Lattice Path Conference

21-25 June 2021

Presentation times for this poster:

Tuesday	6-7 pm
Wednesday	1:30-2:30 pm

Algebraic area enumeration for lattice closed paths

Stéphane Ouvry[a] Alexios P. Polychronakos[b] Shuang Wu[c]

[a]Laboratoire de Physique Théorique et Modèles Statistiques, Université Paris Saclay, France

[b]Physics Department, City College of New York and the Graduate Center of CUNY, USA

[c]Physics Department, Shanghai Jiao Tong University, China

Closed paths on a square lattice

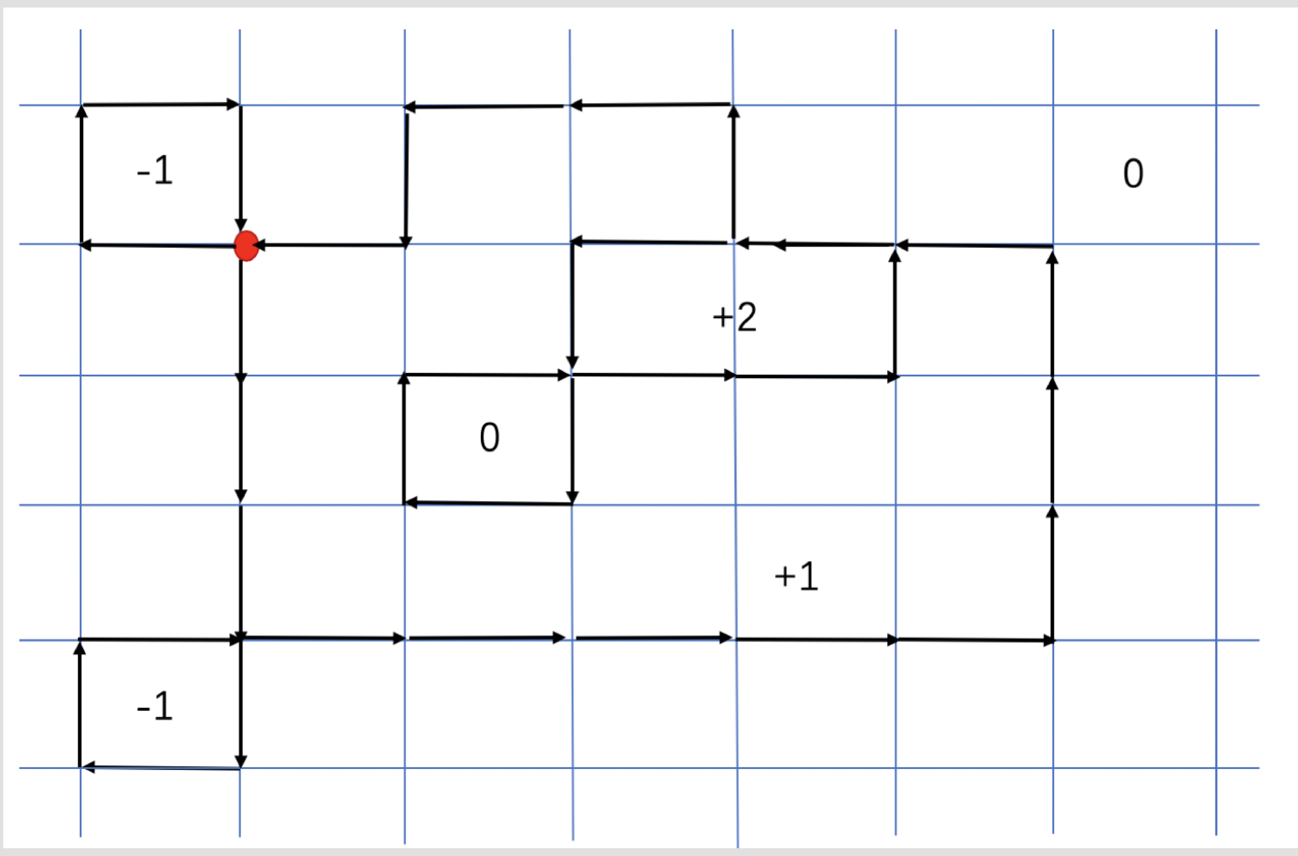
- **n**-steps lattice path
- closed paths : start from and return to the origin $\Rightarrow$ **n** even ($= 2n$)
- $\binom{\mathbf{n}}{\mathbf{n}/2}^2$ such paths

