Lattice Path Conference

21-25 June 2021

Presentation times for this poster:

Monday	1:30-2:30 pm
Tuesday	1:30-2:30 pm
Thursday	6-7 pm

Lattice Paths And Negatively Indexed Weight-Dependent Binomial Coefficients

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Weight-dependent binomial coefficients

For $n,k\in\mathbb{Z}$, we define the weight-dependent binomial coefficient as

$${}_w\begin{bmatrix}n\\0\end{bmatrix} = {}_w\begin{bmatrix}n\\n\end{bmatrix} = 1 \qquad \text{for } n\in\mathbb{Z}\,,$$

and for $n,k\in\mathbb{Z}$, provided that $(n+1,k)\neq(0,0)$,

$${}_w\begin{bmatrix}n+1\\k\end{bmatrix} = {}_w\begin{bmatrix}n\\k\end{bmatrix} + {}_w\begin{bmatrix}n\\k-1\end{bmatrix} W(k,n+1-k),$$

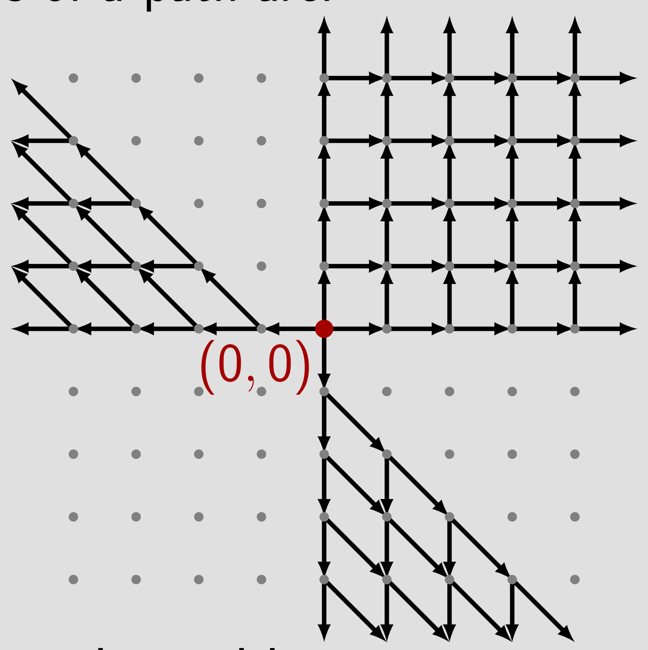
with $W(s,t)=\prod_{j=1}^t w(s,j)$ for a sequence of weights $w(s,t)$ and we define products generally by

$$\prod_{j=1}^t A_j = \begin{cases} A_1A_2\dots A_t & t>0\\ 1 & t=0\\ A_0^{-1}A_{-1}^{-1}\dots A_{t+1}^{-1} & t<0\end{cases}.$$

The lattice path model

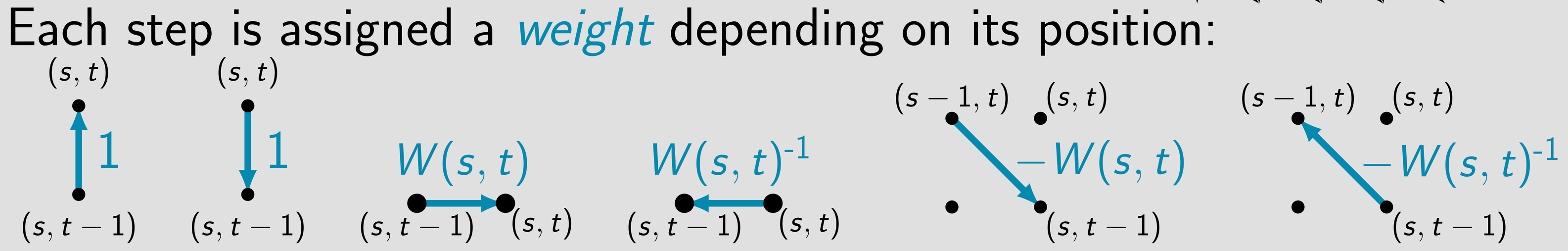
For $m,n\in\mathbb{Z}$, a *hybrid lattice path* is a path from $(0,0)$ to (m,n). Depending on m and n, the possible steps of a path are:

1. $n,m\geq 0$: $\uparrow$ and $\rightarrow$
2. $m<0\leq n$: $\leftarrow$ and $\nwarrow$
3. $n<0\leq m$: $\downarrow$ and $\searrow$
4. $n,m<0$: no allowed steps



Additionally, if $m<0$, the first step has to be $\leftarrow$ and if $n<0$, the first step has to be $\downarrow$.

