Lattice Path Conference

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Presentation times for this poster:

Thursday 1:30-2:30 pm, 6-7 pm

A Continuous Analogue of Lattice Path Enumeration

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Lattice paths

Consider a collection of **admissible directions** $\mathbf{W} = \{\mathbf{w}_1, \dots, \mathbf{w}_N\}$ $\mathbf{w}_i \in \mathbb{Z}^d$ which all lie on the same side of some fixed hyperplane containing the origin. We define a **lattice path** as an ordered $(n+1)$-tuple of integer vectors $(\mathbf{0}, \mathbf{p}_1, \dots, \mathbf{p}_n)$, with each $\mathbf{p}_j \in \mathbb{Z}^d$, where

$$\mathbf{p}_k := \mathbf{p}_{k-1} + \lambda_k \mathbf{w}_{c_k}, \tag{1}$$

for some $\mathbf{w}_{c_k} \in \mathbf{W}$, and $\lambda_k \in \mathbb{Z}_{\geq 0}$.

Directed paths

Consider the set of all directed paths from the origin to a fixed $\mathbf{q} \in \mathbb{R}^d$, using the set of directions from the admissible directions $\mathbf{W}$: we define the **path polytope**

$$P(\mathbf{q}, \mathbf{c}) := \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n_{\geq 0} \mid \lambda_1 \mathbf{w}_{c_1} + \dots + \lambda_n \mathbf{w}_{c_n} = \mathbf{q}\}, \tag{2}$$

for some $\mathbf{w}_{c_1}, \dots, \mathbf{w}_{c_n} \in \mathbf{W}$. We call the collection of indices $\mathbf{c} := (c_1, \dots, c_n)$ a **pattern** for the directed paths.

The set of integer points in $P(\mathbf{q}, \mathbf{c})$ can be interpreted as the set of *lattice paths*, with pattern $\mathbf{c}$, defined by (1): define

$$L(\mathbf{q}, \mathbf{c}) := \{P(\mathbf{q}, \mathbf{c}) \cap \mathbb{Z}^d\}, \tag{3}$$

the set of integer points in $P(\mathbf{q}, \mathbf{c})$, which is also the set of lattice paths (from $\mathbf{0}$ to $\mathbf{q}$) that use the subset $\mathbf{w}_{c_1}, \dots, \mathbf{w}_{c_n}$ of the admissible directions $\mathbf{W}$.

Moduli space

The **moduli space of all directed paths** from the origin to $\mathbf{q} \in \mathbb{R}^d$ is defined in [1] as thefollowing union of polytopes:

$$\mathcal{M}_{\mathbf{W}}(\mathbf{q}) = \coprod_{n=0}^{\infty} \coprod_{\mathbf{c} \in D(n,N)} P(\mathbf{q}, \mathbf{c})$$

and can be endowed with a natural flat metric. This suggests the natural definition for the volume of the moduli space:

$$\mathrm{vol}(\mathcal{M}_{\mathbf{W}}(\mathbf{q})) := \sum_{n=0}^{\infty} \sum_{\mathbf{c} \in D(n,N)} \mathrm{vol}\, P(\mathbf{q}, \mathbf{c}).$$

Example

In dimension $d = 2$ and $\mathbf{W} := \{(1,0), (0,1)\}$, for $\mathbf{q} := (s, x-s) \in \mathbb{R}^2_{\geq 0}$, with $0 < s < x$, the **continuous binomial coefficient**

$$\left\{ \begin{matrix} x \\ s \end{matrix} \right\} := \mathrm{vol}(\mathcal{M}_{\mathbf{W}}(\mathbf{q})) = \sum_{n=0}^{\infty} \sum_{\mathbf{c} \in D(n,2)} \mathrm{vol}\, P(\mathbf{q}, \mathbf{c}). \tag{4}$$

is computed by Cano and Díaz [1] as

$$\left\{ \begin{matrix} x \\ s \end{matrix} \right\} = 2 I_0 \left(2\sqrt{s(x-s)} \right) + \frac{x}{\sqrt{s(x-s)}} I_1 \left(2\sqrt{s(x-s)} \right), \tag{5}$$

where $I_\nu(z)$ denotes the modified Bessel function of the first kind. It satisfies the differential equation

$$\frac{\partial}{\partial x} \frac{\partial}{\partial y} \left\{ \begin{matrix} x+y \\ x \end{matrix} \right\} = \left\{ \begin{matrix} x+y \\ x \end{matrix} \right\}, \tag{6}$$

a continuous analogue of the usual identity

$$\Delta_n \Delta_k \binom{n+k}{k} = \binom{n+k}{k}.$$

The d-dimensional extension

Consider all lattice paths from the origin $\mathbf{0}$ to any $\mathbf{q} \in \mathbb{Z}^d$, using the standard basis as the set of admissible directions $\mathbf{E} := \{\mathbf{e}_1, \dots, \mathbf{e}_d\}$. The number of such lattice paths equals the multinomial coefficient

$$\binom{q_1 + \dots + q_d}{q_1, \dots, q_d} := \frac{(q_1 + \dots + q_d)!}{q_1! \dots q_d!}.$$

