pm

LLT polynomials in a nutshell: on Schur expansion of LLT polynomials

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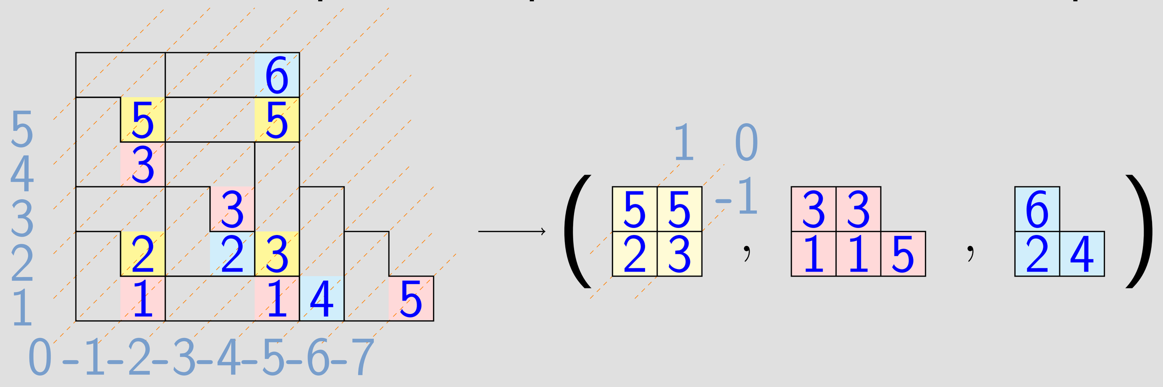
LLT polynomials

- LLT polynomials are a family of symmetric functions introduced by Lascoux, Leclerc and Thibon in '1997.
- The original definition uses **cospin** statistic of ribbon tableaux.
- A ribbon tableau maps to a tuple of skew tableaux via quotient map.



Variant of Bylund and Haiman using inversion statistic:

- Consider a k-tuple of skew diagrams $\boldsymbol{\nu} = (\nu^{(1)}, \dots, \nu^{(k)})$, and let $$\mathrm{SSYT}(\boldsymbol{\nu}) = \mathrm{SSYT}(\nu^{(1)}) \times \cdots \times \mathrm{SSYT}(\nu^{(k)}).$$
- The **content** of a cell $u = (\text{row, column}) = (i,j)$ is the integer $$c(u) = i - j.$$
- Given $T \in (T^{(1)}, \dots, T^{(k)}) \in \mathrm{SSYT}(\boldsymbol{\nu})$, a pair of entries $T^{(i)}(u) > T^{(j)}(v)$ form an **inversion** if either
  - (i) $i < j$ and $c(u) = c(v)$ or
  - (ii) $i > j$ and $c(u) = c(v) + 1$.
- $\mathrm{inv}(T)$ = the number of inversions in T.

Definition of LLT polynomials

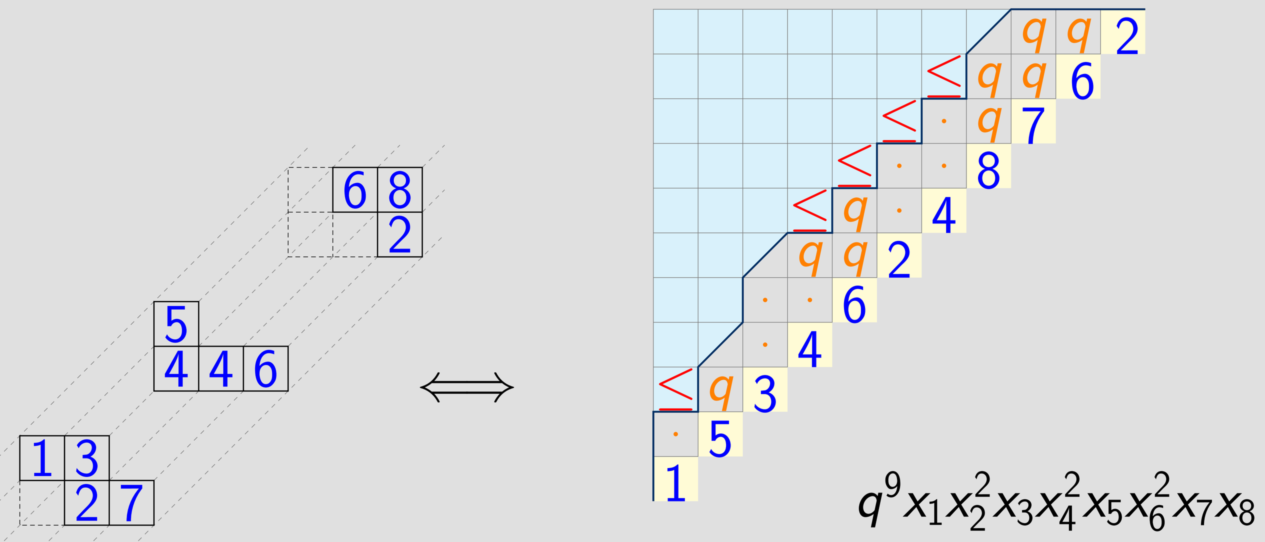
The **LLT polynomial** indexed by $\boldsymbol{\nu} = (\nu^{(1)}, \dots, \nu^{(k)})$ is
$$\mathrm{LLT}_{\boldsymbol{\nu}}(x;q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(T)} x^T.$$

