Lattice Path Conference
21-25 June 2021
Presentation times for this poster:

Tuesday	6-7 pm
Wednesday	1:30-2:30 pm

Coefficientwise total positivity of some matrices defined by linear recurrences

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Coefficientwise total positivity

Equip $\mathbb{R}[\mathbf{x}]$ with the partial coefficientwise order: $P \in \mathbb{R}[\mathbf{x}] \succeq 0 \iff P$ is a polynomial with nonnegative coefficients. A matrix M with entries belonging to $\mathbb{R}[\mathbf{x}]$ is **coefficientwise totally positive** if all of its minors are polynomials with nonnegative coefficients. If all the minors of M of size $\le r$ are polynomials with nonnegative coefficients then M is **coefficientwise totally positive of order** r.

A matrix defined by a general linear recurrence

Let $T(a,c,d,e,f,g) = (T(n,k))_{n,k\ge 0}$ be the matrix with entries belonging to $\mathbb{Z}[a,c,d,e,f,g]$ defined by the recurrence

$$T(n,k) = [a(n-k)+c]\,T(n-1,k-1) + [dk+e]\,T(n-1,k) + [f(n-2)+g]\,T(n-2,k-1)$$

for $n \ge 1$ with initial condition $T(0,k) = \delta_{0,k}$.

- (n,k)-entry of $T(0,1,1,1,0,0)$ (the Stirling subset triangle) counts **partitions of** $[n+1] = \{1,2,\dots,n+1\}$ **into** $k+1$ **nonempty blocks**.
- (n,k)-entry of $T(1,1,0,1,0,0)$ (reversed Stirling subset triangle) counts **partitions of** $[n+1]$ **into** $n-k+1$ **nonempty blocks**.
- (n,k)-entry of $T(1,1,1,1,0,0)$ (the Eulerian triangle) counts **permutations of** $[n+1]$ **with** k **descents (or excedances)**.

$T(a,c,d,e,f,g)$ conjecture

The matrix $T(a,c,d,e,f,g)$ is coefficientwise totally positive in the indeterminates a,c,d,e,f,g.

$T(a,c,0,e,0,0)$ theorem

The matrix $T(a,c,0,e,0,0)$ is coefficientwise totally positive in the indeterminates a,c,e.

Purely $n-$ (or $k-$) dependent recurrences

Let $T = (T(n,k))_{n,k\ge 0}$ be the matrix with entries defined by

$$T(n,k) = a_{n,k}T(n-1,k-1) + b_{n,k}T(n-1,k) + c_{n,k}T(n-2,k-1)$$

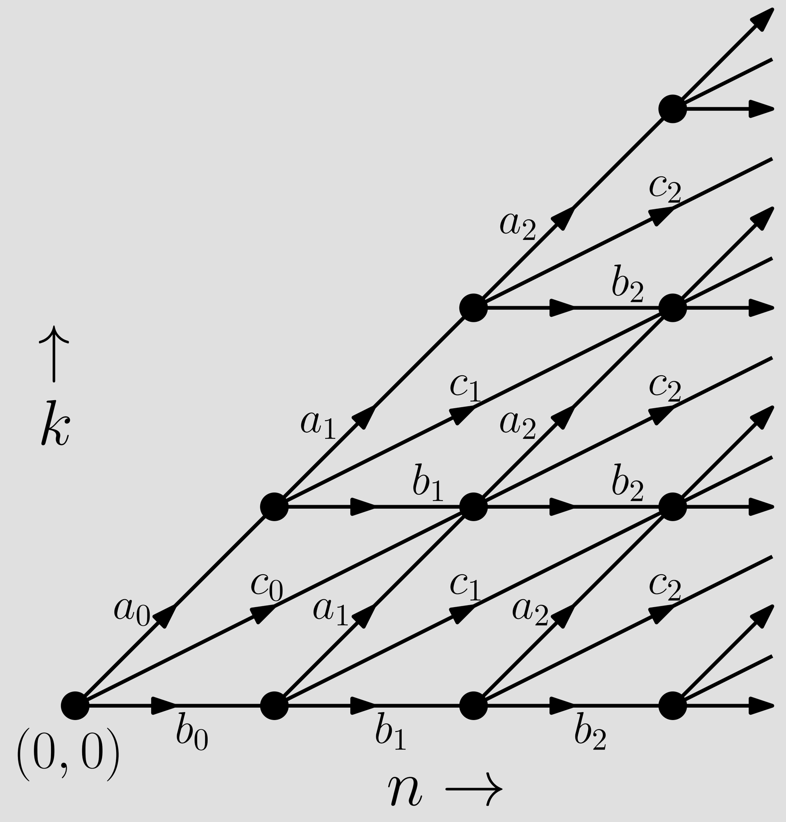
for $n \ge 1$ and $T(0,k) = \delta_{0,k}$ where $\mathbf{a} = \{a_{n,k}\}_{n,k\ge0}$, $\mathbf{b} = \{b_{n,k}\}_{n,k\ge0}$, and $\mathbf{c} = \{c_{n,k}\}_{n,k\ge0}$ are sequences of indeterminates. Brenti showed that if the indeterminates are purely n-dependent or purely k-dependent then T is coefficientwise totally positive in the indeterminates $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

n-dependent recurrence and walks from $(0,0)$ to (n,k)

