Lattice Path Conference
21-25 June 2021
Presentation times for this poster:

Tuesday	1:30-2:30 pm
Thursday	6-7 pm

Enumeration of walks with small steps by winding angle

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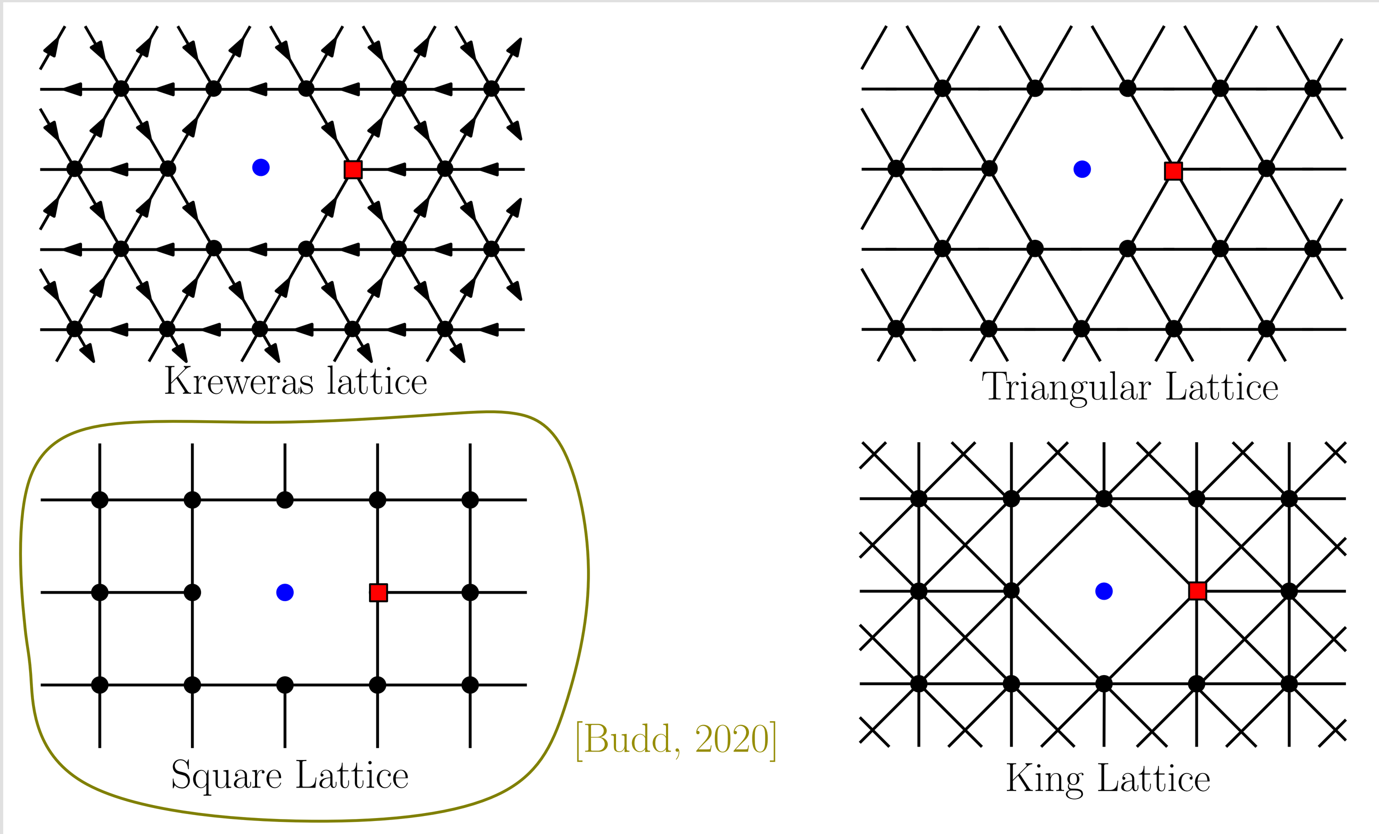
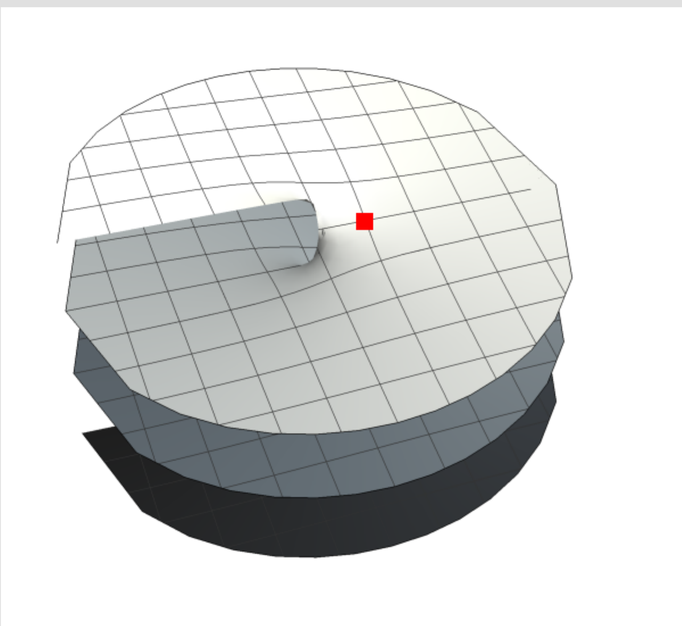
Nice lattices