Properties of LLT polynomials

- LLT polynomials are **symmetric**.
- When $q = 1$, $\mathrm{LLT}_{\boldsymbol{\nu}}(x;1) = \prod_{i=1}^{k} s_{\nu^{(i)}}(x)$.
- Furthermore, they are **Schur positive**. [Grojnowski-Haiman, '2007]
- However, combinatorial description for Schur coefficients are **not** known.
- **Macdonald polynomials** $\tilde{H}_\mu(x;q,t)$ can be expanded positively in terms of LLT polynomials.
- Hence, the Schur expansion of LLT polynomial gives a combinatorial formula for q,t-**Kostka polynomials**.

① LLT polynomials indexed by ribbons

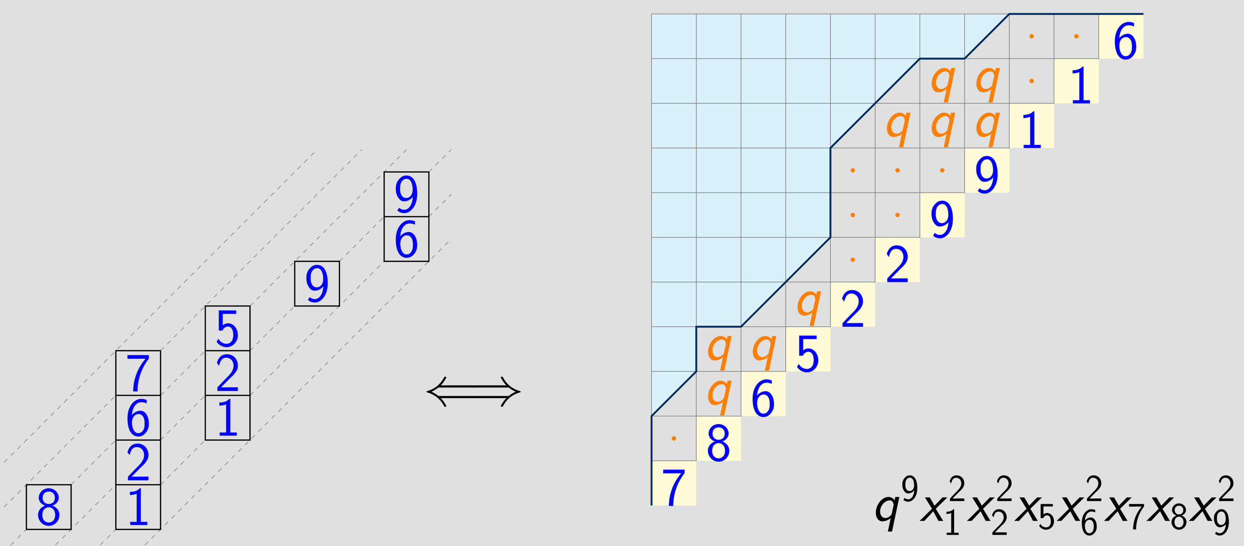
To read the inversion statistic:



Macdonald polynomials can be expanded positively in terms of ribbon LLT's:

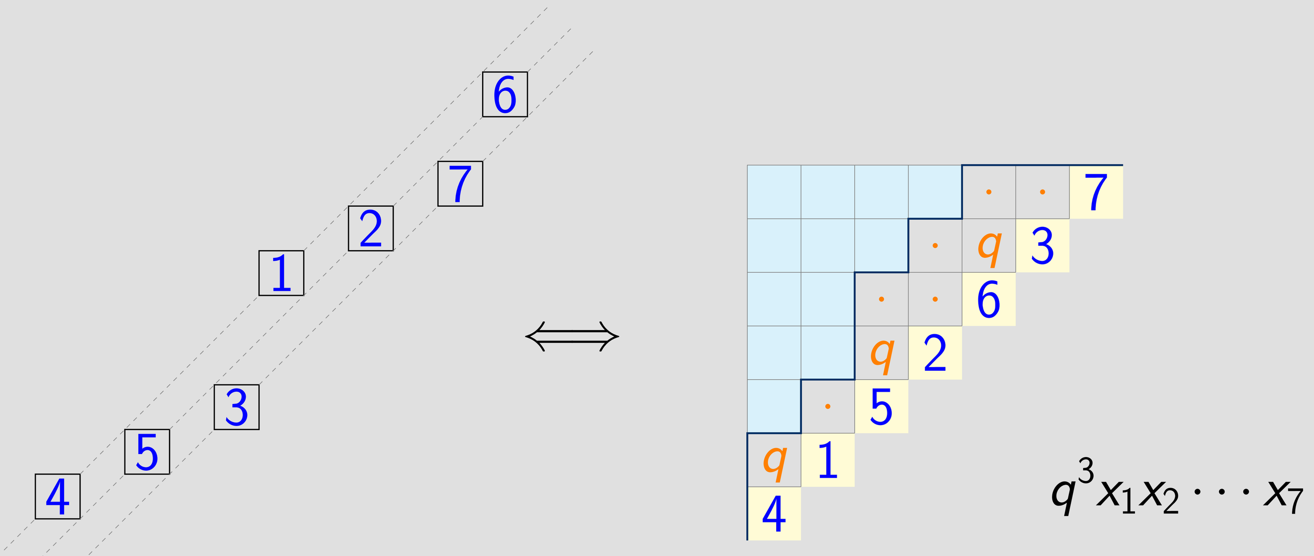
Theorem [Haglund-Haiman-Loehr, '2005]
$$\tilde{H}_\mu(X;q,t) = \sum_{\sigma:\mu\to\mathbb{Z}_+} q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} x^\sigma$$
$$= \sum_{D \subseteq \{(i,j)\in\mu:\, i>1\}} q^{-a(D)} t^{\mathrm{maj}(\sigma)} \mathrm{LLT}_{\boldsymbol{\nu}(\mu,D)}(X;q),$$
where $\boldsymbol{\nu}(\mu,D) = (\nu^{(1)}, \dots, \nu^{(k)})$, a tuple of ribbons.

② LLT polynomials indexed by vertical strips

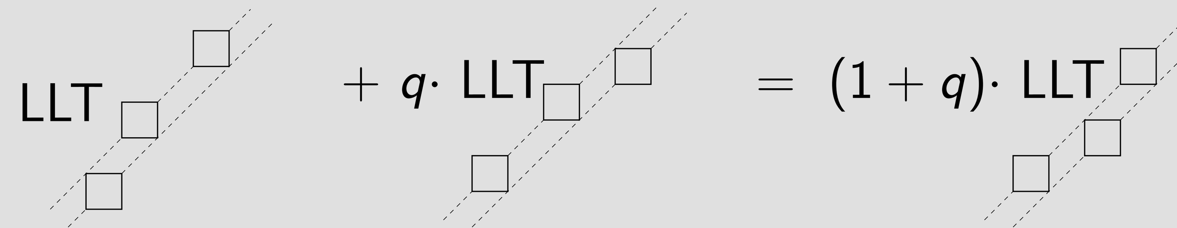


Shuffle Conjecture [HHLRU, '2005]
$\Rightarrow$ **Shuffle Theorem** [Carlsson-Mellit, '2017]
$$\nabla e_n = \sum_{\sigma\in\mathcal{WP}_n} q^{\mathrm{dinv}(\sigma)} t^{\mathrm{area}(\sigma)} X^\sigma$$
$$= \sum_P \mathrm{LLT}_P(X;q),$$
where P's are Schröder paths of size n.

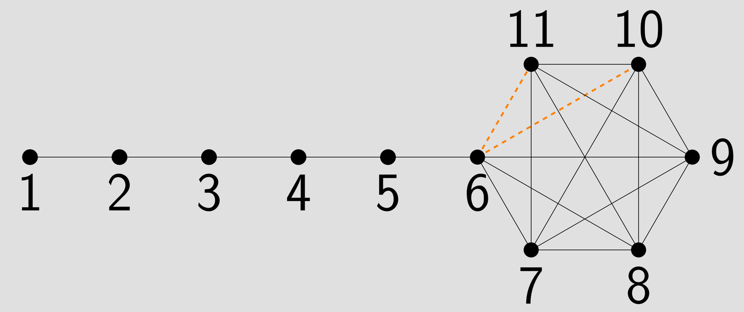
③ Unicellular LLT polynomials



Linear relation [Lee, '2021]:



- [HNY, '2020] In the case of LLT diagrams corresponding to the class of **melting lollipop graphs**, explicit combinatorial formula is given.



- [Lee, '2021] LLT polynomials with bandwidth ≤ 2 is proved to be 2-Schur positive.
- [Miller, '2019] LLT polynomials with bandwidth ≤ 3 is proved to be 3-Schur positive.
  Note. k-Schur positivity is stronger than Schur positivity.

Remark

- [Alexandersson-Sulzgruber, '2018] In the unicellular case, **power-sum** expansion is known.
- [Alexandersson-Sulzgruber, '2020] Unicellular and vertical-strip case, after substituting $q \mapsto q+1$, e-expansion is proved.
- [Novelli-Thibon, '2019] introduced a **non-commutative** lift of the **unicellular LLT polynomials** and prove that they are **positive** in a non-commutative analogue of the Gessel quasisymmetric basis, and the **relation** to chromatic quasisymmetric function via (non-commutative) plethysm.