Lattice Path Conference
21-25 June 2021
Presentation times for this poster:
Wednesday 1:30-2:30 pm
Thursday 6-7 pm

The language of self-avoiding walks

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The connective constant

G — graph
o — some vertex of G
c_n — number of self-avoiding walks of length n in G starting at o

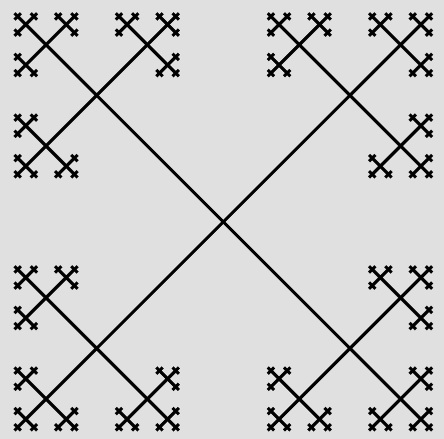
Definition. The connective constant of the graph G is

$$\mu(G) := \lim_{n\to\infty} \sqrt[n]{c_n}$$

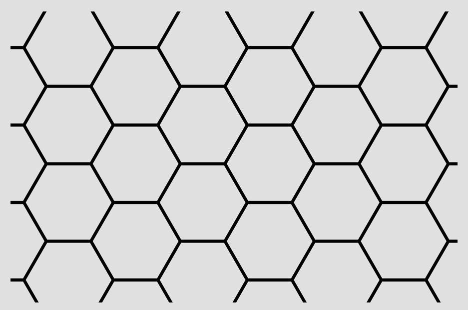
if the limit exists.

Theorem. (Hammersley 1957)
Every quasi-transitive graph has a well-defined connective constant.

Some connective constants

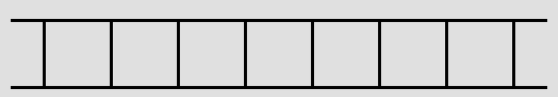


$\mu(G) = 3$



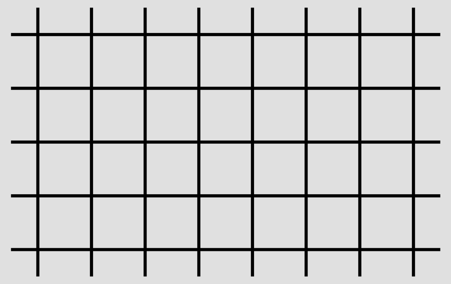
$\mu(G) = \sqrt{2+\sqrt{2}}$

Duminil-Copin & Smirnov 2012



$\mu(G) = \frac{1+\sqrt{5}}{2}$

e.g. Alm & Jansen 1990

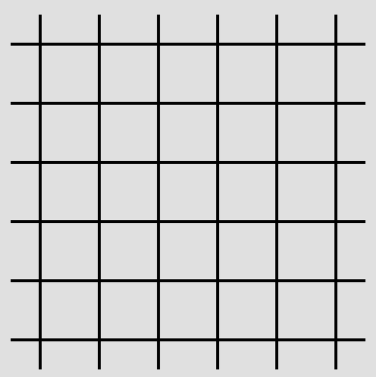


$\mu(G) = ?$

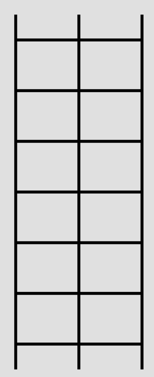
Ends of graphs

Definition. Ends of a graph G are equivalence classes of rays, where two rays are equivalent if they are connected by infinitely many disjoint paths.

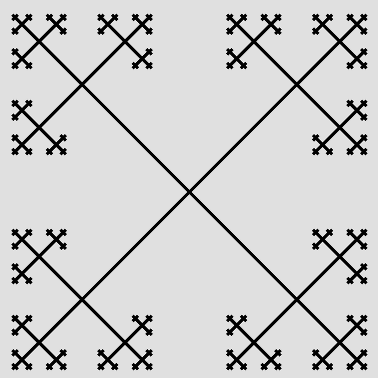
The size of an end is the maximal number of disjoint rays it contains.