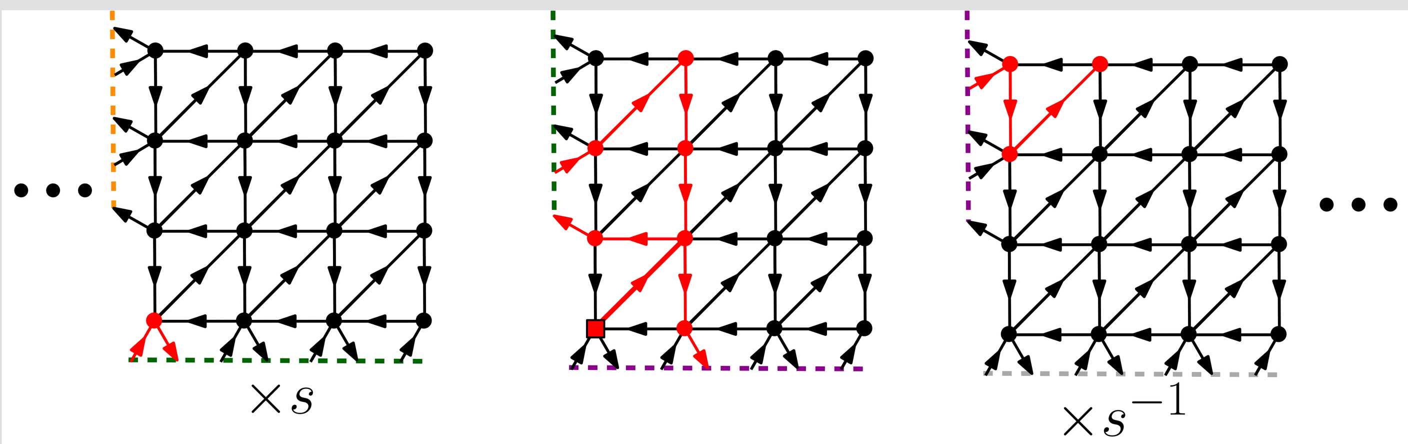
Aim: Count walks starting at ■ by end point and winding angle around •.



Aim: Similar method works for walks using any (non-singular) step set $S \subset \{-1,0,1\}^2$.