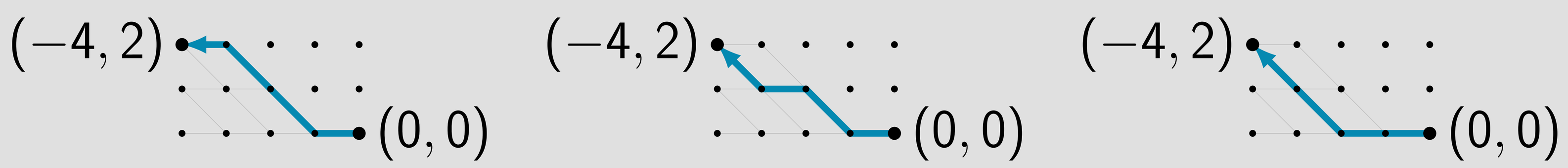
Each step is assigned a *weight* depending on its position:



The *weight of a path* $w(P)$ is the product over the weights of its steps.

Examples

There are three paths from $(0,0)$ to $(-4,2)$:



The weight of the first path is for example

$$\begin{aligned} w(P) &= W(0,0)^{-1}(-W(-1,1)^{-1})(-W(-2,2)^{-1})W(-3,2)^{-1}\\ &= (w(-1,1)w(-2,1)w(-2,2)w(-3,1)w(-3,2))^{-1}\end{aligned}$$

Weighted counting

The weight-dependent binomial coefficients count weighted hybrid lattice paths. For all $n,k\in\mathbb{Z}$,

$${}_w\begin{bmatrix}n\\k\end{bmatrix} = \sum_P w(P),$$

where the sum runs over all paths from $(0,0)$ to $(k,n-k)$.

Reflection formulae

We define the weight-reflections $\breve{w}(s,t)=w(s,1-s-t)^{-1}$ and $\tilde{w}(s,t)=w(1-s-t,t)^{-1}$ to obtain

$$\begin{aligned} {}_w\begin{bmatrix}n\\k\end{bmatrix} &= (-1)^k\operatorname{sgn}(k)\,{}_{\breve{w}}\begin{bmatrix}k-n-1\\k\end{bmatrix}\prod_{j=1}^k W(j,-j)\\ &= (-1)^{n-k}\operatorname{sgn}(n-k)\,{}_{\tilde{w}}\begin{bmatrix}-k-1\\-n-1\end{bmatrix}\prod_{j=1}^{n-k} W(n+1-j,j)^{-1},\end{aligned}$$

where $\operatorname{sgn}(n)$ is 1 for $n\geq 0$ and -1 for $n<0$. These formulas explain the behavior of the weight-dependent binomial coefficient at negative values.

Noncommutative binomial theorem

Let x and y be noncommutative variables satisfying the three relations

$$yx = w(1,1)xy,\quad xw(s,t) = w(s+1,t)x \text{ and } yw(s,t) = w(s,t+1)y,$$

then for all $n\in\mathbb{Z}$:

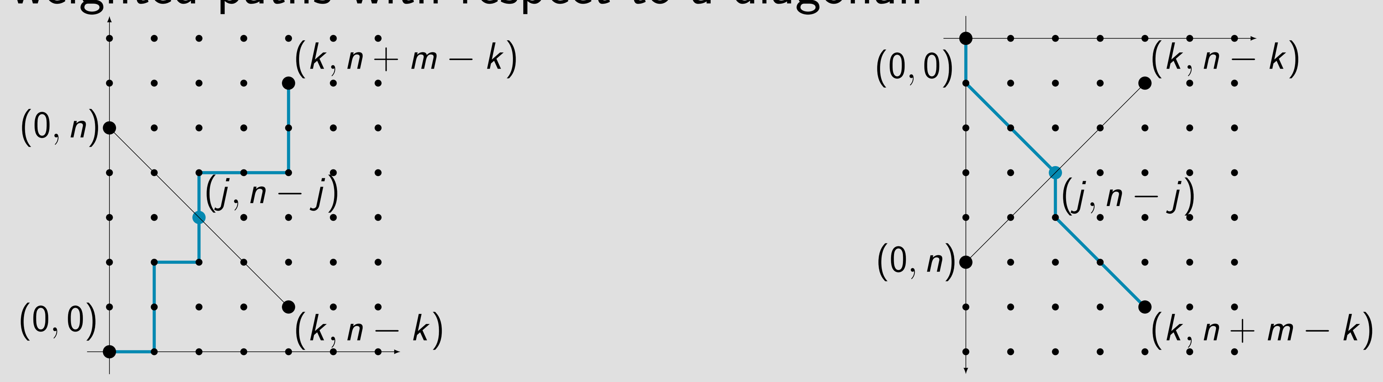
$$(x+y)^n = \sum_{k\geq 0} {}_w\begin{bmatrix}n\\k\end{bmatrix} x^k y^{n-k} = \sum_{k\leq n} {}_w\begin{bmatrix}n\\k\end{bmatrix} x^k y^{n-k}.$$