Algebraic area

algebraic area = area enclosed by the path, weighted by the winding number: if the path moves around a region in counterclockwise (positive) direction, its area counts as positive, otherwise negative; if the path winds around more than once, the area is counted with multiplicity

An example: $n = 36$

- winding numbers: +2,+1,0,-1,-1 (winding number $0 = +1 - 1$ does not contribute)
- number of lattice cells per winding sectors: 2,14,1,1,1
- algebraic area $A = 2 \times 2 + 14 \times 1 + 1 \times (-1) + 1 \times (-1) = 16$



Counting the algebraic area A: hopping operators u and v

u = right v = up

$v\ u = Q\ u\ v$ do not commute

$(u + u^{-1} + v + v^{-1})^{\mathbf{n}} = \sum_A C_{\mathbf{n}}(A)\ Q^A + \ldots$

$\mathbf{Tr}\ (u + u^{-1} + v + v^{-1})^{\mathbf{n}} = \sum_A C_{\mathbf{n}}(A)\ Q^A$

In quantum mechanics: the Hofstadter model

$Q = e^{i2\pi\Phi/\Phi_o}$ where Φ flux external magnetic field through unit lattice cell

$\Rightarrow$ quantum particle hopping on the square lattice coupled to external magnetic field

Hamiltonian $H = u + u^{-1} + v + v^{-1}$: this is the Hofstadter model

when flux is rational $Q = e^{i2\pi p/q}$ p, q coprimes $\Rightarrow$ Harper equation

$$H = \begin{pmatrix} Qe^{ik_y} + Q^{-1}e^{-ik_y} & e^{ik_x} & 0 & \cdots & 0 & e^{-ik_x} \\ e^{-ik_x} & Q^2e^{ik_y} + Q^{-2}e^{-ik_y} & e^{ik_x} & \cdots & 0 & 0 \\ 0 & e^{-ik_x} & \ldots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \ldots & e^{ik_x} \\ e^{ik_x} & 0 & 0 & \cdots & e^{-ik_x} & Q^qe^{ik_y} + Q^{-q}e^{-ik_y} \end{pmatrix}$$

secular determinant

$$\det(1 - zH) = \sum_{n=0}^{\lfloor q/2 \rfloor} (-1)^n Z(n) z^{2n} - 2\big(\cos(qk_x) + \cos(qk_y)\big) z^q$$

Kreft (1993) $\rightarrow$ $Z(n) = \sum_{k_1=1}^{q-2n+2} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} s_{k_1+2n-2} s_{k_2+2n-4} \cdots s_{k_{n-1}+2} s_{k_n}$

$$s_k = (1 - Q^k)(1 - Q^{-k}) = 4\sin^2(\pi k p/q)$$

Algebraic area enumeration

Introduce $b(n)$ via

$$\log\left(\sum_{n=0}^{\infty} Z(n) z^n\right) = \sum_{n=1}^{\infty} b(n) z^n$$

and coefficients $c(l_1, l_2, \ldots, l_j)$ labeled by compositions $l_1, l_2, \ldots, l_j$ of n

$$c(l_1, l_2, \ldots, l_j) = \frac{\binom{l_1+l_2}{l_1}}{l_1 + l_2}\ l_2 \frac{\binom{l_2+l_3}{l_2}}{l_2 + l_3} \cdots\ l_{j-1} \frac{\binom{l_{j-1}+l_j}{l_{j-1}}}{l_{j-1} + l_j}$$

so that $b(n) = (-1)^{n+1} \sum_{\substack{l_1, l_2, \ldots, l_j \\ \text{composition of } n}} c(l_1, l_2, \ldots, l_j) \sum_{k=1}^{q-j+1} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1}$