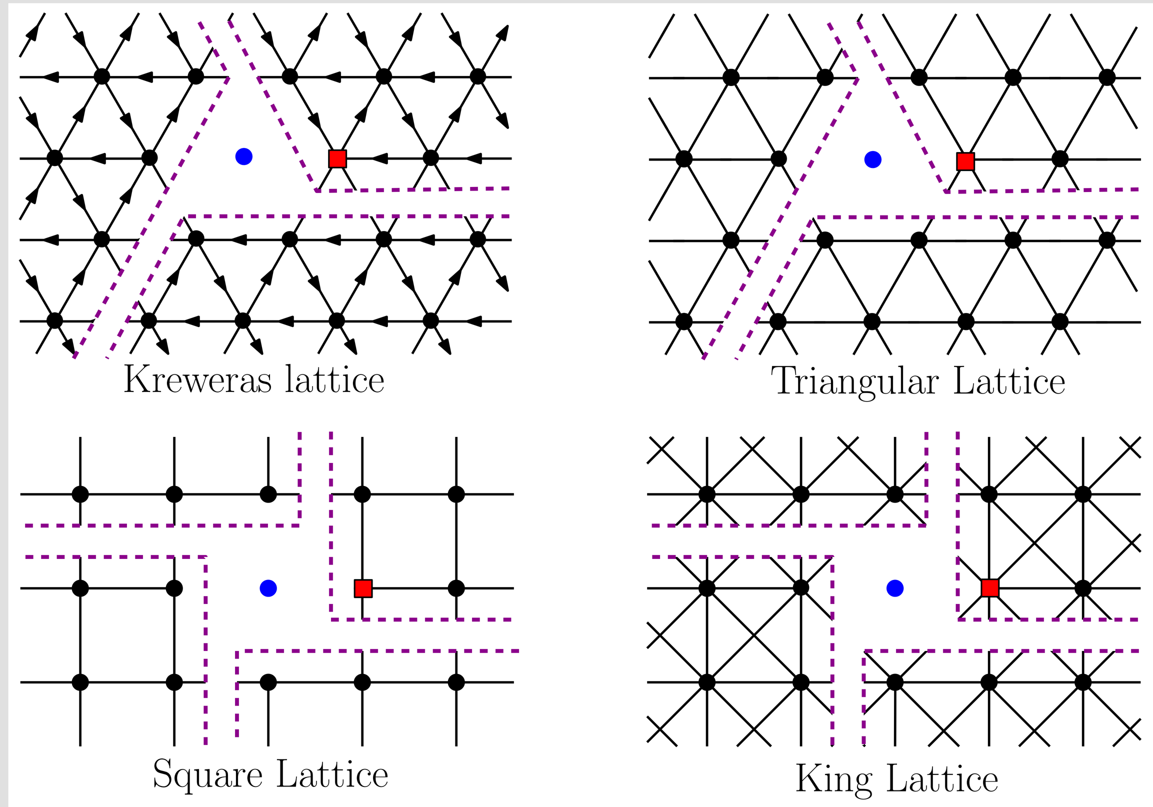
Equivalent formulations





Top left: square lattice by winding angle
Top right: dog walk on brick-work lattice by winding angle
Bottom: Kreweras lattice walk by winding angle (using decomposition into quadrants)

Decomposition into quadrants decompositions



Functional equation (on Kreweras lattice)

Definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} s^{n(p)}$

Example: Red path in bottom left diagram contributes $t^{11}xy^3s^{-1}$.

Functional equation:

$$Q(x,y) = 1 + txyQ(x,y) + t\frac{Q(x,y)-Q(0,y)}{x} + t\frac{Q(x,y)-Q(x,0)}{y} + e^{i\alpha}t\frac{Q(0,x)-Q(0,0)}{x} + s^{-1}ty^2Q(y,0).$$

The gf $E(t,s)$ counting excursions (starting and ending at ■) is

$$E(t,s) = Q(t,s,0,0) + Q(t,se^{\frac{2\pi i}{3}},0,0) + Q(t,se^{\frac{-2\pi i}{3}},0,0)$$

Solution

Define

$$T_k(u,q) = \sum_{n=0}^{\infty}(-1)^n(2n+1)^k q^{n(n+1)/2}(u^{n+1} - (-1)^k u^{-n}).$$