Suppose $\mathbf{a} = \{a_n\}_{n\ge0}$, $\mathbf{b} = \{b_n\}_{n\ge0}$, and $\mathbf{c} = \{c_n\}_{n\ge0}$. Then $T(n,k)$ is the sum of weighted paths from $(0,0)$ to (n,k), where:

- Each step from (n,k) to $(n+1,k+1)$ has weight a_n;
- Each step from (n,k) to $(n+1,k)$ has weight b_n;
- Each step from (n,k) to $(n+2,k+1)$ has weight c_n.

The weight of a path is the product of the weights of its edges.

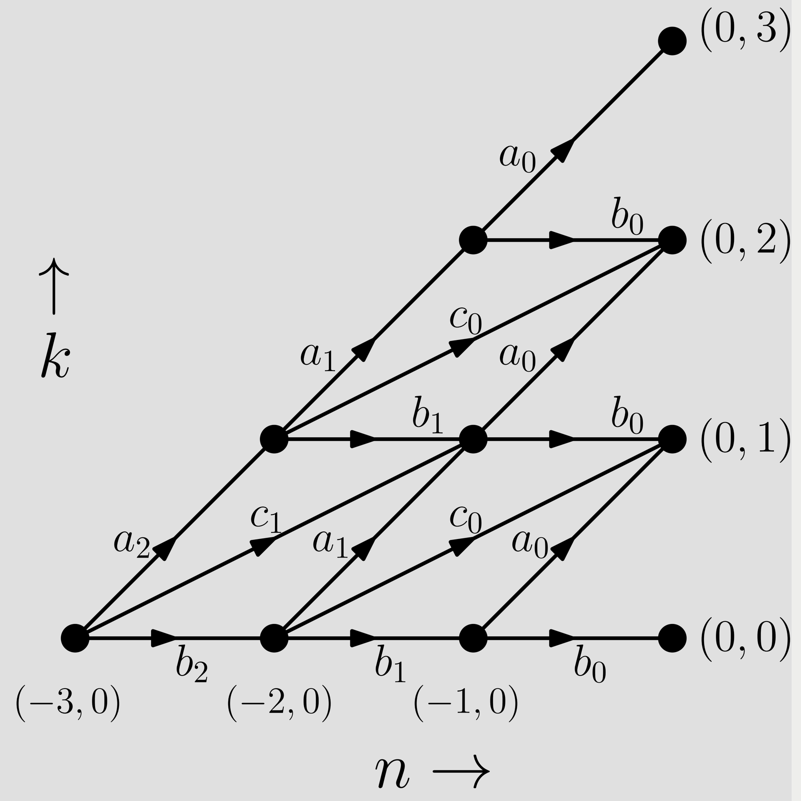


Vertical invariance and a planar network

The (n,k)-entry of T can also be seen as the sum over weighted paths from $(-n,0)$ to $(0,k)$ in the **locally finite acyclic digraph** (LFAD) $\mathcal{N}$ where

- Each step from $(-n,k)$ to $(-n+1,k+1)$ has weight a_{n-1};
- Each step from $(-n,k)$ to $(-n+1,k)$ has weight b_{n-1};
- Each step from $(-n,k)$ to $(-n+2,k+1)$ has weight c_{n-2}.

Let $U := \{(-n,0) : n \ge 0\}$ be the sources of $\mathcal{N}$ and $V := \{(0,k) : k \ge 0\}$ be the sinks. Then U and V are **fully compatible**, and $\mathcal{N}$ is called a **planar network**.