use $\log\det(1 - zH) = \mathbf{Tr}\ \log(1 - zH) \Rightarrow \mathbf{Tr}\ H^{\mathbf{n}=2n} = 2n(-1)^{n+1} b(n)$

so $\mathbf{Tr}\ H^{\mathbf{n}=2n} = 2n \sum_{\substack{l_1, l_2, \ldots, l_j \\ \text{composition of } n}} c(l_1, l_2, \ldots, l_j) \sum_{k=1}^{q-j+1} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1}$

one can also compute the trigonometric sum $\sum_{k=1}^{q-j+1} s_{k+j-1}^{l_j} \cdots s_{k+1}^{l_2} s_k^{l_1}$

The formula: $\mathbf{Tr}\ H^{\mathbf{n}} = \sum_A C_{\mathbf{n}}(A)\ Q^A$

$$C_{\mathbf{n}}(A) = 2n \sum_{\substack{l_1, l_2, \ldots, l_j \\ \text{composition of } n}} \frac{\binom{l_1+l_2}{l_1}}{l_1 + l_2}\ l_2 \frac{\binom{l_2+l_3}{l_2}}{l_2 + l_3} \cdots\ l_{j-1} \frac{\binom{l_{j-1}+l_j}{l_{j-1}}}{l_{j-1} + l_j}$$

$$\sum_{k_3=0}^{2l_3} \sum_{k_4=0}^{2l_4} \cdots \sum_{k_j=0}^{2l_j} \prod_{i=3}^{j} \binom{2l_i}{k_i}$$

$$\binom{2l_1}{l_1 + A + \sum_{i=3}^{j}(i-2)(k_i - l_i)} \binom{2l_2}{l_2 - A - \sum_{i=3}^{j}(i-1)(k_i - l_i)}$$

Why Z_n and b_n ? statistical mechanics and generalization

- $Z(n)$ = the n-body partition function with exclusion statistics $g = 2$ and spectrum $s_k = e^{-\epsilon(k)}$
- $b(n)$ = the n-th cluster coefficient

generalization to g-statistics

$$Z(n) = \sum_{k_1=1}^{q-gn+g} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} s_{k_1+gn-g} s_{k_2+gn-2g} \cdots s_{k_{n-1}+g} s_{k_n}$$

$$b(n)\ :\ \sum_{\substack{l_1, l_2, \ldots, l_n \\ \text{composition of } n}} c(l_1, l_2, \ldots, l_j) \rightarrow \sum_{\substack{l_1, l_2, \ldots, l_n \\ g-\text{composition of } n}} c_g(l_1, l_2, \ldots, l_j)$$

$$c_g(l_1, l_2, \ldots, l_j) = \frac{(l_1 + \cdots + l_{g-1} - 1)!}{l_1! \cdots l_{g-1}!} \prod_{i=1}^{j-g+1} \binom{l_i + \cdots + l_{i+g-1} - 1}{l_{i+g-1}}$$

g- composition obtained by inserting at will inside a given composition no more than $g - 2$ zeroes in succession

example $g = 3$

$n = 3 = 2 + 1 = 1 + 2 = 1 + 1 + 1 = 2 + 0 + 1 = 1 + 0 + 2 = 1 + 0 + 1 + 1 = 1 + 1 + 0 + 1 = 1 + 0 + 1 + 0 + 1$

$\rightarrow$ algebraic area numeration chiral triangular walks on triangular lattice same $s_k = 4\sin^2(\pi k p/q)$

References

[1] S. O. and Shuang Wu, "The algebraic area of closed lattice random walks", J. Phys. A: Math. Theor. 52 (2019) 255201

[2] S.O. and Alexios P. Polychronakos, "Exclusion statistics and lattice random walks", NPB 948 (2019) 114731; "Lattice walk area combinatorics, some remarkable trigonometric sums and Apéry-like numbers", NPB[FS] Volume 960 (2020) 115174