Let $q(t) \equiv q = t^3 + 15t^6 + 279t^9 + \cdots$ satisfy

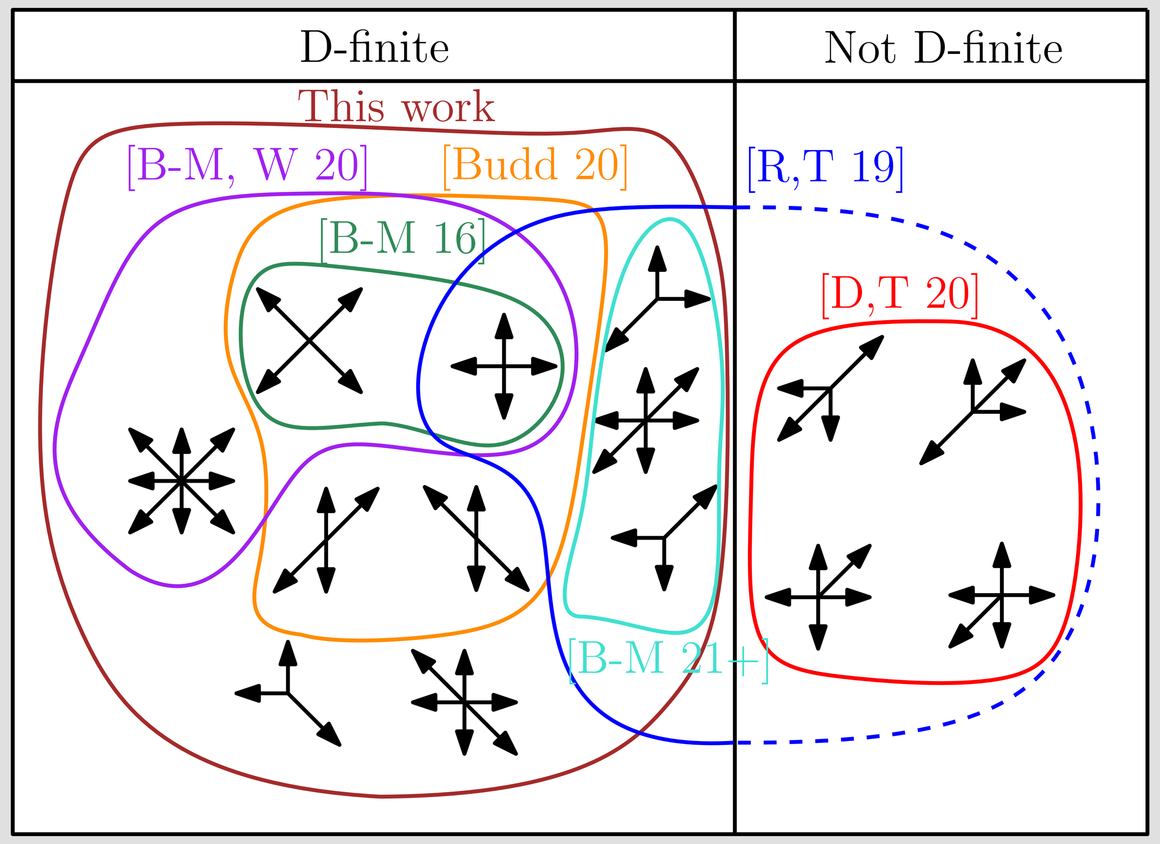
$$t = q^{1/3}\frac{T_1(1,q^3)}{4T_0(q,q^3) + 6T_1(q,q^3)}.$$

The gf for Kreweras-lattice excursions $E(t,s)$ is:

$$E(t,s) = \frac{q^{-\frac{2}{3}}}{3t(1-s^3)}\frac{T_0(q,q^3)^2}{T_1(1,q^3)^2}\left(-s(1-s)\frac{T_2(s,q)}{2T_0(s,q)} - se^{\frac{2\pi i}{3}}(1-se^{\frac{2\pi i}{3}})\frac{T_2(se^{\frac{2\pi i}{3}},q)}{2T_0(se^{\frac{2\pi i}{3}},q)} - se^{-\frac{2\pi i}{3}}(1-se^{-\frac{2\pi i}{3}})\frac{T_2(e^{-\frac{2\pi i}{3}},q)}{2T_0(e^{-\frac{2\pi i}{3}},q)}\right).$$

Corollary: 3/4-plane walks

Using the reflection principal we can count walks in the 3/4-plane using any of 9 different step sets $S \in \{-1,0,1\}^2$



[Bousquet-Mélou 16],[Raschel, Trotignon 19], [Budd 20], [Dreyfus, Trotignon 20], [Bousquet-Mélou, Wallner 20], [Bousquet-Mélou 21+].

Further results

- Method works for walks by winding angle with any non-singular step-set $S \in \{-1,0,1\}^2$. Solutions are no longer so explicit.
- **Coming soon:** Same idea solves walks with any non-singular step-set $S \in \{-1,0,1\}^2$ in the 3/4-plane.

References

[1] Bousquet-Mélou, M.: *Square lattice walks avoiding a quadrant.* Journal of Combinatorial Theory, Series A 144 (2016): 37-79.

[2] Bousquet-Mélou, M.; Wallner, M: *More models of walks avoiding a quadrant.* In Analysis of Algorithms (2020).

[3] Budd, T.: *Winding of simple walks on the square lattice.* Journal of Combinatorial Theory, Series A, 172 (2020), 105191.

[4] Dreyfus, T.; Trotignon, A.: *On the nature of four models of symmetric walks avoiding a quadrant.* Annals of Combinatorics (2021), 1-28.

[5] Raschel, K.; Trotignon, A.: *On Walks Avoiding a Quadrant.* The Electronic Journal of Combinatorics (2019): 3-31.