We fix any $\mathbf{q} \in \mathbb{R}^d_{\geq 0}$, and as above we consider all directed paths between the origin and $\mathbf{q}$. Fixing a pattern $\mathbf{c} := (c_1, \dots, c_n)$, we get a path polytope $P(\mathbf{q}, \mathbf{c})$ with dimension $n - d$. A natural definition for the **continuous multinomial coefficient** would then be:

$$\left\{ \begin{matrix} x_1 + \dots + x_d \\ x_1, \dots, x_d \end{matrix} \right\} := \mathrm{vol}\left(\mathcal{M}_{\mathbf{E}}(\mathbf{x})\right) = \sum_{n=0}^{\infty} \sum_{\mathbf{c} \in D(n,d)} \mathrm{vol}\, P(\mathbf{q}, \mathbf{c}). \tag{7}$$

It can be characterized by its multivariate Borel transform. The Borel transform of a multi-variable analytic function $f(x) = \sum_{i_1, \dots, i_d = 0}^{\infty} k_{i_1 \dots i_d} x^{i_1} \dots x^{i_d}$ is defined as

$$\mathcal{B}(f)(x_1, \dots, x_d) := \sum_{i_1, \dots, i_d = 0}^{\infty} \frac{k_{i_1 \dots i_d}}{i_1! \dots i_d!} x^{i_1} \dots x^{i_d}.$$

Theorem

Let

$$F(x_1, \dots, x_d) := \frac{1}{1 - \left(\frac{x_1}{1+x_1} + \dots + \frac{x_d}{1+x_d} \right)}.$$

Then the continuous multinomial is equal to

$$\left\{ \begin{matrix} x_1 + \dots + x_d \\ x_1, \dots, x_d \end{matrix} \right\} = \frac{\partial}{\partial x_1} \dots \frac{\partial}{\partial x_d} \mathcal{B}(F)(x_1, \dots, x_d).$$

It satisfies thes partial differential equation

$$\prod_{j=1}^{d} \left(1 + \frac{\partial}{\partial x_j}\right) \left\{ \begin{matrix} x_1 + \dots + x_d \\ x_1, \dots, x_d \end{matrix} \right\} = \sum_{i=1}^{d} \prod_{j \neq i} \left(1 + \frac{\partial}{\partial x_j}\right) \left\{ \begin{matrix} x_1 + \dots + x_d \\ x_1, \dots, x_d \end{matrix} \right\}.$$

Recovering discrete objects

The **Todd operator** is defined as

$$\mathrm{Todd}_h := \frac{d/dh}{1 - e^{-d/dh}} = \sum_{k \geq 0} (-1)^k \frac{B_k}{k!} \left(\frac{d}{dh} \right)^k$$

where B_k are the Bernoulli numbers.

We show that the usual (discrete) multinomial coefficients can be recovered from their continuous counterpart using the **Khovanskii-Pukhlikov theorem**: for a unimodular polytope P,

$$\#(P \cap \mathbb{Z}^d) = \mathrm{Todd}_{\mathbf{h}} \, \mathrm{vol}(P(\mathbf{h}))|_{\mathbf{h}=0}.$$

Analytical characterizations

For more results about the continuous multinomial coefficients (such as integral representations, Chu-Vandermonde convolution identity) and the continuous Catalan numbers, see the companion article [3].

References

[1] L. Cano and R. Díaz, Indirect Influences on Directed Manifolds, *Advanced Studies in Contemporary Mathematics*, 28–1, 93–114, 2018 .

[2] T. Wakhare, C. Vignat, Q.-N. Le, and S. Robins, A continuous analogue of lattice path enumeration, The Electronic Journal of Combinatorics, 26–3, P.3–57, 2019

[3] T. Wakhare and C. Vignat, A continuous analog of lattice path enumeration: Part II, *Online Journal of Analytic Combinatorics*, 14, 2019.