**Introduction**

The properties of extremes of a stochastic process are of fundamental importance in describing a plethora of natural and artificial phenomena. Applications can be found in: computer science, finance, climate studies, physics, etc. A natural question in finance is: "How long should one wait between the buying and the selling of a stock?" A good strategy would be to buy when the price is minimal and then to sell when it is maximal. **Main question:** how long does it take to go from the global maximum to the global minimum of a generic one-dimensional stochastic process?

**Problem statement**

- Consider a one-dimensional **Brownian motion** \( x(t) \) during a time interval \([0, T]\). The position of the Brownian particle evolves according to the Langevin equation

  \[
  \dot{x}(t) = \eta(t)
  \]

  where \( \eta(t) \) is Gaussian white noise.

- Define \( t_{\text{max}} \) (or \( t_{\text{min}} \)) as the time at which the global maximum (or minimum) is reached. The probability distribution of \( t_{\text{max}} \) is known in literature:

  \[
  P(t_{\text{max}} | T) = \frac{1}{\pi \sqrt{t_{\text{max}}(T - t_{\text{max})}}
  \]

  \( t_{\text{min}} \) has the same distribution by symmetry.

- **How long does it take to go from the global maximum to the global minimum?**

  \[
  \tau = t_{\text{min}} - t_{\text{max}}
  \]

  What is the distribution of \( \tau \)?

  Note that: \(-T \leq \tau \leq T\).

- **Non-trivial problem:** strong correlations between \( t_{\text{max}} \) and \( t_{\text{min}} \).

- **What is the distribution of \( \tau \) for a Brownian bridge** \( \tau \) (a periodic Brownian motion of period \( T \))?  

**Results**

**Brownian motion** Using a path integral technique we compute

\[
P(\tau | T) = \frac{1}{T} f_{\text{BM}}(\frac{\tau}{T})
\]

where

\[
f_{\text{BM}}(y) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^2} \tan^2 \left( \frac{m\pi}{2} \sqrt{1 - \left| y \right|} \right).
\]

\( f_{\text{BM}}(y) \) is symmetric:

\[
f_{\text{BM}}(y) = f_{\text{BM}}(-y)
\]

(see Fig 2). Asymptotic behaviors:

\[
f_{\text{BM}}(y) \approx \begin{cases} 8 \pi^2 y^2 & y = 0 \\ \frac{1}{2} y^2 & y \to \pm \infty \end{cases}
\]

**Brownian bridge** We show that

\[
f_{\text{BB}}(y) = \sum_{m=1}^{\infty} \frac{3(-1)^{m-1} m^2}{m^2 (1 - y^2)}
\]

which is again symmetric around \( y = 0 \) (see the inset of Fig. 2).

Asymptotic behaviors:

\[
f_{\text{BB}}(y) \approx \begin{cases} \frac{1}{2} y^2 & y = 0 \\ \frac{\sqrt{2\pi}}{1 - y^2} e^{-y^2} & y \to \pm \infty \end{cases}
\]

**Covariance of \( t_{\text{min}} \) and \( t_{\text{max}} \)**

We compute exactly the covariance

\[
\text{cov}(t_{\text{min}}, t_{\text{max}}) = \langle t_{\text{min}} t_{\text{max}} \rangle - \langle t_{\text{min}} \rangle \langle t_{\text{max}} \rangle
\]

In the two cases we get:

\[
\text{cov}_{\text{BM}}(t_{\text{min}}, t_{\text{max}}) = -\frac{7}{32} \pi^2 \approx -0.0754 T^2
\]

\[
\text{cov}_{\text{BB}}(t_{\text{min}}, t_{\text{max}}) = -\frac{9}{36} \pi^2 \approx -0.0241 T^2
\]

In both cases \( t_{\text{min}} \) and \( t_{\text{max}} \) are anti-correlated.

**Applications**

**Random walks** We demonstrate that the scaling function \( f_{\text{BM}} \) is universal in the sense of Central Limit Theorem. Consider a random walk of \( n \) steps

\[
x_k = x_{k-1} + \eta_k, \quad x_0 = 0,
\]

where \( \eta_k \sim p(\eta) \). \( p(\eta) \) is symmetric and it has variance \( \sigma^2 \). We show that

- if \( \sigma^2 \) is finite, the scaling function of \( \tau \) converges to \( f_{\text{BM}} \) in the large \( n \) limit
- if \( \sigma^2 \) diverges, the scaling function of \( \tau \) depends on the tail behavior of \( p(\eta) \)

**Fluctuating interfaces** Consider the stationary state of a Kardar-Parisi-Zhang (KPZ) [3] interface in one-dimension on a finite substrate of size \( L \). Let \( H(x, t) \) be the height of the interface as a function of the position \( x \) and time \( t \). \( H(x, t) \) evolves according to

\[
\frac{\partial H(x, t)}{\partial t} = \frac{\partial^2 H(x, t)}{\partial x^2} + \lambda \left( \frac{\partial H(x, t)}{\partial x} \right)^2 + \eta(x, t)
\]

where \( \lambda \geq 0 \) and \( \eta(x, t) \) is Gaussian white noise. We define \( \tau \) as the position difference between the minimal and maximal heights of the stationary interface \( H(x) \). We show that

- \( P(\tau | L) = \frac{1}{4} f_{\text{BB}}(\eta) \) for FBC
- \( P(\tau | L) = \frac{1}{4} f_{\text{BM}}(\eta) \) for PBC

**References**

