Down-step statistics in generalized Dyck paths

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A family of generalized Dyck paths

$k_r$-Dyck paths

- Let $k \in \mathbb{N}$ and $0 \leq t \leq k - 1$, $t \in \mathbb{Z}$.
- Step set: $S = \{ (1, k), (1, -1) \}$.
- Length $(k + 1)n$: starts at $(0, 0)$, ends at $(0, (k + 1)n)$, $n \in \mathbb{N}$.
- Restrictions: paths stay (weakly) above $y = -t$.

Down-steps between pairs of up-steps

$s_{n,t,r}$ is the total number of down-steps between the $r$-th and the $(r + 1)$-th up-steps in all $k_r$-Dyck paths of length $(k + 1)n$.

**Enumerative results**

Let $0 \leq r \leq n$ be a non-negative integer, then:

1. The total number of down-steps before the first up-step in all $k_r$-Dyck paths of length $(k + 1)n$ is given by

$$s_{n,0,0} = \frac{\sum_{j=0}^{n-k-1} j + 1 \binom{j + 1 n + j + 1}{n}}{n} - \frac{1}{n}.$$

2. The total number of down-steps between the first and second up-step in all $k_r$-Dyck paths is given by

$$s_{n,0,1} = \frac{k (k + 1)(n + 1)}{n}.$$

3. For all positive integers $r$ with $1 \leq r \leq n$, the following recurrence relation holds:

$$s_{n,t,r} = s_{n,t,r-1} + \frac{t + 1}{k} \frac{(k + 1)r + t + 1}{r} \times (s_{n,r-1,0} - t[r = n]).$$

Proofs via bijective and generating function methods.

Asymptotic results (Average/Variance)

As $n \to \infty$,

1. The expected number of down-steps before the first up-step in $k_r$-Dyck paths of length $(k + 1)n$ is

$$E[X_{n,0}] = \frac{k}{t + 1} \left( \frac{k + 1}{(k + 1)^2} - \frac{k + t}{(k + 1)^2} \right) + O \left( \frac{1}{n} \right).$$

2. For fixed $1 \leq r < n$, the expected number of down-steps between the $r$-th and $(r + 1)$-th up-steps in $k_r$-Dyck paths of length $(k + 1)n$ is

$$E[X_{n,t,r}] = \frac{k^{r+1} t}{(k + 1)^2} \sum_{j=0}^{n-k-1} \frac{1}{j+1} \binom{j + 1 n + j + 1}{j} \left( \frac{k^r}{(k + 1)^{k+1}} \right)^j$$

$$+ \frac{k}{t + 1} \frac{(k + 1)^r}{(k + 1)^2} + \frac{k + t}{(k + 1)^2} + O \left( \frac{1}{n} \right).$$

3. The average number of down-steps after the last up-step in $k_r$-Dyck paths of length $(k + 1)n$ is

$$E[X_{n,r,t}] = \frac{(k + 1)^{r+1}}{k^r} - \frac{(t + 1)}{k} + O \left( \frac{1}{n} \right).$$

Given values of $k$, $r$, $t$, $n$, there is an algorithm to determine the variance.

Code: https://gitlab.aau.at/behackl/kt-dyck-downstep-code

Limiting behaviour

The following limiting behaviour is observed:

- $\lim_{k \to \infty} \lim_{n \to \infty} \frac{\sum_{j=0}^{n-k-1} j + 1 \binom{j + 1 n + j + 1}{n}}{n} = \frac{k + 1}{(k + 1)^2}$.
- $\lim_{k \to \infty} \lim_{n \to \infty} \frac{k (k + 1)(n + 1)}{n} = \frac{k + 1}{(k + 1)^2}$.
- $\lim_{k \to \infty} \lim_{n \to \infty} \frac{t + 1}{k} \frac{(k + 1)r + t + 1}{r} = e - \frac{2t(r - r)}{r} + O \left( \frac{1}{n} \right)$.
- $\lim_{k \to \infty} \lim_{n \to \infty} \frac{(k + 1)^r}{(k + 1)^2} + \frac{k + t}{(k + 1)^2} = e - \frac{2t(r - r)}{r} + O \left( \frac{1}{n} \right)$.

Note the relationship of the above to the Lambert $W$ function,

$$W(-x) = \sum_{j=1}^{j \to \infty} \frac{j^{j-1}}{j!}.$$

This is used to explain why for $0 < \beta < 1$, $\lim_{k \to \infty} \lim_{n \to \infty} \frac{k^{r+1} t}{(k + 1)^2} \sum_{j=0}^{n-k-1} \frac{1}{j+1} \binom{j + 1 n + j + 1}{j} \left( \frac{k^r}{(k + 1)^{k+1}} \right)^j = 1$.

Example: $t = 7$, $k = 10$, $r = 40, 400, 1600, 64000$.

![Graph showing limiting behaviour](https://github.com/behackl/kt-dyck-downstep-code)

Conclusions

- A bijective approach to the problem gives a recursive formula for the number of down-steps, while a generating function approach leads to a closed form formula.
- Results from both methods leads to several combinatorial identities.
- Asymptotic results were obtained, and unexpected limiting behaviours explained.
- A relationship between coding theory and $k_r$-Dyck paths was established, along with a Cycle-lemma type result, and a new interpretation for the Catalan numbers.
- Further work, i.e. $k \leq t$ and asymptotics, is in progress.

References