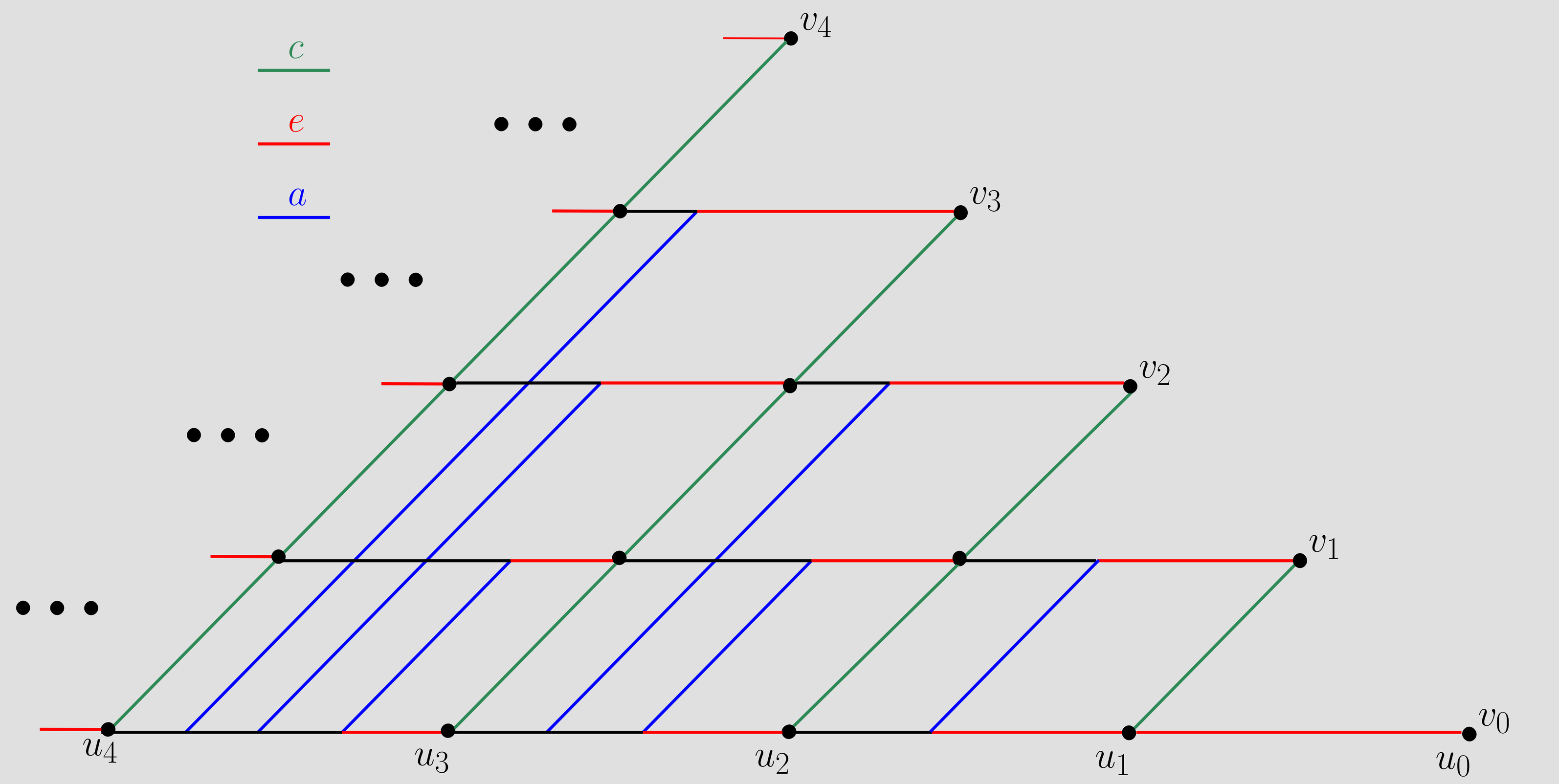
The Lindström-Gessel-Viennot (LGV) lemma and total positivity

- Suppose D is a LFAD with sources U, sinks V, and edge weights belonging to some commutative ring.
- Let $P_D := (P(u_n \to v_k))_{0\le n,k\le N}$ be the matrix with (n,k)-entry given by the sum over weighted paths from u_n to v_k.
- LGV lemma states that the weighted sum over families of nonintersecting paths from U to V is $|\det(P_D)|$.
- If D is fully compatible then P_D is coefficientwise totally positive.

A matrix with entries dependent on both n and k

- Entries of $T(a,c,0,e,0,0) = (T(n,k))_{n,k\ge0}$ satisfy a recurrence dependent on **both** n **and** k:
$$T(n,k) = [a(n-k)+c]\,T(n-1,k-1) + eT(n-1,k)$$
- The entries satisfy an alternative recurrence:
$$T(n,k) = cT(n-1,k-1) + \sum_{m=0}^{n-1}\binom{n-1}{m} a^m e T(n-1-m,k-1)$$
for $n \ge 1$, where $T(n,k) = 0$ if $n < 0$ or $k < 0$.
- Each (n,k)-entry counts **weighted partitions** of $[n+1]$ into $n-k+1$ **nonempty blocks**.

A planar network for $T(a,c,0,e,0,0)$



The matrix $T(a,c,0,e,0,0)$ is the path matrix corresponding to the above planar network.

Open problem

- The $T(a,c,d,e,f,g)$ conjecture is still wide open, we have tested it to 12×12 (this took 109 days of CPU time).

References

[1] X. Chen, B. Deb, A. Dyachenko, T. Gilmore, and A. Sokal: *Coefficientwise total positivity of some matrices defined by linear recurrences*, to appear in the proceedings of FPSAC2021.

[2] F. Brenti: *The applications of total positivity to combinatorics, and conversely*. In: Total Positivity and its Applications, edited by M. Gasca and C.A. Micchelli (Kluwer, Dordrecht, 1996), pp. 451–473.