1 end, size ∞



2 ends, size 3



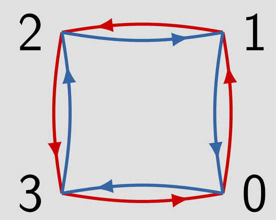
∞ ends, size 1

Cayley graphs

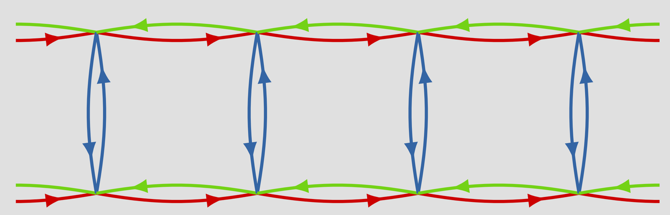
Γ — finitely generated group
S — finite generating set with $S = S^{-1}$

The Cayley graph $\mathrm{Cay}(\Gamma, S)$ has

- vertex set Γ,
- a directed edge from g to gs with label s for every $g \in G$ and $s \in S$.

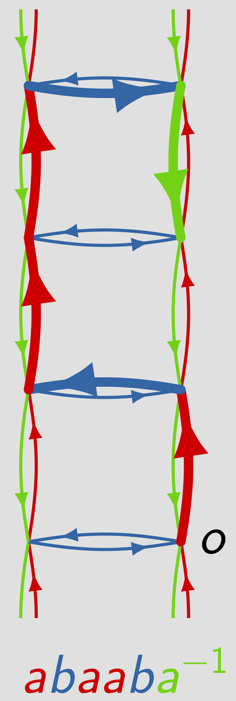


$\mathrm{Cay}(\mathbb{Z}_4, \{1, -1\})$



$\mathrm{Cay}(\mathbb{Z} \times \mathbb{Z}_2, \{(1,0), (-1,0), (0,1)\})$

Languages of walks in Cayley graphs



- walk on a Cayley graph $\leftrightarrow$ word in the generators
- family of walks $\leftrightarrow$ set of words (formal language)

Example. The family consisting of all closed walks leads to the well-studied word problem:
'Does a given word represent the identity element?'

Definition. The language L_{SAW} consists of all words corresponding to self-avoiding walks starting at o.

Tree decompositions of Cayley graphs

Theorem.
Γ is virtually free

$\Updownarrow$ (Karrass, Pietrowski, Solitar 1973)

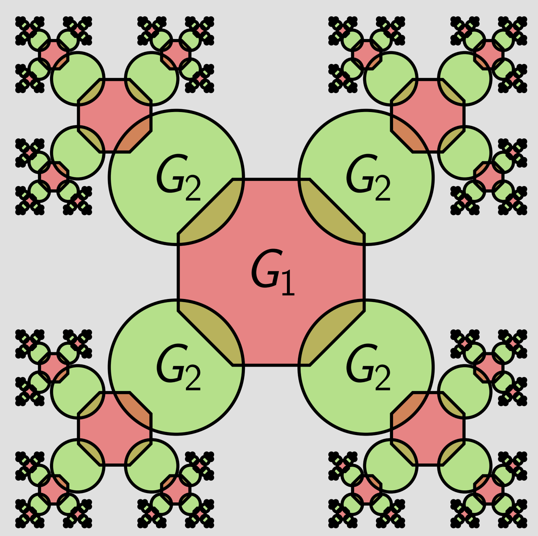
Γ splits as a finite graph of finite groups

$\Updownarrow$ (Woess 1989)

all ends of any Cayley graph have finite size

$\Updownarrow$ (Dunwoody & Krön 2015; Hamann et. al. 2020+)

