**Lattice paths**

Consider a collection of admissible directions \( W = \{ w_1, \ldots, w_k \} \), where \( w_i \in \mathbb{Z}^d \) which all lie on the same side of some fixed hyperplane containing the origin. We define a **lattice path** as an ordered \((n + 1)\)-tuple of integer vectors \((0, p_1, \ldots, p_n)\), with each \( p_i \in \mathbb{Z}^d \), where \[ p_k := p_{k-1} + \lambda_k w_{k}, \] for some \( w_{k} \in W \), and \( \lambda_k \in \mathbb{Z}_{\geq 0} \).

**Directed paths**

Consider the set of all directed paths from the origin to a fixed \( q \in \mathbb{R}^d \), using the set of directions from the admissible directions \( W \): we define the **path polytope**

\[ P(q, c) := \{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}_{\geq 0}^n \mid \lambda_1 w_1 + \cdots + \lambda_n w_n = q \}, \]

for some \( w_1, \ldots, w_n \in W \). We call the collection of indices \( c := (c_1, \ldots, c_n) \) a **pattern** for the directed paths. The set of integer points in \( P(q, c) \) can be interpreted as the set of lattice paths, with pattern \( c \), defined by (1) define

\[ I(q, c) := \{ P(q, c) \cap \mathbb{Z}^d \}, \]

the set of integer points in \( P(q, c) \), which is also the set of lattice paths (from 0 to \( q \)) that use the subset \( w_1, \ldots, w_n \) of the admissible directions \( W \).

**Moduli space**

The **moduli space of all directed paths** from the origin to \( q \in \mathbb{R}^d \) is defined in [1] as the following union of polytopes:

\[ M_W(q) = \bigcap_{n=0}^{\infty} \bigcap_{c \in D(n, M)} P(q, c), \]

and can be endowed with a natural flat metric. This suggests the natural definition for the volume of the moduli space:

\[ \text{vol}(M_W(q)) := \sum_{n=0}^{\infty} \sum_{c \in D(n, M)} \text{vol}(P(q, c)) . \]

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**Example**

In dimension \( d = 2 \) and \( W := \{(1, 0), (0, 1)\} \), for \( q := (s, x - s) \in \mathbb{R}^2 \), with \( 0 < s < x \), the continuous binomial coefficient

\[ \left\{ \binom{x}{s} \right\} := \text{vol}(M_W(q)) = \sum_{n=0}^{\infty} \sum_{0 \leq c \in D(n, M)} \text{vol}(P(q, c)) . \]

is computed by Cano and Díaz [1] as

\[ \left\{ \binom{x}{s} \right\} = 2^{n} \binom{2\sqrt{s} (x - s)}{\sqrt{s} (x - s)} + \frac{1}{\sqrt{s} (x - s)} (2 \sqrt{s} (x - s))^{n}, \]

where \( J_n(z) \) denotes the modified Bessel function of the first kind. It satisfies the differential equation

\[ \frac{d}{d x} \binom{x + y}{x} = \binom{x + y}{x}, \]

a continuous analogue of the usual identity

\[ \Delta_n \binom{n + k}{k} = \binom{n + k}{k}. \]

**The \( d \)-dimensional extension**

Consider all lattice paths from the origin 0 to any \( q \in \mathbb{Z}^d \), using the standard basis as the set of admissible directions \( E := \{ e_1, \ldots, e_d \} \). The number of such lattice paths equals the multinomial coefficient

\[ \binom{q_1 + \cdots + q_d}{q_1, \ldots, q_d} := \binom{q_1 + \cdots + q_d}{q_1! \cdots q_d!} . \]

We fix any \( q \in \mathbb{R}^d \), and as above we consider all directed paths between the origin and \( q \). Fixing a pattern \( c := (c_1, \ldots, c_n) \), we get a path polytope \( P(q, c) \) with dimension \( n - d \). A natural definition for the continuous multinomial coefficient would then be:

\[ \binom{q_1 + \cdots + q_d}{q_1, \ldots, q_d} := \text{vol}(M_E(x)) = \sum_{n=0}^{\infty} \sum_{0 \leq c \in D(n, M)} \text{vol}(P(q, c)). \]

It can be characterized by its multivariate Borel transform. The Borel transform of a multi-variable analytic function \( f(x) = \sum_{0 \leq d \leq q} k_{d, 1} \cdots k_{d, q} x^1 \cdots x^q \) is defined as

\[ B(f)(x_1, \ldots, x_q) := \sum_{0 \leq d \leq q} \frac{k_{d, 1} \cdots k_{d, q}}{q!} x_1^1 \cdots x_q^q . \]

**Theorem**

Let

\[ F(x_1, \ldots, x_q) := \frac{1}{1 - \frac{x_1}{1 + \cdots + x_d}} . \]

Then the continuous multinomial is equal to

\[ \binom{q_1 + \cdots + q_d}{q_1, \ldots, q_d} = \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_d} B(F)(x_1, \ldots, x_d). \]

It satisfies the partial differential equation

\[ \prod_{j=1}^{d} \left( 1 + \frac{\partial}{\partial x_j} \right) \binom{x_1 + \cdots + x_d}{x_1, \ldots, x_d} = \sum_{i=1}^{d} \prod_{j \neq i}^{d} \left( 1 + \frac{\partial}{\partial x_j} \right) \binom{x_1 + \cdots + x_d}{x_1, \ldots, x_d} . \]

**Recovering discrete objects**

The **Todd operator** is defined as

\[ T_{d,k} := \frac{d^k}{d^k \log} = \sum_{k \geq 0} (-1)^k \frac{B_k}{k!} \left( \frac{d}{dh} \right)^k \]

where \( B_k \) are the Bernoulli numbers. We show that the usual (discrete) multinomial coefficients can be recovered from their continuous counterpart using the **Khovanskii-Pukhlikov theorem**: for a unimodular polytope \( P \),

\[ \#(P \cap \mathbb{Z}^d) = T_{d,k} \text{vol}(P(h))|_{h=0} . \]

**Analytical characterizations**

For more results about the continuous multinomial coefficients (such as integral representations, Chu-Vandermonde convolution identity) and the continuous Catalan numbers, see the companion article [3].

**References**