Convolution formula

Let x,y be noncommutative as before, $n,m\in\mathbb{Z}$ and $k\geq 0$, then

$${}_w\begin{bmatrix}n+m\\k\end{bmatrix} = \sum_{j=0}^k {}_w\begin{bmatrix}n\\j\end{bmatrix}\left(x^jy^{n-j}\,{}_w\begin{bmatrix}m\\k-j\end{bmatrix}y^{j-n}x^{-j}\right)\prod_{i=1}^{k-j} W(i+j,n-j).$$

For $m,n>0$ or $m,n<0$, this identity can be interpreted as convolution over weighted paths with respect to a diagonal.



Specializations

Binomial coefficient

For $w(s,t)=1$ we obtain the ordinary *binomial coefficient* $\binom{n}{k}$ studied for arbitrary integer values by Loeb [1].

Gaussian binomial coefficient

For $w(s,t)=q$ we obtain the *q-binomial coefficient* $\begin{bmatrix}n\\k\end{bmatrix}_q$ studied for arbitrary integer values by Formichella and Straub [2].

Elliptic binomial coefficient

For

$$w(s,t) = \frac{\theta(aq^{s+2t}, bq^{2s+t-2}, aq^{t-s-1}/b;p)}{\theta(aq^{s+2t-2}, bq^{2s+t}, aq^{t-s+1}/b;p)}q$$

we obtain the *elliptic binomial coefficient* [3]

$${}_w\begin{bmatrix}n\\k\end{bmatrix} = \begin{bmatrix}n\\k\end{bmatrix}_{a,b;q,p} := \frac{(q^{1+k}, aq^{1+k}, bq^{1+k}, aq^{1-k}/b;q,p)_{n-k}}{(q,aq,bq^{1+2k},aq/b;q,p)_{n-k}},$$

where $\theta(x;p)=\prod_{k=0}^\infty((1-xp^k)(1-p^{k+1}/x))$ is the *modified Jacobi theta function* and $(a;q,p)_k=\prod_{i=0}^{k-1}\theta(aq^i;p)$ is the *theta shifted factorial*.

Symmetric functions

For $w(s,t)=\frac{a_{s+t}}{a_{s+t-1}}$ we obtain

$${}_w\begin{bmatrix}n\\k\end{bmatrix} = \begin{cases} e_k(a_1,a_2,\dots,a_n)\prod_{i=1}^k a_i^{-1}, & 0\leq k\leq n\\ h_k(-a_0,-a_{-1},\dots,-a_{n+1})\prod_{i=1}^k a_i^{-1}, & n<0\leq k\\ h_{n-k}(-a_0^{-1},-a_{-1}^{-1},\dots,-a_{n+1}^{-1})\prod_{i=k+1}^n a_i, & k\leq n<0\end{cases}$$

where e_k is the *elementary symmetric function* and h_k is the *complete homogeneous symmetric function* of order k.

Conclusion

- Many results from [1] and [2] can be generalized to the weighted case with its specializations.
- In [1] and [2] binomial coefficients were interpreted with hybrid sets. Hybrid lattice paths can be translated to the corresponding hybrid sets.

References

[1] Loeb, D. E.: *A generalization of the binomial coefficients.* Discrete Math., 105(1-3):143–156, 1992.

[2] Formichella, S.; Straub, A.: *Gaussian binomial coefficients with negative arguments.* Ann. Comb., 23(3-4):725–748, 2019.

[3] Schlosser, M. J.: *A noncommutative weight-dependent generalization of the binomial theorem.* Sém. Lothar. Combin., 81:Art. B81j, 24, 2020.