Lattice Path Conference
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Presentation times for this poster:

Tuesday	6:00-7:00 pm
Friday	1:30-2:30 pm

The Exponential-Dual Matrix Method: Applications to Markov Chain Analysis

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This poster corresponds to [1], to appear in 2021.

Uniformization

The problem

- We are given a continuous time Markov chain (CTMC) X on a discrete state space S. Initial distribution: α; generator: A.
- The goal is to find a closed-form of the transient distribution of X, which, seen as a row vector p, is $p(t) = \alpha \exp(At)$ (assumed to exist).
- Typical application area: queuing systems.

Uniformization

- Uniformization moves the continuous time previous problem to discrete time (differentials to differences).
- Assume $\sup_i |A_{i,i}| \le \Lambda < \infty$ ($\implies \exp(A)$ exists); let $U = I + A/\Lambda$.
- Let q = transient distribution of the discrete time Markov chain on S with initial distribution α and transition probability matrix U. Then,

$$p(t) = \sum_{n \ge 0} \exp(-\Lambda t) \frac{(\Lambda t)^n}{n!} q(n). \quad (1)$$

Finding q using Lattice Path Combinatorics is often pretty hard.

Stochastic Siegmund duality [2], [3]

Monotonicity

- We need a complete order on S, so, assume $S = \mathbb{N}$.
- Let $P(t) = \exp(At)$. Process X is monotone iff for all i, k, t,

$$\mathbb{P}_i(X(t) \ge k) \le \mathbb{P}_{i+1}(X(t) \ge k). \quad (2)$$

Duality

- If X is monotone, then there exists X^* Markov, on S, also monotone, such that for all i, j, t,

$$\mathbb{P}_i(X^*(t) \le j) = \mathbb{P}_j(X(t) \ge i). \quad (3)$$

- If A^* is the generator of X^*, we have similar linear mappings between $P(t)$ and $P^*(t) = \exp(A^*t)$, and between A and A^*. For instance,

$$A^*_{i,j} = \sum_{k=0}^{i-1} \Big[A_{j-1,k} - A_{j,k} \Big], \qquad A_{i,j} = \sum_{k=0}^{i} \Big[A^*_{j,k} - A^*_{j+1,k} \Big]. \quad (4)$$

Interest

- Given other properties of duality, finding transients on the Uniformized version of X^* is sometimes much simpler than on the original X.
- This comes from the fact that, typically, X is irreducible and its dual X^* is absorbing.
- For instance, in the $M/M/1$ queue, X^* has one absorbing state. In the $M/M/1/H$, X^* has two absorbing states.
- For more details and examples, see for instance [4], [5].

Problems with duality

- The monotonicity condition makes that in many cases, there is no dual (example: any circular chain, whatever the transition rates).
- Duality depends on the specific ordering of the states.
- In some cases the dual exists only if the transition rates of X satisfy specific inequalities.
- Other problems can arise in the infinite state space case.
- All these issues are illustrated and discussed in [1].

The exponential-dual matrix

- Given an arbitrary real or complex matrix A, we define the *exponential-dual* of A, A^*, by means of the left equality in (4).
- This transformation needs no condition on A, and when A is the generator of a CTMC X *and X's dual exists*, then the Siegmund dual X^* of X has generator A^*.
- A^* is not any matrix.
  - First of all, when A is finite, the dimension $\dim(A^*)$ of A^* is $= \dim(A) + 1$.
  - The rows of A^* sum up to 0.
  - The last row of A^* is only composed of 0's.

Examples

First example: $A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mapsto A^* = \begin{pmatrix} -1 & 0 & 1 \\ -2 & -2 & 4 \\ 0 & 0 & 0 \end{pmatrix}$.

Second example:

$$A = \begin{pmatrix} 1 & i & 0 \\ 1+i & 2 & -i \\ i & 0 & 1 \end{pmatrix} \mapsto A^* = \begin{pmatrix} 1+i & 2-i & i-2 & -1-i \\ i & 2-2i & i-1 & -1 \\ 0 & -i & 1+i & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The main property of the exponential-dual matrix

- If we know $\exp(A^*t) = E^*(t)$, then denoting $E(t) = \exp(At)$ we have:

$$E_{i,j}(t) = \sum_{k=0}^{i} \Big[E^*_{j,k}(t) - E^*_{j+1,k}(t) \Big]. \quad (5)$$

- In words, from $\exp(A^*t)$ we can find $\exp(At)$ using a simple linear transformation. This coincides with the right equality in (4), but it has now an algebraic proof (deterministic result here).

Solving process

Let's summarize some main points in the solving process:

- Uniformization moves the problem of transient analysis in CTMCs to discrete time.
- To obtain closed-forms, one important family of techniques is Lattice Path Combinatorics (reflection principles, roots of unity, etc.).
- This works both for the stochastic dual and for the exponential-dual when the former doesn't exist. Duality transformations allow to work with a different structure, and sometimes this is simpler.
- Typical examples where it happens are the $M/M/1$, the $M/M/1/H$, the $M/M/2$, the $M/M/1$ or the $M/M/1/H$ with catastrophes, etc.

Conclusions

- The concept of exponential-dual solves all the existence problems of Sigmund duality.
- Remaining tasks:
  - to develop the equivalent of the exponential-dual in discrete time and the connexions with the continuous time setting;
  - to analyze the use of Lattice Path Combinatorics tools when the original A is not a generator.

References

[1] G. Rubino and A. Krinik, *The Exponential-Dual Matrix Method: Applications to Markov Chain Analysis*, in *Stochastic Processes and Functional Analysis: New Perspectives*, edited by R. Swift, A. Krinik, J. Switkes and J. Park, Contemporary Mathematics 774, ISSN 0271-4132, Amer. Math. Soc., Providence, RI, to appear in 2021.

[2] D. Siegmund, *The equivalence of absorbing and reflecting barrier problems for stochastically monotone Markov processes*, Ann. Prob. **6** (1976), 914–924.

[3] W.J. Anderson, *Continuous-time Markov chains: an applications-oriented approach*, Springer, New York, 1991.

[4] A. Krinik, C. Mortensen, and G. Rubino, *Connections between birth-death processes*, Stochastic Processes and Functional Analysis (Alan C. Krinik and Randall J. Swift, eds.), DOI: 10.1201/9780203913574.ch12, Marcel Dekker, 2004, pp. 219–240.

[5] A. Krinik, G. Rubino, D. Marcus, R. J. Swift, H. Kasfy, and H. Lam, *Dual processes to solve single server systems*, Journal of Statistical Planning and Inference **135** (2005), no. 1, 702–713.