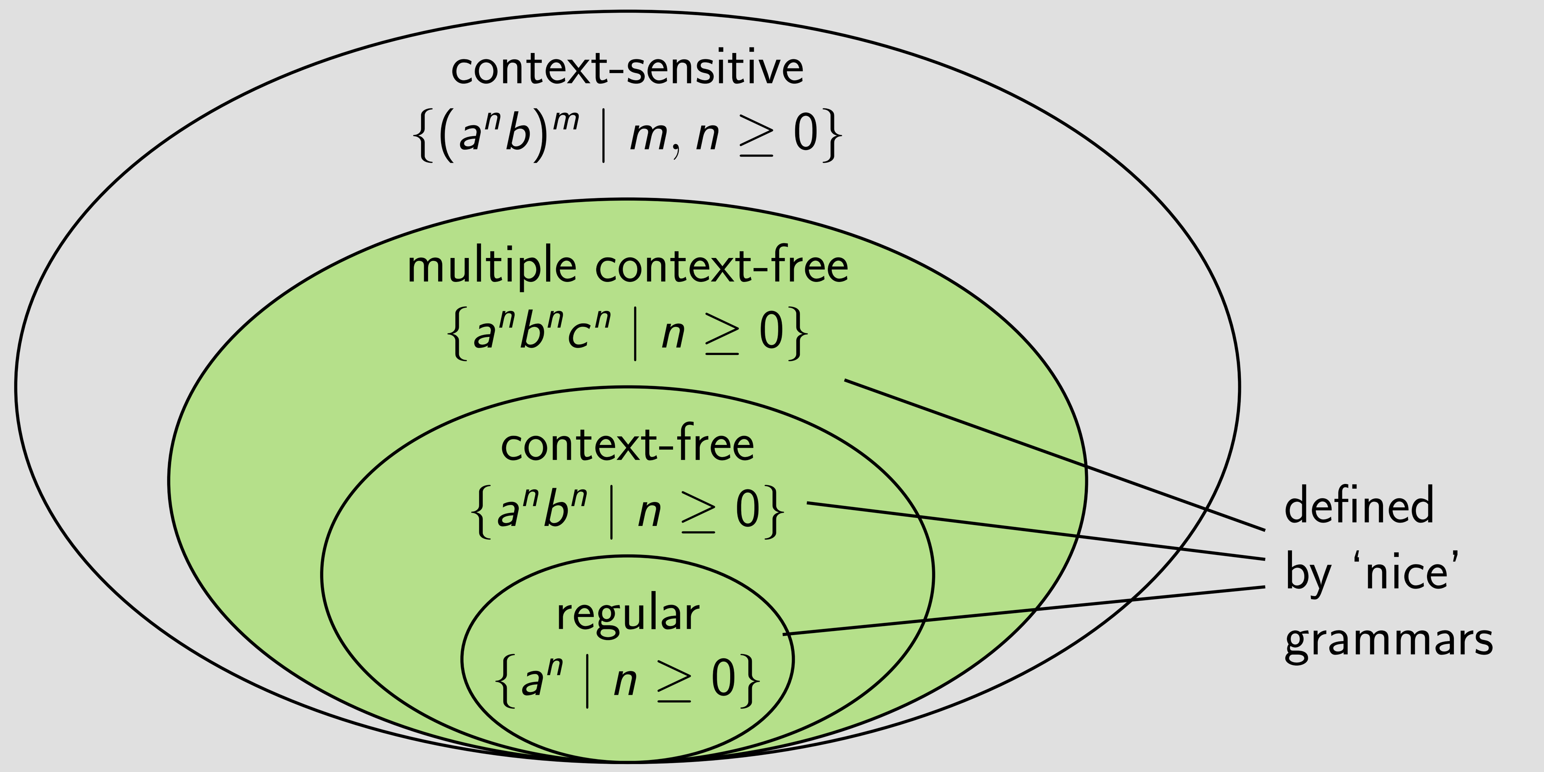
any Cayley graph has a Γ-invariant tree decomposition into finite parts



General idea

Combining self avoiding walks on parts of the tree decomposition gives a grammar for L_{SAW}.

Some classes and examples of formal languages



Results

Theorem. (Lindorfer & Woess 2020, Lehner & Lindorfer 2021+)

- L_{SAW} is regular $\iff$ all ends of $\mathrm{Cay}(\Gamma, S)$ have size 1.
- L_{SAW} is context-free $\iff$ all ends have size at most 2.
- L_{SAW} is multiple context-free $\iff$ all ends have finite size.

Using the Chomsky-Schützenberger enumeration theorem we obtain:

Corollary.
If Γ is virtually free, then $F_{\mathrm{SAW}} := \sum_{n\geq 0} c_n z^n$ is algebraic over $\mathbb{Q}(z)$.
In particular, $\mu(\mathrm{Cay}(\Gamma, S))$ is an algebraic number.

- this also works for quasi-transitive graphs
- work in progress: study properties of random self avoiding walks

References

[1] Lindorfer, C.; Woess, W.: *The Language of Self-Avoiding Walks.* Combinatorica, 40 (5), p. 691-720, 2020.

[2] Lehner, F.; Lindorfer, C.: *Self-avoiding walks and multiple context-free languages.* arXiv preprint, 2021+.